

ON MULTIPOINT PADÉ APPROXIMANTS WHOSE POLES ACCUMULATE ON CONTOURS THAT SEPARATE THE PLANE

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ABSTRACT. In this note we consider asymptotics of the multipoint Padé approximants to Cauchy integrals of analytic non-vanishing densities defined on a Jordan arc connecting -1 and 1 . We allow for the situation where the (symmetric) contour attracting the poles of the approximants does separate the plane.

1. INTRODUCTION AND MAIN RESULTS

Let L be a smooth Jordan arc joining -1 and 1 (and oriented in this way), and

$$w_L(z) := \sqrt{z^2 - 1}, \quad w_L(z) = z + \mathcal{O}(1), \quad \text{as } z \rightarrow \infty,$$

be the branch holomorphic in $\mathbb{C} \setminus L$. Let $\rho(z)$ be a function analytic in a “sufficiently large”, see Assumption 2 further below, neighborhood of L and

$$(1) \quad \hat{\rho}_L(z) := \frac{1}{2\pi i} \int_L \frac{\rho(s)}{s - z} \frac{ds}{w_{L+}(s)}, \quad z \notin L,$$

where $w_{L+}(s)$ is the trace of $w_L(z)$ on the left-hand side of L . Further, let $\{E_i\}_{i=1}^\infty$ be an *interpolation scheme*, i.e., a sequence of multi-sets in $D_L := \overline{\mathbb{C}} \setminus L$ such that each E_i consists of i not necessarily distinct nor finite points from D_L . The *multipoint Padé approximant to $\hat{\rho}_L(z)$ of type (m, n) associated with E_{m+n}* is a rational function $[m/n; E_{m+n}] \hat{\rho}_L(z) = (p_{m,n}/q_{m,n})(z)$ such that $\deg(p_{m,n}) \leq m$, $\deg(q_{m,n}) \leq n$, $q_{m,n}(z) \neq 0$, and

$$(2) \quad \frac{q_{m,n}(z) \hat{\rho}_L(z) - p_{m,n}(z)}{v_{m,n}(z)} = \mathcal{O}\left(z^{-\min\{m,n\}-1}\right) \quad \text{as } z \rightarrow \infty$$

as well as is analytic in D_L , where $v_{m,n}(z)$ is the monic polynomial vanishing at the finite elements of E_{m+n} according to their multiplicity¹, i.e.,

$$v_{m,n}(z) := \prod_{e \in E_{m+n}, |e| < \infty} (z - e).$$

It is well-known that linear system (2) is always solvable and that the solution corresponding to monic $q_{m,n}(z)$ of minimal degree is always unique. In what follows, we understand that $q_{m,n}(z)$ and $p_{m,n}(z)$ come from this unique solution. We shall call an approximant *diagonal* if $m = n$, in which case we simply write $p_n(z)$, $q_n(z)$, and $v_n(z)$.

The goal of this note is to explain the numerical results shown on Figure 1. To accomplish it, let us set

$$\varphi_L(z) := z - w_L(z), \quad z \in D_L,$$

which is an analytic function in D_L with a simple zero at infinity that satisfies $\varphi_{L+}(s)\varphi_{L-}(s) \equiv 1$, $s \in L$. Given an interpolation scheme $\{E_i\}_{i=1}^\infty$, define

$$(3) \quad B_i(\zeta) := \prod_{e \in E_i} \frac{\zeta - \varphi_L(e)}{1 - \zeta \varphi_L(e)}.$$

Clearly, each $B_i(\zeta)$ is a rational function with i zeros and i poles in $\overline{\mathbb{C}}$, counting multiplicities. All the zeros of $B_i(\zeta)$ belong to the interior of a Jordan curve $J^{-1}(L)$, where $J(\zeta) := (\zeta + 1/\zeta)/2$ is the Joukowski map ($\varphi_L(z)$ maps D_L conformally onto this domain and $J(\varphi_L(z)) = z$ for $z \in D_L$). Notice that if a multi-set E_i is conjugate-symmetric (i.e., $e \in E_i$ if and only if $\bar{e} \in E_i$), then $B_i(\zeta)$ is

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¹This definition yields one additional interpolation condition at infinity.

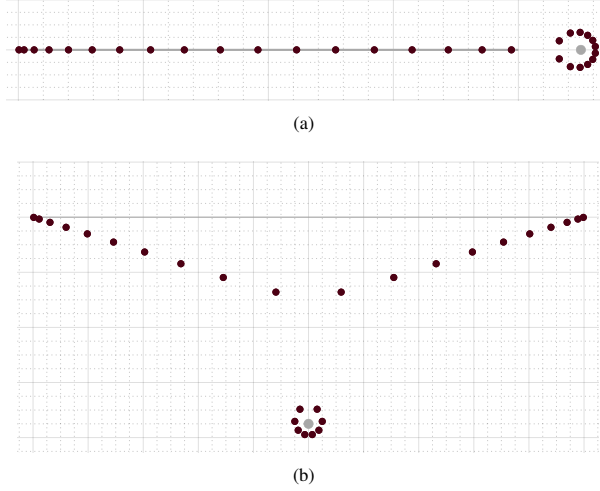


FIGURE 1. Poles of multipoint Padé approximants to $1/w_L(z)$ where panel (a): L is an arc connecting -1 to some $x_* > 5/4$ through the upper half-plane and then x_* to 1 through the lower half-plane and there are 48 interpolation conditions at infinity and 10 conditions at $5/4$; panel (b): L is a lower unit semi-circle and there are 48 interpolation conditions at infinity and 8 conditions at $-3i/4$.

a Blaschke product. Moreover, if $L = [-1, 1]$, then all the zeros of $B_i(\zeta)$, that is, points $\varphi_{[-1,1]}(e)$, $e \in E_i$, belong to the unit disk \mathbb{D} .

We shall consider only those interpolation schemes for which the level-lines $\{|B_i(\tau)| = 1\}$ have a definite asymptotic behavior. That is, we assume the following.

Assumption 1

Arc L and interpolation scheme $\{E_i\}_{i=1}^\infty$ in D_L are such that

- (i) there exist $M > 0$ and an integer $l_* \geq 0$ such that the functions $B_i(\zeta)$ from (3) satisfy

$$|B_i(\tau)| \leq M, \quad \tau \in \Gamma := \bigcup_{l=-l_*}^{l_*} \Gamma_l, \quad i \in \mathbb{N},$$

where Γ_l are pairwise disjoint smooth Jordan curves, Γ_0 contains ± 1 , Γ_l , $l > 0$, belong to the interior of Γ_0 , and $\Gamma_{-l} = \{\tau : \tau^{-1} \in \Gamma_l\}$, $0 \leq l \leq l_*$;

- (ii) in each connected component of the complement of Γ it holds locally uniformly that either $|B_i(\zeta)| \rightarrow 0$ as $i \rightarrow \infty$ or $|B_i(\zeta)| \rightarrow \infty$ as $i \rightarrow \infty$.

It follows from (3) that $B_i(1/\zeta) = 1/B_i(\zeta)$ and therefore $|B_i(\tau)| = 1$ if and only if $|B_i(1/\tau)| = 1$. This explains why $\pm 1 \in \Gamma$ and why each connected component of Γ must be either invariant under the map $\tau \mapsto 1/\tau$ (there is only one such component by the assumption, namely, Γ_0), or is mapped into another connected component by this map. This symmetry also shows that $|B_i(\tau)| \geq M^{-1} > 0$, $\tau \in \Gamma$, $i \in \mathbb{N}$, and of course $M \geq 1$.

Let $\Delta := J(\Gamma)$. We shall say that Δ is a *symmetric contour corresponding to the interpolation scheme* $\{E_i\}_{i=1}^\infty$. We write $\Delta = \bigcup_{l=0}^{l_*} \Delta_l$, where $\Delta_l := J(\Gamma_l)$. Notice that Δ_0 is a Jordan arc connecting -1 and 1 , and each Δ_l , $l > 0$, is a Jordan curve that contains Δ_0 in its exterior.

One way to think about the symmetric contours defined above is through the level lines of functions $B_i(\zeta)$ from (3). Given a contour L , fix $E_N := \{e_1, \dots, e_N\}$, a collection of not necessarily distinct points in D_L . Define $E_0 := \emptyset$, $E_i := \{e_1, \dots, e_i\}$ for $i < N$, and $E_{kN+i} = (kE_N) \cup E_i$, $k \geq 1$, $0 \leq i < N$, where kE_N is the multi-set consisting of k copies of E_N . Clearly, it holds that

$$B_{kN+i}(\zeta) = B_N^k(\zeta) B_i(\zeta)$$

for any $k \geq 0$ and $0 \leq i < N$. Let $\Gamma := \{\tau : |B_N(\tau)| = 1\}$ be the 1-level line of $B_N(\zeta)$. As mentioned before, it follows immediately from (3) that Γ is invariant under the map $\tau \mapsto 1/\tau$. It also

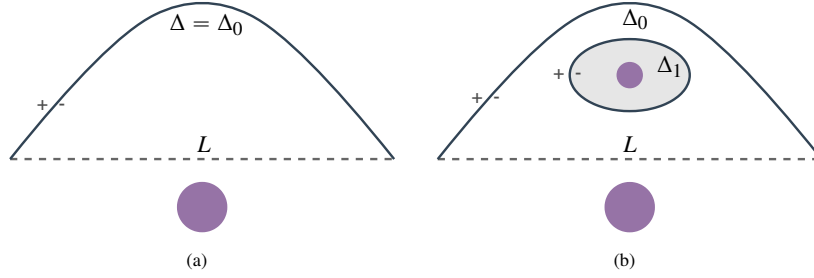


FIGURE 2. Darker filled circles represent interpolation points (bigger circle represents more interpolation conditions at the point), dashed lines represent L , solid lines represent Δ , and lightly shaded regions represent D_{Δ}^{∞} . Panel (a): interpolation points create an external field that pushes L up to Δ . Panel (b): interpolation points below L push it up, interpolation points above L push it down, but create weaker external field resulting in L going through them while simultaneously forming a barrier Δ_1 .

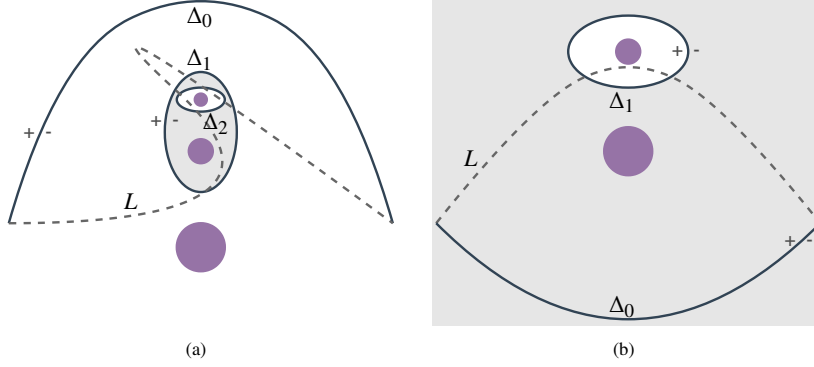


FIGURE 3. This is a continuation of Figure 2. Panel (a): top and bottom groups of interpolation points create an external field that pushes L up while the middle group pushes L down, due to different strength of the components of the external field generated by these groups, two barriers are created. Panel (b): interpolation points below L create an external field that pushes L up all the way through ∞ to the displayed position of Δ_0 , interpolation points above L create a weaker external field that results in a barrier Δ_1 .

follows from the choice of the sets E_i , $i \leq N$, that there exists $M \geq 1$ such that $M^{-1} \leq |B_i(\tau)| \leq M$ for $\tau \in \Gamma$ and $i < N$. Thus, it necessarily holds that $M^{-1} \leq |B_i(\tau)| \leq M$ for all $i \in \mathbb{N}$. It is also clear that condition (ii) of Assumption 1 is satisfied. Hence, if Γ consists of *non-intersecting* Jordan curves, it satisfies all the requirements of Assumption 1. The projection of Γ by the Joukovsky map $J(\zeta)$ is then an admissible symmetric contour Δ that we work with in this note. In general, there are no reasons to exclude the cases of contours Γ with points of self-intersection. We do it here only to simplify the analysis of the asymptotic behavior of the approximants. In particular, Figures 4 and 5 depict possible contours Δ obtained from the above construction for sets E_N containing only two distinct element in different proportions (one of them being infinity). In general, sets Γ from Assumption 1 are not level lines of any particular rational function of the form (3), but certain limits of such level lines.

Another way to think about the symmetric contours defined above is through heuristics of potential theory. One can think of L as an elastic band that can freely move in the complex plane except for the fixed points ± 1 . Assume also that, say positive, unit charge can freely float along L . Somewhat unrealistically suppose also that the band can attach to itself to form a loop and then snap into disjoint pieces at the point where the loop was formed. Distribute a unit, also positive, charge among the points of the set E_i , where each point gets the amount of charge proportional to its multiplicity in E_i . These charges create an external field that makes L change its location. If L settles down to some

collection of bands in the limit as $i \rightarrow \infty$, this is exactly a symmetric contour Δ defined above. This heuristics is the one used in captions to Figures 2 and 3. In particular, if L is the lower unit semicircle, then the unit charge at infinity will move it to $\Delta = [-1, 1]$. If we place one fifth of the charge at point $-3i/4$ and keep the rest of it at infinity, then we will get Δ as on Figure 5(a). By decreasing the charge at $-3i/4$, we will make L wrap around $-3i/4$ until it forms a loop around $-3i/4$ when $1/6$ the charge is placed there, Figure 5(b). If the charge at $-3i/4$ is further decreased, the band L snaps into two pieces, one band Δ_0 connecting -1 and 1 and one loop Δ_1 around $-3i/4$, see Figure 5(c).

Let us now continue with the assumption on the weight function $\rho(s)$ in (1). To this end, we introduce an orientation for all the connected components of Δ . We orient Δ_0 from -1 to 1 . To orient $\Delta_l, l > 0$, put

$$(4) \quad b_i(z) := \prod_{e \in E_i} \frac{\varphi_{\Delta_0}(z) - \varphi_L(e)}{1 - \varphi_{\Delta_0}(z)\varphi_L(e)} = B_i(\zeta),$$

where $z = J(\zeta)$ and ζ belongs to the interior domain of Γ_0 . Let $D_\Delta := \overline{\mathbb{C}} \setminus \Delta$ and denote by D_Δ^0 and D_Δ^∞ the parts of D_Δ where the functions $|b_i(z)|$ converge to 0 and diverge to ∞ , respectively ($D_\Delta = D_\Delta^0 \cup D_\Delta^\infty$), see Figures 2 and 3. Notice that each curve $\Delta_l, l > 0$, is a part of the boundary of both D_Δ^0 and D_Δ^∞ and we orient it so that D_Δ^0 remains on the left as Δ_l is traversed in the positive direction (this orientation allows us to distinguish + side (left) and - side (right) of Δ_l in a manner convenient for our purposes; our choice is not related to the location of the interior domain of Δ_l , it might end up on any side of the curve, see, for example, Figure 3). Let us write $\overline{\mathbb{C}} \setminus (L \cup \Delta_0) =: U_u \cup U_b$, where U_u is the unbounded component of $\overline{\mathbb{C}} \setminus (L \cup \Delta_0)$. We shall also set $U_{l,b}$ to be the interior of $\Delta_l, l > 0$.

Assumption 2

It is assumed that $\rho(z)$ is analytic in a domain that contains each $\overline{U_{l,b}}, l > 0$, as well as $\overline{U_b}$ (resp. $\overline{U_u}$) when $\infty \in D_\Delta^0$ (resp. $\infty \in D_\Delta^\infty$). It is also assumed that $\rho(z)$ is non-vanishing on Δ_0 and $\cup_{l>0} \overline{U_{l,b}}$ (thus, it has zero winding number on each $\Delta_l, l > 0$).

The situation $\infty \in D_\Delta^0$ is schematically depicted on Figures 2(a,b) and 3(a). It means that the unbounded component of the complement of D_Δ is a part of D_Δ^0 . The situation $\infty \in D_\Delta^\infty$ is schematically depicted on Figure 3(b) and represents geometries where the unbounded component of the complement of D_Δ is a part of D_Δ^∞ . Under Assumption 2 the following proposition holds.

Proposition 1

For all i large enough each element of E_i has a neighborhood in which $\hat{\rho}_L(z) = \hat{\rho}_\Delta(z)$, where

$$(5) \quad \hat{\rho}_\Delta(z) := \frac{1}{2\pi i} \int_\Delta \frac{\rho(s) ds}{s - z w(s)}, \quad z \in D_\Delta,$$

we set $\varsigma := 1$ when $\infty \in D_\Delta^0$ and $\varsigma := -1$ when $\infty \in D_\Delta^\infty$, as well as

$$(6) \quad w(s) := \begin{cases} \varsigma w_{\Delta_0+}(s), & s \in \Delta_0, \\ w_{\Delta_0}(s), & s \in \cup_{l>0} \Delta_l. \end{cases}$$

In particular, $[m/n; E_{m+n}]_{\hat{\rho}_L}(z) = [m/n; E_{m+n}]_{\hat{\rho}_\Delta}(z)$ for all $m + n$ large enough.

In view of the above proposition we can now define functions of the second kind as

$$(7) \quad R_{m,n}(z) := \frac{q_{m,n}(z)\hat{\rho}_\Delta(z) - p_{m,n}(z)}{v_{m,n}(z)}, \quad z \in D_\Delta$$

(in the diagonal case we shall simply denote them by $R_n(z)$). It follows from (2) and Proposition 1 that $R_{m,n}(z)$ is analytic in the domain of its definition and vanishes at infinity with order at least $\min\{m, n\} + 1$.

Proposition 2

With $w(s)$ given by (6), it holds that

$$R_{m,n}(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{q_{m,n}(s)}{v_{m,n}(s)} \frac{\rho(s)}{w(s)} \frac{ds}{s-z}, \quad z \in D_{\Delta}.$$

Recall that $\rho(s)$ is assumed to be non-vanishing on Δ with a zero winding number on each Δ_l , $l > 0$. Thus, we can define $\log \rho(s)$ continuously on Δ . Let

$$(8) \quad S_{\rho}(z) := \exp \left\{ \frac{w_{\Delta_0}(z)}{2\pi i} \int_{\Delta} \frac{\log \rho(s)}{z-s} \frac{ds}{w(s)} \right\}, \quad z \in D_{\Delta},$$

be the Szegő function of $\rho(s)$. That is, $S_{\rho}(z)$ is holomorphic and non-vanishing in D_{Δ} with continuous traces on Δ that satisfy

$$(9) \quad \begin{cases} S_{\rho+}(s)S_{\rho-}(s) = \rho^{-\zeta}(s), & s \in \Delta_0, \\ S_{\rho+}(s) = S_{\rho-}(s)/\rho(s), & s \in \Delta_l, l > 0. \end{cases}$$

Proposition 3

Let $b_{2n}(z)$ be given by (4) and $S_{\rho}(z)$ be given by (8). Then there exist choices of the branch of the square root so that

$$(10) \quad \Psi_n(z) := \begin{cases} \sqrt{v_n(z)/b_{2n}(z)} S_{\rho}(z), & z \in D_{\Delta}^0, \\ \sqrt{v_n(z)b_{2n}(z)}/S_{\rho}(z), & z \in D_{\Delta}^{\infty}, \end{cases}$$

is an analytic and non-vanishing function in D_{Δ} that satisfies

$$(11) \quad \begin{cases} \Psi_n(z) = \gamma_n^{-1} z^n + O(z^{n-1}) & \text{as } z \rightarrow \infty, \\ \Psi_{n+}(s)\Psi_{n-}(s) = (v_n/\rho)(s), & s \in \Delta, \end{cases}$$

for some non-zero constant γ_n .

The following theorem constitutes the main result of this note.

Theorem 1

Let L , $\{E_i\}_{i=1}^{\infty}$, and $\rho(z)$ be such that Assumptions 1 and 2 are satisfied. Further, let Δ be the symmetric contour corresponding to the interpolation scheme $\{E_i\}_{i=1}^{\infty}$ and $p_n(z)/q_n(z) = [n/n; E_{2n}]_{\hat{\rho}_L}(z)$ be the sequence of diagonal multipoint Padé approximants of $\hat{\rho}_L(z)$ associated with $\{E_i\}_{i=1}^{\infty}$, see (2). Then it holds uniformly on closed subsets of D_{Δ} that

$$(12) \quad q_n(z) = \gamma_n(1 + o(1))\Psi_n(z)$$

and

$$(13) \quad R_n(z) = \frac{\gamma_n(1 + o(1))}{w_{\Delta_0}(z)\Psi_n(z)} \begin{cases} 1, & z \in D_{\Delta}^0, \\ -1, & z \in D_{\Delta}^{\infty}. \end{cases}$$

In particular, it holds uniformly on closed subsets of D_{Δ} that

$$(14) \quad \hat{\rho}_{\Delta}(z) - [n/n; E_{2n}]_{\hat{\rho}_L}(z) = \frac{1 + o(1)}{w_{\Delta_0}(z)} \begin{cases} b_{2n}(z)S_{\rho}^{-2}(z), & z \in D_{\Delta}^0, \\ -b_{2n}^{-1}(z)S_{\rho}^2(z), & z \in D_{\Delta}^{\infty}. \end{cases}$$

Let us place the above theorem, which covers contours on Figures 4(d) and 5(c), in the context of recent and not so distant results on attracting curves of the poles of Padé approximants. When the approximants are classical (i.e., all interpolation points are at infinity) and the approximated functions are of hyperelliptic type (Cauchy integrals of densities of the form $\rho(s)/w_{L+}(s)$, where $w_L^2(s)$ is a polynomial of even degree), it was recognized by Nuttall and Singh [13] that the pole attracting contour Δ is a projection of a certain level line on the Riemann surface of $w_L(z)$ to the complex plane (such Δ 's do not separate the plane). In this case an analog of Theorem 1 was obtained in [23] by

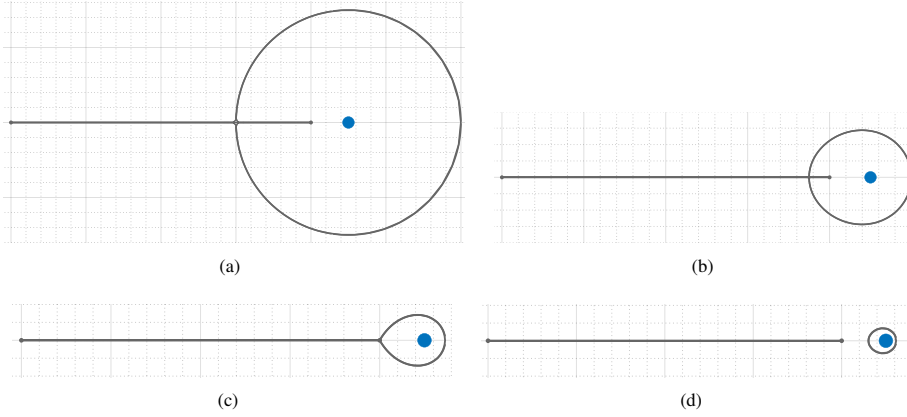


FIGURE 4. Symmetric contours Δ that correspond to L that connects -1 to some x_* $> 5/4$ through the upper half-plane and then x_* to 1 through the lower half-plane and interpolation schemes where the interpolation conditions are equally distributed between ∞ and $5/4$ (panel (a)) or there are twice (panel (b)), three times (panel (c)), or four times (panel (d)) more interpolation conditions at ∞ than at $5/4$ (the disk on all of the figures). The plots are obtained by numerically plotting the level-line $\{|B_i(\tau)| = 1\}$.

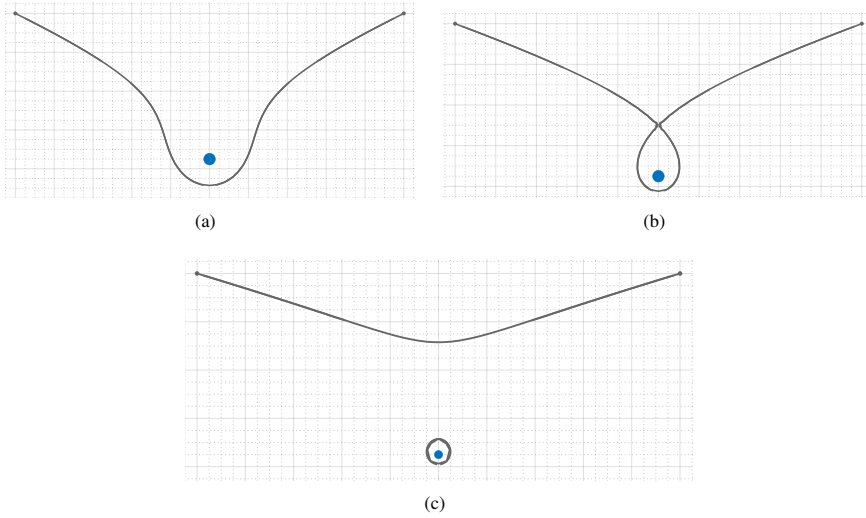


FIGURE 5. Symmetric contours Δ that correspond to L being a lower unit semi-circle and interpolation schemes where there are four (panel (a)), five (panel (b)), or six (panel (c)) times more interpolation conditions at ∞ than at $-3i/4$ (the disk on all of the figures).

the author (see also [18, 19, 3]). This approach was later applied to multipoint Padé approximants by Baratchart and the author in [2] when Δ is Jordan arc, which covers the contour on Figure 5(a) (see also [6, 17]) and by the author [21] in the full generality of [13] for contours that do not separate the plane. So, this work is a natural continuation of [2] (Δ_0 is a Jordan arc in Theorem 1) and the author is currently preparing generalization of [21] to the case of contours that do disconnect the plane. When the approximated function is multi-valued but not hyperelliptic, the approach of [13] no longer works. However, in a series of pathbreaking papers [14, 15, 16] Stahl showed that attracting contours for classical Padé approximants can be identified as the ones having minimal logarithmic capacity among all possible branch cuts for the approximated function. Strong asymptotics of classical Padé approximants for "generic" geometries then was obtained by Aptekarev and the author [1]. Stahl's approach was extended to the set up that includes multipoint Padé approximants by Gonchar and

Rakhmanov [12] (contours are not allowed to separate the plane). Further extension of Stahl's ideas to contours that are allowed to separate the plane was obtained by Buslaev [4, 5]. Buslaev's existence theorem covers all possible contours arising from interpolation schemes with finitely many points (in particular, it covers Figures 4 and 5). However, his convergence results consider only those geometries where each component of $\mathbb{C} \setminus \Delta$ contains a branch point of the approximated function as on Figure 4(a,b) (they are also concerned with weak asymptotics). Some strong asymptotics analogs of Buslaev convergence results were recently obtained by the author [22]. In this sense the current note is a continuation of the work done in [22].

2. PROOFS

Let us recall one of the Plemelj-Sokhotski formulae [11, Section I.4.2] that states that

$$\Phi_+(s) - \Phi_-(s) = \phi(s), \quad s \in I^\circ, \quad \Phi(z) := \frac{1}{2\pi i} \int_I \frac{\phi(s) ds}{s - z},$$

where I is a smooth oriented Jordan arc (closed or open), I° is its interior, and $\phi(s)$ is a function that is Hölder smooth on any subarc of I° . It also holds that $\Phi_\pm(s)$ are Hölder smooth functions on any subarc of I° . Recall further Yamashita-Bagemihl analytic continuation principle, see [20, Theorem 3]: if domains D_1, D_2 are disjoint, I is an open analytic arc that belongs to the boundary of both domains, $f_i(z)$ belongs to the Hardy space $H_1(D_i)$, $i \in \{1, 2\}$ (in particular, bounded in D_i), and

$$\lim_{L_1(s) \ni z \rightarrow s} f_1(z) = \lim_{L_2(s) \ni z \rightarrow s} f_2(z) \neq \infty$$

for almost every $s \in I$, where $L_i(s)$ is some arc in D_i terminating at s , then $f_2(z)$ is the analytic continuation of $f_1(z)$ across L from D_1 into D_2 .

Proof of Proposition 1. Denote the unbounded component of D_Δ by Ω . We shall consider only those indices i that are large enough so that $b_i(e) = 0$ when $e \in E_i \cap D_\Delta^0$ and $b_i(e) = \infty$ when $e \in E_i \cap D_\Delta^\infty$. Fix such an index i . Since E_i is finite and L can be perturbed in (1) without changing the relevant values of $\hat{\rho}_L(z)$ in (2), we can assume that L and Δ_0 have at most finitely many points in common and these arcs intersect at those points transversally.

According to Plemelj-Sokhotski formulae it follows from (1) that

$$\hat{\rho}_{L+}(s) - \hat{\rho}_{L-}(s) = \rho(s)/w_{L+}(s), \quad s \in L.$$

Therefore, the analytic continuation of $\hat{\rho}_L(z)$ across L is given by $\hat{\rho}_L(z) - \rho(z)/w_L(z)$. Define $\hat{\rho}_{\Delta_0}(z)$ as in (1) with L replaced by Δ_0 . Note that $w_L(z) = -w_{\Delta_0}(z)$ for $z \in U_b$ and $w_L(z) = w_{\Delta_0}(z)$ for $z \in U_u$. Respectively, it also holds that $\varphi_{\Delta_0}(z) = 1/\varphi_L(z)$ for $z \in U_b$ and $\varphi_{\Delta_0}(z) = \varphi_L(z)$ for $z \in U_u$.

Assume that $\Omega \subseteq D_\Delta^0$. Then

$$(15) \quad \hat{\rho}_{\Delta_0}(z) = \begin{cases} \hat{\rho}_L(z) + \rho(z)/w_{\Delta_0}(z), & z \in U_b, \\ \hat{\rho}_L(z), & z \in U_u. \end{cases}$$

Indeed, the right- and left-hand sides of (15) are analytic in $\overline{\mathbb{C}} \setminus \Delta_0$, vanish at infinity, and have smooth traces on Δ_0 . The jump of the right-hand side across Δ_0 is equal to $\rho(s)/w_{\Delta_0+}(s)$ when U_b lies to the left of Δ_0 and again $-\rho(s)/w_{\Delta_0-}(s) = \rho(s)/w_{\Delta_0+}(s)$ when U_b lies to the right of Δ_0 . That is, in any case it matches the jump of $\hat{\rho}_{\Delta_0}(z)$ across Δ_0 . Hence, the difference of the left- and right-hand sides of (15) is analytic in $\mathbb{C} \setminus \{\pm 1\}$ by the analytic continuation principle. As it vanishes at infinity and can have at most square root singularities at ± 1 by [11, Section I.8.4], it must be identically zero. Let $e \in E_i \cap \Omega$. As $\Omega \subseteq D_\Delta^0$, $b_i(e) = 0$ and therefore $e \in U_u$. In this case

$$(16) \quad \hat{\rho}_\Delta(z) = \hat{\rho}_{\Delta_0}(z) = \hat{\rho}_L(z)$$

for z around e as claimed, where the first conclusion follows from Assumption 2 and Cauchy theorem. Let now $e \in E_i \cap D_\Delta^0$, $e \notin \Omega$. Again, it must hold that $e \in U_u$. Since each curve Δ_l , $l > 0$, lies on the border of both D_Δ^0 and D_Δ^∞ , e is an interior point of even number of such curves. As orientations of these curves alternate between clockwise and counter-clockwise and

$$(17) \quad \hat{\rho}_{\Delta_l}(z) := \frac{1}{2\pi i} \int_{\Delta_l} \frac{\rho(s)}{s - z} \frac{ds}{w_{\Delta_0}(s)} = \pm \frac{\rho(z)}{w_{\Delta_0}(z)}, \quad z \in U_{l,b},$$

(the choice of the sign depends on the orientation of Δ_l) by Assumption 2 and Cauchy integral formula, we again get that (16) holds. Now, when $e \in E_i \cap D_\Delta^\infty$, $b_i(z)$ must have a pole at e , that

is, $\varphi_{\Delta_0}(e) = 1/\varphi_L(e)$, which implies that $e \in U_b$. Let Δ_{l_e} be the curve that borders Ω and contains e in its interior. Then Δ_{l_e} is clockwise oriented. There could be other curves Δ_l containing e in their interior, but they are even in number and their net contribution to $\hat{\rho}_\Delta(z)$ around e is zero due to alternating orientations and (17). Hence,

$$\hat{\rho}_\Delta(z) = \hat{\rho}_{\Delta_0}(z) + \hat{\rho}_{\Delta_{l_e}}(z) = \hat{\rho}_L(z) + \rho(z)/w_{\Delta_0}(z) + \hat{\rho}_{\Delta_{l_e}}(z) = \hat{\rho}_L(z)$$

for z around e as claimed, where we used (15) and (17) one more time.

Assume now that $\Omega \subseteq D_\Delta^\infty$. Since $w_L(z) = w_{\Delta_0}(z)$ for $z \in U_u$, reasoning as in (15) now implies that

$$(18) \quad -\hat{\rho}_{\Delta_0}(z) = \begin{cases} \hat{\rho}_L(z), & z \in U_b, \\ \hat{\rho}_L(z) - \rho(z)/w_{\Delta_0}(z), & z \in U_u. \end{cases}$$

Let $e \in E_i \cap \Omega$. Since $\Omega \subseteq D_\Delta^\infty$, $b_i(z)$ has a pole at e and therefore $e \in U_b$. Recall (6). Since $\rho(z)$ is analytic on each $\bar{U}_{l,b}$, we get as in (16) that

$$(19) \quad \hat{\rho}_\Delta(z) = -\hat{\rho}_{\Delta_0}(z) = \hat{\rho}_L(z)$$

for z around e . If $e \in E_i \cap (D_\Delta^\infty \setminus \Omega)$, then $e \in U_b$ and it lies interior to even number of curves Δ_l , $l > 0$, with consecutively alternating orientations. Then the net contribution to $\hat{\rho}_\Delta(z)$ by these curves is zero and (19) remains valid. Finally, if $e \in E_i \cap D_\Delta^0$, then $e \in U_u$ and it lies interior to odd number of curves Δ_l , $l > 0$, with consecutively alternating orientations starting (and ending) with the counter-clockwise one. The desired claim now follows from (17) and (18). \square

Proof of Proposition 2. It follows from (5), (7), Plemelj-Sokhotski formulae, and the analytic continuation principle that the difference

$$R_{m,n}(z) - \frac{1}{2\pi i} \int_\Delta \frac{q_{m,n}(s) \rho(s)}{v_{m,n}(s) w(s)} \frac{ds}{s-z}$$

is analytic in $\bar{\mathbb{C}} \setminus \{\pm 1\}$ and is vanishing at infinity. It is also follows from the known behavior of Cauchy integrals around the endpoints of contours of integration [11, Section I.8.4] that it can have at most square root singularity at ± 1 . Hence, it must be identically zero. \square

Proof of Proposition 3. Write $v_n(z) = v_{n,0}(z)v_{n,\infty}(z)$, where $v_{n,0}(z)$ and $v_{n,\infty}(z)$ are monic polynomials that vanish at the zeros and poles of $b_{2n}(z)$ in \mathbb{C} , respectively. It follows from (4) that

$$b_{2n}(z) \frac{v_{n,\infty}(z)}{v_{n,0}(z)} = \left(c_n \zeta^{n-\deg(v_{n,\infty})} \frac{\prod_{e \in E_{2n,\infty}} (\zeta - \varphi_L(e))}{\prod_{e \in E_{2n,0}} (1 - \zeta \varphi_L(e))} \right)^2 := T_n^2(\zeta),$$

where $z = J(\zeta)$, ζ belongs to the interior domain of Γ_0 , $E_{2n,0}$ and $E_{2n,\infty}$ are the multi-sets of zeros of $v_{n,0}(z)$ and $v_{n,\infty}(z)$ (i.e., counting multiplicities), and

$$c_n^2 := (-2)^{\deg(v_{n,0}) - \deg(v_{n,\infty})} \prod_{e \in E_{2n,0}} \varphi_L(e) \prod_{e \in E_{2n,\infty}} \varphi_L^{-1}(e)$$

whose square root we fix arbitrarily. Observe that $T_n(\zeta)$ is a non-vanishing holomorphic function in the interior of Γ_0 except for a possible zero or pole at the origin. In any case, we can define

$$\sqrt{(b_{2n}v_{n,\infty}/v_{n,0})(z)} := T_n(\zeta),$$

where $z = J(\zeta)$ and ζ belongs to the interior domain of Γ_0 , which is a holomorphic and non-vanishing function in $\mathbb{C} \setminus \Delta_0$ that has either a pole or a zero at infinity. Then we have that

$$(20) \quad \sqrt{(b_{2n}v_{n,\infty}/v_{n,0})(s)}_+ \sqrt{(b_{2n}v_{n,\infty}/v_{n,0})(s)}_- = T_n(\tau)T_n(1/\tau) = (v_{n,\infty}/v_{n,0})(s)$$

for $s = J(\tau) \in \Delta_0$, i.e., $\tau \in \Gamma_0$. Now we can define

$$(21) \quad \sqrt{v_n(z)/b_{2n}(z)} := v_{n,\infty}(z)T_n^{-1}(\zeta) \quad \text{and} \quad \sqrt{v_n(z)b_{2n}(z)} := v_{n,0}(z)T_n(\zeta),$$

where $z = J(\zeta)$ and ζ belongs to the interior domain of Γ_0 . Relations (11) now easily follow from (9), (20), and (21) (observe that it necessarily holds that $E_{2n} \subset \mathbb{C}$ when $\infty \in D_\Delta^\infty$, in which case $2n = \deg(v_n) = \deg(v_{n,0}) + \deg(v_{n,\infty})$). \square

Proof of Theorem 1. For brevity, let us set

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To prove the theorem, we follow by now classical approach of Fokas, Its, and Kitaev [9, 10] connecting orthogonal polynomials to matrix Riemann-Hilbert problems and then utilizing the non-linear steepest descent method of Deift and Zhou [8].

Step 1. Consider the following 2×2 Riemann-Hilbert problem (RHP- \mathbf{Y}):

- (a) $\mathbf{Y}(z)$ is analytic in $\overline{\mathbb{C}} \setminus \Delta$ and $\lim_{z \rightarrow \infty} \mathbf{Y}(z) z^{-n\sigma_3} = \mathbf{I}$;
 (b) $\mathbf{Y}(z)$ has continuous traces on $\Delta \setminus \{\pm 1\}$ that satisfy

$$\mathbf{Y}_+(s) = \mathbf{Y}_-(s) \begin{pmatrix} 1 & (\rho/(v_n w))(s) \\ 0 & 1 \end{pmatrix}, \quad s \in \Delta \setminus \{\pm 1\},$$

where $w(s)$ was defined in (6);

- (c) it holds that $\mathbf{Y}(z) = \mathcal{O} \begin{pmatrix} 1 & |z - e|^{-1/2} \\ 1 & |z - e|^{-1/2} \end{pmatrix}$ as $D_\Delta \ni z \rightarrow e \in \{\pm 1\}$.

To connect RHP- \mathbf{Y} to the polynomials $q_n(z)$, recall (7). If it holds that

$$(22) \quad \deg(q_n) = n \quad \text{and} \quad R_{n+1, n-1}(z) = k_n^{-1} z^{-n} [1 + o(1)] \quad \text{as} \quad z \rightarrow \infty$$

for a non-zero finite constant k_n , then RHP- \mathbf{Y} is solved by

$$(23) \quad \mathbf{Y}(z) = \begin{pmatrix} q_n(z) & R_n(z) \\ k_n q_{n+1, n-1}(z) & k_n R_{n+1, n-1}(z) \end{pmatrix}.$$

Conversely, if RHP- \mathbf{Y} is solvable, then its solution necessarily has form (23) and (22) is satisfied. Indeed, if (22) holds, then so is RHP- \mathbf{Y} (a) by (2) and since $\deg(q_{n+1, n-1}) < n$. The fact that RHP- \mathbf{Y} (b) is fulfilled follows from Proposition 2 and Plemelj-Sokhotski formulae. Finally, RHP- \mathbf{Y} (c) holds due to the known behavior of Cauchy integrals around endpoints of contours of integration [11, Section I.8.4]. Conversely, if $\mathbf{Y}(z)$ is a solution of RHP- \mathbf{Y} , then its unique (if $\mathbf{Y}_1(z)$ and $\mathbf{Y}_2(z)$ are solutions, then their determinants are identically equal to 1 and $\mathbf{Y}_1(z)\mathbf{Y}_2^{-1}(z)$ must be entire and equal to \mathbf{I} at infinity, i.e., equal to \mathbf{I} everywhere). Furthermore, $q(z)$, the $(1, 1)$ -entry of $\mathbf{Y}(z)$, must be a monic polynomial of degree exactly n by RHP- \mathbf{Y} (a,b) and $R(z)$, the $(1, 2)$ -entry, must have an integral representation as in Proposition 2 with $q_n(z)$ replaced by $q(z)$ by RHP- \mathbf{Y} (b,c) and the analytic continuation principle. Replace the first row of $\mathbf{Y}(z)$ with

$$(q(z) + q_n(z) \quad R(z) + R_n(z)) \quad \text{or} \quad ((q(z) + q_n(z))/2 \quad (R(z) + R_n(z))/2)$$

depending on whether $\deg(q_n) < n$ or $\deg(q_n) = n$. Thus obtained matrix is still a solution of RHP- \mathbf{Y} and therefore must be equal to $\mathbf{Y}(z)$. That is, $q_n(z) = q(z)$ as claimed. The form of the second row can be deduced similarly.

Step 2. Let J_0 be a smooth Jordan curve encircling Δ_0 such that $\rho(z)$ is non-vanishing and analytic on the closure of Ω_0 , the interior domain of J_0 with Δ_0 removed. We orient J_0 clockwise when $\infty \in D_\Delta^0$ and counter-clockwise otherwise. Further, let $J_l^+ \subset D_\Delta^0$ and $J_l^- \subset D_\Delta^\infty$, $l > 0$, be Jordan curves such that Δ_l contains one of them in its interior and another one in its exterior, and $\rho(z)$ is non-vanishing and analytic on the closures of Ω_l^+ and Ω_l^- , the annular domains with boundaries $J_l^+ \cup \Delta_l$ and $J_l^- \cup \Delta_l$, respectively. We orient J_l^\pm in the direction of Δ_l . Set $J := J_0 \cup \cup_{l>0} (J_l^+ \cup J_l^-)$. Assume also that the closures of Ω_0 and $\Omega_l^+ \cup \Omega_l^-$, $l > 0$, are pairwise disjoint.

Recall the definition of ζ in Proposition 1. If $\mathbf{Y}(z)$ is a solution of RHP- \mathbf{Y} , let

$$(24) \quad \mathbf{X}(z) := \mathbf{Y}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -\zeta(v_n w_{\Delta_0}/\rho)(z) & 1 \end{pmatrix}, & z \in \Omega_0, \\ \begin{pmatrix} 1 & 0 \\ \mp(v_n w_{\Delta_0}/\rho)(z) & 1 \end{pmatrix}, & z \in \Omega_l^\pm, \\ \mathbf{I}, & \text{otherwise.} \end{cases}$$

Then the matrix function $\mathbf{X}(z)$ solves the following Riemann-Hilbert problem (RHP- \mathbf{X}):

- (a) $\mathbf{X}(z)$ is analytic in $\overline{\mathbb{C}} \setminus (\Delta \cup J)$ and $\lim_{z \rightarrow \infty} \mathbf{X}(z)z^{-n\sigma_3} = \mathbf{I}$;
 (b) $\mathbf{X}(z)$ has continuous traces on $(\Delta \cup J) \setminus \{\pm 1\}$ that satisfy

$$\mathbf{X}_+(s) = \mathbf{X}_-(s) \begin{cases} \begin{pmatrix} 0 & (\rho/(v_n w))(s) \\ -(v_n w/\rho)(s) & 1 \end{pmatrix}, & s \in \Delta, \\ \begin{pmatrix} 1 & 0 \\ (v_n w_{\Delta_0}/\rho)(s) & 1 \end{pmatrix}, & s \in J; \end{cases}$$

- (c) it holds that $\mathbf{X}(z) = \mathcal{O} \begin{pmatrix} 1 & |z - e|^{-1/2} \\ 1 & |z - e|^{-1/2} \end{pmatrix}$ as $D_\Delta \ni z \rightarrow e \in \{\pm 1\}$.

In fact, it is quite easy to see that **RHP-X** is solvable if and only if **RHP-Y** is solvable and the solutions are connected by (24) (all matrices in (24) have determinants identically equal to 1 and therefore are invertible).

Step 3. Let $\Psi_n(z)$ be given by (10) and $\varphi(z) = z - \varsigma w_{\Delta_0}(z)$. Define

$$(25) \quad \mathbf{M}(z) := \begin{cases} \begin{pmatrix} \varsigma \Psi_n(z) & \varsigma/(w_{\Delta_0} \Psi_n)(z) \\ \varsigma(\varphi \Psi_n)(z) & \varsigma/(\varphi w_{\Delta_0} \Psi_n)(z) \end{pmatrix}, & z \in D_\Delta^0, \\ \begin{pmatrix} \varsigma \Psi_n(z) & -\varsigma/(w_{\Delta_0} \Psi_n)(z) \\ \varsigma(\Psi_n/\varphi)(z) & -\varsigma(\varphi/(w_{\Delta_0} \Psi_n))(z) \end{pmatrix}, & z \in D_\Delta^\infty. \end{cases}$$

Using (11) and the fact that $\varphi_+(s)\varphi_-(s) \equiv 1$ for $s \in \Delta_0$, one can readily check that $\mathbf{M}(z)$ solves the following Riemann-Hilbert problem (**RHP-M**):

- (a) $\mathbf{M}(z)$ is analytic in $\overline{\mathbb{C}} \setminus \Delta$ and $\mathbf{C}^{-1} := \lim_{z \rightarrow \infty} \mathbf{M}(z)z^{-n\sigma_3}$ is a diagonal matrix with non-zero entries;
 (b) $\mathbf{M}(z)$ has continuous traces on $\Delta \setminus \{\pm 1\}$ that satisfy

$$\mathbf{M}_+(s) = \mathbf{M}_-(s) \begin{pmatrix} 0 & (\rho/(v_n w))(s) \\ -(v_n w/\rho)(s) & 0 \end{pmatrix}, \quad s \in \Delta \setminus \{\pm 1\};$$

- (c) it holds that $\mathbf{M}(z) = \mathcal{O} \begin{pmatrix} 1 & |z - e|^{-1/2} \\ 1 & |z - e|^{-1/2} \end{pmatrix}$ as $D_\Delta \ni z \rightarrow e \in \{\pm 1\}$.

Observe also that

$$(26) \quad \det(\mathbf{M}(z)) = \frac{1}{(w_{\Delta_0} \varphi)(z)} - \frac{\varphi(z)}{w_{\Delta_0}(z)} \equiv 2\varsigma, \quad z \in \mathbb{C}.$$

Step 4. Consider the following Riemann-Hilbert problem (**RHP-Z**):

- (a) $\mathbf{Z}(z)$ is analytic in $\overline{\mathbb{C}} \setminus J$ and $\mathbf{Z}(\infty) = \mathbf{I}$;
 (b) $\mathbf{Z}(z)$ has continuous traces on J that satisfy

$$\mathbf{Z}_+(s) = \mathbf{Z}_-(s) \mathbf{M}(s) \begin{pmatrix} 1 & 0 \\ (v_n w_{\Delta_0}/\rho)(s) & 1 \end{pmatrix} \mathbf{M}^{-1}(s), \quad s \in J.$$

Denote by $\mathbf{J}(s)$ the jump matrix in **RHP-Z**(b), i.e., $\mathbf{Z}_+(s) = \mathbf{Z}_-(s)\mathbf{J}(s)$, $s \in J$. It follows from (10), (25), (26), and a straightforward computation that

$$\mathbf{J}(s) = \mathbf{I} + b_{2n}(s) \frac{\varsigma}{2(w_{\Delta_0} \rho S_\rho^2)(s)} \begin{pmatrix} \varphi^{-1}(s) & -1 \\ \varphi^{-2}(s) & -\varphi^{-1}(s) \end{pmatrix} = \mathbf{I} + \mathbf{o}(1)$$

uniformly on $J \cap D_\Delta^0$ as well as

$$\mathbf{J}(s) = \mathbf{I} + \frac{1}{b_{2n}(s)} \frac{\varsigma S_\rho^2(s)}{2(w_{\Delta_0} \rho)(s)} \begin{pmatrix} \varphi(s) & -1 \\ \varphi^2(s) & -\varphi(s) \end{pmatrix} = \mathbf{I} + \mathbf{o}(1)$$

uniformly on $J \cap D_\Delta^\infty$, where we used Assumption 1(ii) and the very definition of D_Δ^0 and D_Δ^∞ . The above relations and [7, Corollary 7.108] yield that **RHP-Z** is solvable for all n large enough and the solution satisfies $\mathbf{Z}(z) = \mathbf{I} + \mathbf{o}(1)$ uniformly in $\overline{\mathbb{C}}$.

It now can be quite readily checked that the solution of RHP- X is given by $X(z) = CZ(z)M(z)$. Observe that $(1, 1)$ -entry of C is $\zeta\gamma_n$, see (11) and (25). Given a closed set $K \subset D_\Delta$, the arcs comprising J can be chosen so that K does not intersect the closures of Ω_0 and $\Omega_l^+ \cup \Omega_l^-$, $l > 0$. Then it follows from (24) that $X(z) = Y(z)$ for $z \in K$. Hence,

$$q_n(z) = \gamma_n \Psi_n(z) \begin{cases} 1 + o(1) + o(1)\varphi(z), & z \in D_\Delta^0, \\ 1 + o(1) + o(1)\varphi^{-1}(z), & z \in D_\Delta^\infty, \end{cases}$$

uniformly on K . Since $\varphi^S(\infty) = 0$, relations (12) follow. Similarly, (13) holds since

$$R_n(z) = \frac{\gamma_n}{w_{\Delta_0}(z)\Psi_n(z)} \begin{cases} 1 + o(1) + o(1)\varphi^{-1}(z), & z \in D_\Delta^0, \\ -(1 + o(1) + o(1)\varphi(z)), & z \in D_\Delta^\infty, \end{cases}$$

uniformly on K , where one needs to observe that the error terms $o(1)$ must vanish at infinity. Asymptotic formula (14) now follows from (7), (10), (12), and (13). \square

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