AN ASYMPTOTIC EXPANSION FOR THE EXPECTED NUMBER OF REAL ZEROS OF KAC-GERONIMUS POLYNOMIALS

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ABSTRACT. Let $\{\varphi_i(z;\alpha)\}_{i=0}^\infty$, corresponding to $\alpha\in(-1,1)$, be orthonormal Geronimus polynomials. We study asymptotic behavior of the expected number of real zeros, say $\mathbb{E}_n(\alpha)$, of random polynomials

$$P_n(z) := \sum_{i=0}^n \eta_i\varphi_i(z;\alpha),$$

where $\eta_0,\dots,\eta_n$ are i.i.d. standard Gaussian random variables. When $\alpha=0$, $\varphi_i(z;0)=z^i$ and $P_n(z)$ are called Kac polynomials. In this case it was shown by Wilkins that $\mathbb{E}_n(0)$ admits an asymptotic expansion of the form

$$\mathbb{E}_n(0)\sim\frac{2}{\pi}\log(n+1)+\sum_{p=0}^\infty A_p(n+1)^{-p}$$

(Kac himself obtained the leading term of this expansion). In this work we obtain a similar expansion of $\mathbb{E}(\alpha)$ for $\alpha\neq 0$. As it turns out, the leading term of the asymptotics in this case is $(1/\pi)\log(n+1)$.

1. INTRODUCTION AND MAIN RESULTS

Random polynomials is a relatively old subject with initial contributions by Bloch and Pólya, Littlewood and Offord, Erdös and Offord, Arnold, Kac, and many other authors. An interested reader can find a well referenced early history of the subject in the books by Bharucha-Reid and Sambandham [3], and by Farahmand [12]. In [15], Kac considered random polynomials

$$P_n(z)=\eta_0+\eta_1 z+\cdots+\eta_n z^n, \tag{1}$$

where η_i are i.i.d. standard real Gaussian random variables. He has shown that $\mathbb{E}_n(\Omega)$, the expected number of zeros of $P_n(z)$ on a measurable set $\Omega\subset\mathbb{R}$, is equal to

$$\mathbb{E}_n(\Omega)=\frac{1}{\pi}\int_\Omega\frac{\sqrt{1-h_{n+1}^2(x)}}{|1-x^2|}\mathrm{d}x,\quad h_{n+1}(x)=\frac{(n+1)x^n(1-x^2)}{1-x^{2n+2}}, \tag{2}$$

from which he proceeded with an asymptotic formula

$$\mathbb{E}_n(\mathbb{R})=\frac{2+o(1)}{\pi}\log(n+1)\quad\text{as}\quad n\to\infty. \tag{3}$$

It was shown by Wilkins [25], after some intermediate results cited in [25], that there exist constants A_p, $p\geq 0$, such that $\mathbb{E}_n(\mathbb{R})$ has an asymptotic expansion of the form

$$\mathbb{E}_n(\mathbb{R})\sim\frac{2}{\pi}\log(n+1)+\sum_{p=0}^\infty A_p(n+1)^{-p}, \tag{4}$$

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where

$$A_0 = \frac{2}{\pi}\left(\log 2 + \int_0^1 \frac{f(t)}{t}\mathrm{d}t + \int_1^\infty \frac{f(t)-1}{t}\mathrm{d}t\right), \quad f(t) := \sqrt{1-\left(\frac{2t}{e^t-e^{-t}}\right)^2}. \tag{5}$$

Many subsequent results on random polynomials are concerned with relaxing the conditions on random coefficients, see, for example, [13, 18, 10], or the behavior of the counting measures of zeros of random polynomials as in [21, 6, 14, 5, 19, 2, 20, 17, 4, 9]. Our primary interest lies in studying the expected number of real zeros when the basis is a family of orthogonal polynomials in the spirit of [7, 8, 26, 16]. More precisely, Edelman and Kostlan [11] considered random functions of the form

$$P_n(z) = \eta_0 f_0(z) + \eta_1 f_1(z) + \cdots + \eta_n f_n(z), \tag{6}$$

where η_i are certain real random variables and $f_m(z)$ are arbitrary functions on the complex plane that are real on the real line. Using a beautiful and simple geometrical argument they have shown[1] that if $\eta_0,\dots,\eta_n$ are elements of a multivariate real normal distribution with mean zero and covariance matrix C and the functions $f_m(z)$ are differentiable on the real line, then

$$\mathbb{E}_n(\Omega) = \int_\Omega \rho_n(x)\mathrm{d}x, \quad \rho_n(x) = \frac{1}{\pi}\frac{\partial^2}{\partial s\partial t}\log\left(v(s)^\mathsf{T} C v(t)\right)\Big|_{t=s=x},$$

where $v(x) = \big(f_0(x),\dots,f_n(x)\big)^\mathsf{T}$. If random variables η_i in (6) are again i.i.d. standard real Gaussians, then the above expression for $\rho_n(x)$ specializes to

$$\rho_n(x) = \frac{1}{\pi}\frac{\sqrt{K_{n+1}(x,x)K_{n+1}^{(1,1)}(x,x) - K_{n+1}^{(1,0)}(x,x)^2}}{K_{n+1}(x,x)} \tag{7}$$

(this formula was also independently rederived in [16, Proposition 1.1] and [24, Theorem 1.2]), where $K_{n+1}(x,y) := K_{n+1}^{(0,0)}(x,y)$ and

$$K_{n+1}^{(l,k)}(x,y) := \sum_{i=0}^n f_i^{(l)}(x)\overline{f_i^{(k)}(y)}.$$

We are interested in the case where the spanning functions in (6) are taken to be orthonormal polynomials on the unit circle. Recall [23, Theorem 1.5.2] that monic orthogonal polynomials, say $\Phi_m(z)$, satisfy the recurrence relations

$$\begin{cases} \Phi_{m+1}(z) = z\Phi_m(z) - \overline{\alpha}_m\Phi_m^*(z), \\ \Phi_{m+1}^*(z) = \Phi_m^*(z) - \alpha_m z\Phi_m(z), \end{cases} \tag{8}$$

where the recurrence coefficients $\{\alpha_m\}$ belong to the unit disk $\mathbb{D}$ and are uniquely determined by the measure of orthogonality. Furthermore, the orthonormal polynomials, which we denote by $\varphi_m(z)$, are given by

$$\varphi_m(z) = \rho_m^{-1}\Phi_m(z), \quad \rho_m := \prod_{i=0}^{m-1}\sqrt{1-|\alpha_i|^2}. \tag{9}$$

Since the functions $f_m(z)$ in (6) must be real-valued on the real line, we are only interested in real recurrence coefficients, i.e., $\alpha_m \in (-1,1)$ for all $m \geq 0$. It is known [27] that when $m^p|\alpha_m|$ is a bounded sequence for some $p > 3/2$, estimate (3) remains valid for random polynomials (6) with $f_m(z) = \varphi_m(z)$ given by (8)–(9). Moreover, if the recurrence coefficients decay exponentially, it was shown by the authors in [1] that the expected number of real zeros has a full asymptotic expansion of the form (4) with the constant term still given by (5).

[1] In fact, Edelman and Kostlan derive an expression for the real intensity function for any random vector $(\eta_0,\dots,\eta_n)$ in terms of its joint probability density function and of $v(x)$.

The previous works suggest that the constant $\pi/2$ in front of $\log(n+1)$ in (3) and (4) might change if the recurrence coefficients decay slowly or do not decay at all. In this note we support this guess by considering random polynomials of the form

$$P_n(z) = \eta_0\varphi_0(z;\alpha) + \eta_1\varphi_1(z;\alpha) + \cdots + \eta_n\varphi_n(z;\alpha), \tag{10}$$

which we call Kac-Geronimus polynomials, where η_i are i.i.d. standard real Gaussian random variables and

$$\varphi_m(z;\alpha) = \rho^{-m}\Phi_m(z;\alpha), \quad \rho := \sqrt{1-\alpha^2}, \tag{11}$$

are real Geronimus polynomials, that is, polynomials $\Phi_m(z;\alpha)$ satisfying (8) with $\alpha_m = \alpha \in (-1,1)$ for all $m \geq 0$. The measure of orthogonality for general Geronimus polynomials, i.e., $\alpha_m = \alpha \in \mathbb{D}$, is explicitly known, see [23, Section 1.6], and is supported by

$$\Delta_\alpha := \left\{ e^{i\theta} : 2\arcsin(|\alpha|) \leq \theta \leq 2\pi - 2\arcsin(|\alpha|) \right\}$$

with a possible pure mass point, which is present if and only if $|\alpha + 1/2| > 1/2$. When $\alpha = 0$, one can clearly see from (8) that $\Phi_m(z;0) = z^m$ and therefore Kac-Geronimus polynomials (10) specialize to Kac polynomials (1).

For random polynomials (6) with $f_m(z) = \varphi_m(z)$ given by (8)–(9) it can be easily shown using the Christoffel-Darboux formula, see [27, Theorem 1.1], that (7) can be rewritten as

$$\rho_n(x) = \frac{1}{\pi}\frac{\sqrt{1-h_{n+1}^2(x)}}{|1-x^2|}, \quad h_{n+1}(x) := \frac{(1-x^2)b'_{n+1}(x)}{1-b_{n+1}^2(x)}, \quad b_{n+1}(x) := \frac{\varphi_{n+1}(x)}{\varphi_{n+1}^*(x)}, \tag{12}$$

where $\varphi_{n+1}^*(x) := x^{n+1}\varphi_{n+1}(1/x)$ is the reciprocal polynomial (there is no need for conjugation as all the coefficients are real).

Theorem 1. *Let $P_n(z)$ be given by (10)–(11) with $\alpha \in (-1,0)\cup(0,1)$. Define*

$$r(z) := \sqrt{(z-1)^2 + 4\alpha^2 z} \tag{13}$$

to be the branch holomorphic in $\mathbb{C}\setminus\Delta_\alpha$ such that $r(z)/z \to 1$ as $z \to \infty$. Then it holds that

$$\lim_{n\to\infty} b_{n+1}(z) = \frac{-2\alpha}{r(z)+1-z} \tag{14}$$

locally uniformly in $\mathbb{D}$. Moreover, it holds that

$$h_{n+1}(x) = -\alpha\frac{x+1}{r(x)}\left(1 + \mathcal{O}\left((1-x)^2(n+1)e^{-\sqrt{n+1}/\rho}\right)\right), \tag{15}$$

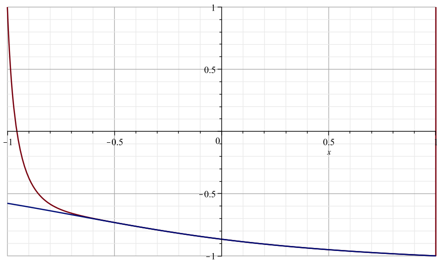
for $-1+(n+1)^{-1/2} \leq x \leq 1-\delta_\alpha^{n+1}$, where $\mathcal{O}(\cdot)$ does not depend on n and $\delta_\alpha := 0$ when $\alpha < 0$ while $\delta_\alpha := ((1-\alpha)/(1+\alpha))^{1/3}$ when $\alpha > 0$.

Observe that $b_{n+1}(1) = h_{n+1}(1) = 1$ for all n and these equalities remain true in the limit when $\alpha < 0$. However, $b(1) = h(1) = -1$ when $\alpha > 0$. This change is due to a single zero of $\varphi_m(z;\alpha)$ that approaches 1 as $m \to \infty$ for every fixed $\alpha > 0$, see Figure 1, and is the reason we need to introduce δ_α in (15).

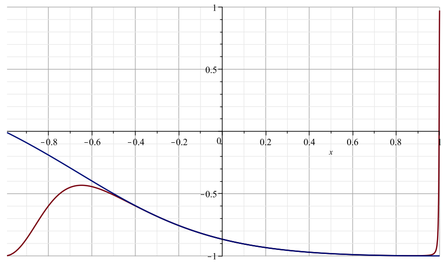
Let $\mathbb{E}_n(\alpha)$ be the expected number of real zeros of random polynomials (10)–(11). It is easy to see that $b_m(1/x) = 1/b_m(x)$ and therefore $b'_m(1/x) = x^2 b'_m(x)/b_m^2(x)$. Thus, we get from (12) that $h_m(1/x) = h_m(x)$ and therefore

$$\mathbb{E}_n(\alpha) = \frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x. \tag{16}$$

Using this formula we can prove the following theorem that constitutes the main result of this work.



(a) $b_4(x)$ and $b(x)$



(b) $h_4(x)$ and $h(x)$

FIGURE 1. The graphs of $b_4(x)$ and $b(x)$ (panel (a)) and $h_4(x)$ and $h(x)$ (panel (b)) on $[-1,1]$ for $\alpha=\sqrt{3}/2$.

Theorem 2. *Let $P_n(z)$ be random polynomials given by (10)–(11) with $\alpha\in(-1,0)\cup(0,1)$. Then there exist constants $A_p^{\alpha,(-1)^n}$, $p\geq 1$, that do depend on the parity of n, such that $\mathbb{E}_n(\alpha)$, the expected number of real zeros of $P_n(z)$, satisfies*

$$\mathbb{E}_n(\alpha)=\frac{1}{\pi}\log(n+1)+A_0^\alpha+\sum_{p=1}^{N-1}A_p^{\alpha,(-1)^n}(n+1)^{-p}+\mathcal{O}_N\left((n+1)^{-N}\right)$$

for any integer N, all n large, where $\mathcal{O}_N(\cdot)$ depends on N, but is independent of n, and

$$A_0^\alpha=\frac{A_0+1+\operatorname{sgn}(\alpha)}{2}+\frac{1}{\pi}\log\frac{2}{|\alpha|}$$

with A_0 given by (5) and $\operatorname{sgn}(\alpha):=\alpha/|\alpha|$.

Notice that $A_0^{|\alpha|}=A_0^{-|\alpha|}+1$. This is due to the fact that polynomials $\varphi_m(x;|\alpha|)$ have a zero exponentially close to 1 while polynomials $\varphi_m(x;-|\alpha|)$ do not have such a zero.

2. PROOF OF THEOREM 1

Lemma 1. *It holds that*

$$b_{n+1}(z)=\frac{\phi(z)-2(1+\alpha)-\varepsilon^{n+1}(z)(\psi(z)-2(1+\alpha))}{\phi(z)-2(1+\alpha)z-\varepsilon^{n+1}(z)(\psi(z)-2(1+\alpha)z)} \tag{17}$$

where $\phi(z):=z+1+r(z)$, $\psi(z):=z+1-r(z)$, $\varepsilon(z):=\psi(z)/\phi(z)$, and $r(z)$ was defined in (13). In particular, (14) takes place.

Proof. Let $U_m(y)$ be the degree m Chebyshëv polynomial of the second kind, that is,

$$U_m(y)=\frac{\left(y+\sqrt{y^2-1}\right)^{m+1}-\left(y-\sqrt{y^2-1}\right)^{m+1}}{2\sqrt{y^2-1}},$$

where for definiteness we take the branch $\sqrt{y^2-1}=y+\mathcal{O}(1)$ as $y\to\infty$ with the cut along $[-1,1]$. It has been shown in [22, Theorem 3.1] that

$$\begin{cases}\varphi_m(z;\alpha)=z^{m/2}\left(U_m\left(\dfrac{z+1}{2\rho\sqrt{z}}\right)-\dfrac{1+\overline{\alpha}}{\rho\sqrt{z}}U_{m-1}\left(\dfrac{z+1}{2\rho\sqrt{z}}\right)\right),\\ \varphi_m^*(z;\alpha)=z^{m/2}\left(U_m\left(\dfrac{z+1}{2\rho\sqrt{z}}\right)-\dfrac{\sqrt{z}(1+\alpha)}{\rho}U_{m-1}\left(\dfrac{z+1}{2\rho\sqrt{z}}\right)\right),\end{cases} \tag{18}$$

where $U_{-1}(y)\equiv 0$ and we take the branch $\sqrt{z}$ that is positive for positive reals (of course, in our case $\overline{\alpha}=\alpha$). Observe that the map

$$y(z)=(z+1)/(2\rho\sqrt{z})$$

takes $\mathbb{D}$ into $\{\mathrm{Re}(z)>0\}\setminus[0,1/\rho]$, the right half-plane with the real segment $[0,1/\rho]$ removed, and its boundary values on Δ_α cover the real interval $[0,1]$ twice. Therefore,

$$\sqrt{y(z)^2-1}=r(z)/(2\rho\sqrt{z}),\quad z\in\mathbb{D}.$$

In particular, it follows from (18) that (17) holds. Observe that

$$|\varepsilon(z)|=\left|\frac{y-\sqrt{y^2-1}}{y+\sqrt{y^2-1}}\right|=\left|y+\sqrt{y^2-1}\right|^{-2}<1 \tag{19}$$

for $|z|<1$. Hence, $b_{n+1}(z)$ converges pointwise and therefore locally uniformly ($|b_{n+1}(z)|<1$ for $z\in\mathbb{D}$) to

$$\frac{z-(1+2\alpha)+r(z)}{1-(1+2\alpha)z+r(z)}=\frac{z-(1+2\alpha)+r(z)}{1-(1+2\alpha)z+r(z)}\frac{z-(1+2\alpha)-r(z)}{z-(1+2\alpha)-r(z)}=\frac{-2\alpha}{r(z)+1-z}.\qquad\square$$

Lemma 2. *Let $h(x):=-\alpha(x+1)/r(x)$. It holds that*

$$h_{n+1}(x)=h(x)\left(1-\varepsilon^{n+1}(x)\frac{\frac{n+1}{\alpha}\frac{(1-x)^2}{x}r(x)+2R(x)(1-\varepsilon^{n+1}(x))}{(1-\varepsilon^{n+1}(x))(S(x)+R(x)\varepsilon^{n+1}(x))}\right), \tag{20}$$

where $R(x):=r(x)+\alpha(1+x)$ and $S(x):=r(x)-\alpha(1+x)$.

Proof. It follows from (17) that

$$b_{n+1}(x)=1-\lambda\frac{(1-x)(1-\varepsilon^{n+1}(x))}{D(x)},$$

where $\lambda:=2(1+\alpha)$ and $D(x):=\phi(x)-\lambda x-\varepsilon^{n+1}(x)(\psi(x)-\lambda x)$. It can be readily checked that

$$1-b_{n+1}^2(x)=2\lambda\frac{(1-x)(1-\varepsilon^{n+1}(x))(S(x)+R(x)\varepsilon^{n+1}(x))}{D^2(x)}.$$

Observe that

$$D'(x)=\phi'(x)-\lambda-(n+1)\varepsilon^n(x)\varepsilon'(x)(\psi(x)-\lambda x)-\varepsilon^{n+1}(x)(\psi'(x)-\lambda).$$

It further holds that

$$\begin{aligned}b'_{n+1}(x)&=\lambda\frac{D(x)(1-\varepsilon^{n+1}(x)+(n+1)(1-x)\varepsilon^n(x)\varepsilon'(x))+D'(x)(1-x)(1-\varepsilon^{n+1}(x))}{D^2(x)}\\&=:\lambda\frac{N_1(x)+(n+1)(1-x)\varepsilon^n(x)\varepsilon'(x)N_2(x)+N_3(x)\varepsilon^{n+1}(x)+N_4(x)\varepsilon^{2(n+1)}(x)}{D^2(x)},\end{aligned}$$

where $N_3(x),N_4(x)$ do not contain terms with $\varepsilon'(x)$. We have that

$$\begin{aligned}N_1(x)&=\phi(x)-\lambda x+(1-x)(\phi'(x)-\lambda)=-2\alpha+r(x)+r'(x)(1-x)\\&=-2\alpha+2\alpha^2(1+x)/r(x)=-2\alpha S(x)/r(x).\end{aligned}$$

Furthermore, we have that

$$N_2(x)=D(x)-(\psi(x)-\lambda x)(1-\varepsilon^{n+1}(x))=2r(x)=R(x)+S(x).$$

It also holds that

$$N_3(x)=-(\phi(x)-\lambda x)-(\psi(x)-\lambda x)-(1-x)(\psi'(x)-\lambda+\phi'(x)-\lambda)=4\alpha.$$

Finally, similarly to $N_1(x)$, we have that

$$N_4(x)=\psi(x)-\lambda x+(1-x)(\psi'(x)-\lambda)=-2\alpha\big(R(x)/r(x)\big).$$

Since

$$\varepsilon'(x)=\big((1-x)/x\big)\big(\varepsilon(x)/r(x)\big), \tag{21}$$

it follows from (12) that

$$h_{n+1}(x) = h(x)\frac{(1-\varepsilon^{n+1}(x))(S(x)-R(x)\varepsilon^{n+1}(x)) - \frac{n+1}{\alpha}\frac{(1-x)^2}{x}r(x)\varepsilon^{n+1}(x)}{(1-\varepsilon^{n+1}(x))(S(x)+R(x)\varepsilon^{n+1}(x))}$$

from which the desired claim easily follows. $\square$

Lemma 3. *Formula* (15) *takes place.*

Proof. It can be readily checked that the function $|y+\sqrt{y^2-1}|$ is an increasing function of t for $y=t$, $t\in[1,\infty)$ and $y=\pm\mathrm{i}t$, $t\in[0,\infty)$. Since $\varepsilon(1)=(1-|\alpha|)/(1+|\alpha|)$, it therefore holds that

$$\begin{aligned} \max_{x\in[-1+(n+1)^{-1/2},1]}|\varepsilon(x)|^n &= \left|\varepsilon(-1+(n+1)^{-1/2})\right|^n \\ &= \left(1-(n+1)^{-1/2}/\rho+\mathcal{O}\left((n+1)^{-1}\right)\right)^n \leq C_1 e^{-\sqrt{n+1}/\rho} \end{aligned} \tag{22}$$

for some absolute constant $C_1>0$.

Assume that $\alpha<0$. Then $|S(x)|\geq r(x)\geq 2|\alpha|\rho$ for $x\in[-1,1]$. Also, since $|h(x)|$ is an increasing function on $[-1,1]$, we have that $|h(x)|\leq 1$ for $x\in[-1,1]$. Thus, we get from (20) and (22) that

$$\begin{aligned} |h_{n+1}(x)-h(x)| &\leq C_2(n+1)e^{-\sqrt{n+1}/\rho}\left((1-x)^2+|R(x)|\right) \\ &\leq C_3(1-x)^2(n+1)e^{-\sqrt{n+1}/\rho} \end{aligned} \tag{23}$$

for some absolute constants C_2, C_3, where one needs to observe that $\varepsilon(0)=0$ and

$$S(x)R(x)=\rho^2(1-x)^2. \tag{24}$$

This proves the lemma in the case $\alpha<0$.

Suppose that $\alpha>0$. It is quite easy to see that estimate (23) remains valid on $[-1+(n+1)^{-1/2},0]$. Observe also that $\varepsilon(x)>0$ and is increasing for $x\in(0,1]$, see (21), and $0<R(x)<4$ on $[-1,1]$. Then by using (24) again, we get that

$$\begin{aligned} (1-\varepsilon^{n+1}(x))(S(x)+R(x)\varepsilon^{n+1}(x)) &\geq S(x)-R(x)\varepsilon^{2(n+1)}(x) \\ &\geq (\rho^2/4)(1-x)^2-4\varepsilon^{2(n+1)}(1) \end{aligned}$$

for $x\in[0,1]$. Notice $\delta_\alpha=\varepsilon^{1/3}(1)$. Then

$$(\rho^2/4)(1-x)^2-4\varepsilon^{2(n+1)}(1)>(\rho^2/8)\delta_\alpha^{2(n+1)}$$

for $x\in\left[0,1-\delta_\alpha^{(n+1)}\right]$ and n sufficiently large. Therefore, similarly to (23), it again follows from (24) that there exists a constant C_4 such that

$$|h_{n+1}(x)-h(x)|\leq C_4(1-x)^2(n+1)\left(\varepsilon(1)/\delta_\alpha^2\right)^{n+1}=C_4(1-x)^2(n+1)\varepsilon^{2(n+1)/3}(1)$$

for $x\in\left[0,1-\delta_\alpha^{(n+1)}\right]$. Since $\varepsilon(1)<1$, the desired estimates follows. $\square$

3. Proof of Theorem 2

To prove Theorem 2 we shall use the following straightforward facts. If $F(y)$ is analytic around the origin, then

$$F\left(\frac{t}{n+1}\right)=\sum_{p=0}^{N-1}\frac{F_p t^p}{(n+1)^p}+\frac{\widetilde{F}_N(t)t^N}{(n+1)^N}, \quad \left|\widetilde{F}_N(t)\right|\leq C_F^{N+1}, \tag{25}$$

for $t\in I_n:=\left[0,\sqrt{n+1}\right]$ and all $n\geq n_F$, where $F_p=F^{(p)}(0)/p!$, the last estimate follows from the extended Cauchy integral formula, and C_F is independent of n,N. Further, if functions $u(t),v(t)$ satisfy

$$g(t)=\sum_{p=0}^{N-1}\frac{B_p(g;t)}{(n+1)^p}+\frac{\widetilde{B}_N(g;t)}{(n+1)^N}, \tag{26}$$

with $g \in \{u, v\}$, then so does their product and

$$B_p(uv;t) = \sum_{k=0}^{p} B_k(u;t) B_{p-k}(v;t) \tag{27}$$

for $p \le N-1$, while

$$\widetilde{B}_N(uv;t) = \sum_{l=0}^{N} \frac{1}{(n+1)^l} \sum_{k+m=N+l,\ k,m\le N} B_{N,k}(u;t) B_{N,m}(v;t) \tag{28}$$

with $B_{N,k}(g;t) = B_k(g;t)$ for $k < N$ and $B_{N,N}(t) = \widetilde{B}_N(g;t)$. Finally, let $F(y)$ be as in (25) and $g(t)$ be as in (26) with $B_0(g;t) = 0$. Assume that the values of $g(t)$ lie the domain of holomorphy of $F(y)$ for all $n \ge n_g$. Then

$$F(g(t)) = F(0) + \sum_{p=1}^{N-1} \frac{B_p(F\circ g;t)}{(n+1)^p} + \frac{\widetilde{B}_N(F\circ g;t)}{(n+1)^N}, \tag{29}$$

with

$$B_p(F\circ g;t) = \sum \frac{F^{(m)}(0)}{m_1!\cdots m_{N-1}!} \prod_{k=1}^{N-1} B_k^{m_k}(g;t) \tag{30}$$

where $m = m_1 + \cdots + m_{N-1}$ and the sum is taken over all partitions $p = \sum_{i=1}^{N-1} i m_i$, $m_i \ge 0$, and

$$\widetilde{B}_N(F\circ g;t) = \sum_{l=0}^{N(N-1)} \frac{1}{(n+1)^l} \sum \frac{F^{(m)}(0)}{m_1!\cdots m_N!} \prod_{k=1}^{N} B_{N,k}^{m_k}(g;t) \tag{31}$$

where $m = m_1 + \cdots + m_N$, the inner sum is taken over all partitions $l + N = \sum_{i=1}^{N} i m_i$, $m_i \ge 0$, and $B_{N,k}(g;t)$ has the same meaning as in (28).

Lemma 4. *Let $t \in I_n = \left[0, \sqrt{n+1}\right]$. Then it holds for all $N \ge 1$ that*

$$r\left(-1 + \frac{t}{n+1}\right) = 2\rho \left(\sum_{p=0}^{N-1} \frac{r_p t^p}{(n+1)^p} + \frac{\tilde{r}_N(t) t^N}{(n+1)^N} \right) \tag{32}$$

for some constants r_p and functions $\tilde{r}_N(t)$ that obey estimate in (25). In particular, $r_0 = 1$, $r_1 = -1/2$, $r_2 = (1-\rho^2)/(8\rho^2)$. Moreover, for $\varepsilon(z)$, defined in Lemma 1, it holds that

$$\varepsilon^{n+1}\left(-1 + \frac{t}{n+1}\right) = (-1)^{n+1} e^{-t/\rho} \left(1 + \sum_{p=1}^{N-1} \frac{t^{p+1} e_p(t)}{(n+1)^p} + \frac{t^{N+1} \tilde{e}_N(t)}{(n+1)^N} \right), \tag{33}$$

where $e_p(t)$ is a polynomial of degree $p-1$ independent of n, N, in particular, $e_1(t) \equiv -1/(2\rho)$, and $|\tilde{e}_N(t)|$ is bounded above on I_n by a polynomial of degree $N-1$ whose coefficients depend only on N.

Proof. Observe that for $y > 0$ it follows from (13) and the choice of the branch of $r(z)$ that

$$r(-1+y) = 2\rho\sqrt{1 - y + y^2/(4\rho^2)},$$

where the root in right-hand side of the above equality is principal. Since the right-hand side above is analytic around the origin, expansion (32) follows from (25). An absolutely analogous argument yields the expansion

$$\log\left(-\varepsilon\left(-1 + \frac{t}{n+1}\right)\right) = \sum_{p=1}^{N} \frac{\varepsilon_p t^p}{(n+1)^p} + \frac{\tilde{\varepsilon}_{N+1}(t) t^{N+1}}{(n+1)^{N+1}}, \quad \varepsilon_1 = -\frac{1}{\rho},\ \varepsilon_2 = -\frac{1}{2\rho},$$

where $|\tilde{\varepsilon}_{N+1}(t)|$ has an upper bound as in (25). Since we can write

$$\varepsilon^{n+1}\left(-1 + \frac{t}{n+1}\right) = \frac{(-1)^{n+1}}{e^{t/\rho}} \exp\left\{ (n+1)\left(\log\left(-\varepsilon\left(-1 + \frac{t}{n+1}\right)\right) + \frac{1}{\rho}\frac{t}{n+1} \right) \right\},$$

it follows from (29)–(31) that (33) holds, where $e_p(t)$ is a polynomial of degree $p-1$ independent of n,N (notice that always $m \leq p$ in (30)) and $|\tilde{e}_N(t)|$ is bounded above on I_n by a polynomial of degree $N-1$ whose coefficients depend only on N (again, we use that $m \leq l+N$ in (31) and that $t^{2l} \leq (n+1)^l$ on I_n). $\square$

Lemma 5. *Set $\gamma(s) := 2s/(e^s - e^{-s})$ and let $x = -1 + t/(n+1)$, $t \in I_n$. It holds that*

$$h_{n+1}(x) = h(x) - (-1)^{n+1}\frac{(1-x)^2}{4}\gamma(t/\rho)(1+\Gamma_{n+1}(t)) \tag{34}$$

with $\Gamma_{n+1}(t)$ having an expansion of the form

$$\Gamma_{n+1}(t) = \sum_{p=1}^{N-1}\frac{H_p(t)}{(n+1)^p} + \frac{\widetilde{H}_N(t)}{(n+1)^N}, \tag{35}$$

for any $N \geq 2$, where $H_1(t) = t - (-1)^{n+1}(\alpha/2\rho)t + \mathscr{O}(t^2)$, $H_p(t) = \mathscr{O}(t^2)$, $p \geq 2$, and $\widetilde{H}_N(t) = \mathscr{O}(t^2)$ as $t \to 0$, $|H_p(t)|$ is bounded above by a polynomial of degree $2p$ independent of n,N, while $|\widetilde{H}_N(t)|$ is bounded above on I_n by a polynomial of degree $2N$ whose coefficients depend on N but not on n.

Proof. Recall (20). Notice that

$$\big(1-\varepsilon^{n+1}(x)\big)\big(S(x)+R(x)\varepsilon^{n+1}(x)\big) = S(x) + 2\alpha(x+1)\varepsilon^{n+1}(x) - R(x)\varepsilon^{2(n+1)}(x). \tag{36}$$

It follows from (32) that $S(x)$ and $R(x)$ have expansions as in (26) with

$$B_p(S;t) = B_p(R;t) = 2\rho r_p t^p,\ p \neq 1, \quad B_1(S;t) = -(\alpha+\rho)t,\ B_1(R;t) = (\alpha-\rho)t,$$

and $\widetilde{B}_N(S;t) = \widetilde{B}_N(R;t) = 2\rho\tilde{r}_N(t)$ for any $N \geq 2$. Therefore, we get from (27)–(28) and (33) that

$$R(x)\varepsilon^{2(n+1)}(x) = 2\rho e^{-2t/\rho}\left(1 + \sum_{p=1}^{N-1}\frac{C_p(t)t^p}{(n+1)^p} + \frac{\widetilde{C}_N(t)t^N}{(n+1)^N}\right)$$

for any $N \geq 2$, where $C_1(t) = (\alpha-\rho-2t)/(2\rho)$, $C_p(t) = r_p + tq_p(t)$ for some polynomial $q_p(t)$ of degree $p-1$ when $p \geq 2$, and $|\widetilde{C}_N(t)|$ is bounded above on I_n by a polynomial of degree N independent of n. Consequently, we get that the expression in (36) has an expansion

$$2\rho\left(1-e^{-2t/\rho}\right)\left(1 + \frac{D_1(t)}{n+1} + \sum_{p=2}^{N-1}\frac{D_p(t)t^p}{(n+1)^p} + \frac{\widetilde{D}_N(t)t^N}{(n+1)^N}\right)$$

for all $N \geq 2$, where

$$\begin{aligned} D_1(t) &= -\alpha\left(\frac{1-(-1)^{n+1}e^{-t/\rho}}{2}\right)^2\frac{2t/\rho}{1-e^{-2t/\rho}} + \frac{t}{2}\left(\frac{2t/\rho}{e^{2t/\rho}-1} - 1\right) \\ &= -\alpha\frac{1-(-1)^{n+1}}{2} + \mathscr{O}\left(t^2\right) \quad \text{as} \quad t \to 0, \end{aligned}$$

with $|D_1(t)|$ bounded above by a linear function independent of n,N, and

$$D_p(t) = r_p + \gamma(t/\rho)\left((-1)^{n+1}\alpha e_{p-1}(t) - \rho e^{-t/\rho}q_p(t)\right)/2$$

for all $p \geq 2$, with $|D_p(t)|$ being bounded above on $[0,\infty)$, and $|\widetilde{D}_N(t)|$ that is bounded on I_n by a constant that depends on N but not on n. In particular, we have that

$$\left|\frac{D_1(t)}{n+1} + \sum_{p=2}^{N-1}\frac{D_p(t)t^p}{(n+1)^p} + \frac{\widetilde{D}_N(t)t^N}{(n+1)^N}\right| = \left|\frac{D_1(t)}{n+1} + \frac{\widetilde{D}_2(t)t^2}{(n+1)^2}\right| < \frac{c_N}{\sqrt{n+1}} < 1$$

for $t \in I_n$ and all $n \geq n_N$, where c_N, n_N are constants dependent only on N. Thus, it follows from (29)–(31) with $F(y) = 1/(1+y)$ that the reciprocal of (36) has an expansion

$$\frac{1}{2\rho}\frac{1}{1-e^{-2t/\rho}}\left(1+\sum_{p=1}^{N-1}\frac{E_p(t)}{(n+1)^p}+\frac{\widetilde{E}_N(t)}{(n+1)^N}\right), \tag{37}$$

for all $N \geq 2$, where $E_1(t) = -D_1(t)$ and more generally

$$E_p(t) = (-1)^p D_1^p(t) + \mathscr{O}\left(t^2\right) = \alpha^p \frac{1-(-1)^{n+1}}{2} + \mathscr{O}\left(t^2\right) \tag{38}$$

as $t \to 0$ with $|E_p(t)|$ bounded above by a polynomial of degree p independent of n, N, while $|\widetilde{E}_N(t)|$ is bounded above on I_n by a polynomial of degree N whose coefficients depend on N but not on n. Furthermore, observe that

$$-h(x)\varepsilon^{n+1}(x)\left(\frac{n+1}{\alpha}\frac{(1-x)^2}{x}r(x)+2R(x)\left(1-\varepsilon^{n+1}(x)\right)\right) =$$
$$-(n+1)(1+x)(1-x)^2\varepsilon^{n+1}(x)\left(-\frac{1}{x}-\frac{2\alpha}{n+1}\frac{R(x)}{r(x)}\frac{1-\varepsilon^{n+1}(x)}{(1-x)^2}\right).$$

It follows from an argument similar to the one given in the first part of the lemma that the above expression has an expansion of the form

$$-(1-x)^2(-1)^{n+1}te^{-t/\rho}\left(1+\sum_{p=1}^{N-1}\frac{G_p(t)t^{p-1}}{(n+1)^p}+\frac{\widetilde{G}_N(t)t^{N-1}}{(n+1)^N}\right), \tag{39}$$

for any $N \geq 3$, where

$$G_1(t) = -\alpha\frac{1-(-1)^{n+1}}{2} + \left(1-(-1)^{n+1}\frac{\alpha}{2\rho}\right)t + \mathscr{O}\left(t^2\right) \tag{40}$$

and

$$G_2(t) = -\alpha\frac{1-(-1)^{n+1}}{2}\left(1+\frac{\alpha}{2\rho}\right) + \mathscr{O}(t) \tag{41}$$

as $t \to 0$, $|G_p(t)|$ is bounded above by a polynomial of degree $p+1$ independent of n, N, while $|\widetilde{G}_N(t)|$ is bounded above on I_n by a polynomial of degree $N+1$ whose coefficients depend on N but not on n. We now get from (20), (37), and (39), that (34) and (35) do hold for $N \geq 3$ and functions $H_p(t)$ and $\widetilde{H}_N(t)$ that can be computed via (27)–(28) and whose moduli satisfy the described bounds. The vanishing of $H_p(t)$ as $t \to 0$ can be verified by using (27), (38), (40), and (41). To see that $\widetilde{H}_N(t) = \mathscr{O}(t^2)$, observe that

$$h_{n+1}(x) = -(-1)^{n+1} - (-1)^{n+1}(1-t/(n+1))\widetilde{H}_N(t)(n+1)^{-N} + \mathscr{O}\left(t^2\right)$$

by what precedes. Thus, we need to show that $h_{n+1}(x) + (-1)^{n+1}$ is divisible by $(1+x)^2$ (of course, if this were not true, formula (16) would not have made sense). Since $h_{n+1}(-1) = -(-1)^{n+1}$, it must hold that $h'_{n+1}(-1) = 0$. As was mentioned before (16), $h_{n+1}(x) = h_{n+1}(1/x)$ and therefore $x^2 h'_{n+1}(x) = -h'_{n+1}(1/x)$, which yields the desired claim. Finally, since $\widetilde{H}_2(t) = H_2(t) + \widetilde{H}_3(t)(n+1)^{-1}$, we can take $N = 2$ in (35) as well. □

Lemma 6. *let $x = -1 + t/(n+1)$, $t \in I_n$. It holds that*

$$\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x} = \frac{\rho f(t/\rho)}{r(x)}\left(1+\gamma(t/\rho)\sum_{p=1}^{N-1}\frac{K_p(t)}{(n+1)^p}+\gamma(t/\rho)\frac{\widetilde{K}_N(t)}{(n+1)^N}\right) \tag{42}$$

for any $N \geq 2$, where $|K_p(t)|$ is bounded above by a polynomial of degree $2p$ independent of n, N while $|\widetilde{K}_N(t)|$ is bounded above on I_n by a polynomial of degree $2N$ whose coefficients depend on N but not on n.

Proof. Observe that $1-h^2(x)=\rho^2(1-x)^2r^{-2}(x)$. Then it follows from (34) that

$$\frac{1-h_{n+1}^2(x)}{1-h^2(x)}=1+(-1)^{n+1}\gamma(t/\rho)\big(1+\Gamma_{n+1}(t)\big)h(x)\frac{r^2(x)}{2\rho^2}-\gamma(t/\rho)^2\big(1+\Gamma_{n+1}(t)\big)^2\frac{(1-x)^2}{4}\frac{r^2(x)}{4\rho^2}.$$

Since $h(x)r(x)=-\alpha(1+x)$, expansions (32), (35) and formulae (27)–(28) yield that

$$(-1)^{n+1}\big(1+\Gamma_{n+1}(t)\big)h(x)\frac{r^2(x)}{2\rho^2}=\sum_{p=1}^{N-1}\frac{H_p^*(t)}{(n+1)^p}+\frac{\widetilde H_N^*(t)}{(n+1)^N},$$

for any $N\geq 2$, where $H_1^*(t)=-(-1)^{n+1}(\alpha/\rho)t$, $H_p^*(t)=\mathscr{O}(t^2)$, $p\geq 2$, and $\widetilde H_N^*(t)=\mathscr{O}(t^2)$ as $t\to 0$, while $|H_p^*(t)|$ and $|\widetilde H_N^*(t)|$ have similar bounds to $|H_p(t)|$ and $|\widetilde H_N(t)|$. Furthermore, it clearly holds that

$$\frac{(1-x)^2}{4}=1-\frac{t}{n+1}+\frac14\frac{t^2}{(n+1)^2}\quad\text{and}\quad\frac{r^2(x)}{4\rho^2}=1-\frac{t}{n+1}+\frac{1}{4\rho^2}\frac{t^2}{(n+1)^2}.$$

Therefore, we again get from (27)–(28) that

$$\big(1+\Gamma_{n+1}(t)\big)^2\frac{(1-x)^2}{4}\frac{r^2(x)}{4\rho^2}=1+\sum_{p=1}^{N-1}\frac{H_p^{**}(t)}{(n+1)^p}+\frac{\widetilde H_N^{**}(t)}{(n+1)^N},$$

for any $N\geq 2$, where $H_1^{**}(t)=-(-1)^{n+1}(\alpha/\rho)t+\mathscr{O}(t^2)$, $H_p^{**}(t)=\mathscr{O}(t^2)$, $p\geq 2$, and $\widetilde H_N^{**}(t)=\mathscr{O}(t^2)$ as $t\to 0$ while $|H_p^{**}(t)|$ and $|\widetilde H_N^{**}(t)|$ have similar bounds to $|H_p(t)|$ and $|\widetilde H_N(t)|$. Altogether, it holds that

$$\frac{1-h_{n+1}^2(x)}{1-h^2(x)}=f^2(t/\rho)\left(1+\gamma(t/\rho)\sum_{p=1}^{N-1}\frac{J_p(t)}{(n+1)^p}+\gamma(t/\rho)\frac{\widetilde J_N(t)}{(n+1)^N}\right),$$

where $J_p(t)=f^{-2}(t/\rho)\big(H_p^*(t)-\gamma(t/\rho)H_p^{**}(t)\big)$ and a similar formula holds for $\widetilde J_N(t)$. Observe that $f^2(s)$ is a positive function for $s>0$ that tends to 1 as $s\to\infty$ and such that $f^2(s)=s^2/3+\mathscr{O}(s^4)$ as $s\to 0$. Therefore, it follows from the corresponding properties of $H_p^*(t)$, $H_p^{**}(t)$, $\widetilde H_N^*(t)$, and $\widetilde H_N^{**}(t)$ that $J_p(t)$ and $\widetilde J_N(t)$ have finite value at the origin and have moduli that satisfy similar bounds to $|H_p(t)|$ and $|\widetilde H_N(t)|$. Observe also that there exist n_N and $c_N<1$ such that

$$\left|\gamma(t/\rho)\sum_{p=1}^{N-1}\frac{J_p(t)}{(n+1)^p}+\gamma(t/\rho)\frac{\widetilde J_N(t)}{(n+1)^N}\right|<c_N$$

for all $n\geq n_N$. Therefore, the claim of the lemma now follows from (29)–(31) applied with $F(y)=\sqrt{1+y}$. □

Lemma 7. *Let $x=-1+t/(n+1)$, $t\in I_n$. There exist constants O_p, $p\geq 1$, such that*

$$\frac{2\rho}{\pi}\int_0^{\sqrt{n+1}}\frac{f(t/\rho)}{tr(x)}\mathrm{d}t=\frac{1}{2\pi}\log(n+1)+\frac{A_0}{2}-\frac1\pi\log(2\rho)+\frac1\pi\mathscr{L}\left(-1+\frac{1}{\sqrt{n+1}}\right)+\sum_{p=1}^{N-1}\frac{O_p}{(n+1)^p}+\mathscr{O}_N\left((n+1)^{-N}\right),$$

for any $N\geq 1$, where $\mathscr{O}_N(\cdot)$ does not depend on n and

$$\mathscr{L}(x):=\log\left(\frac{4\rho}{\rho(1-x)+r(x)}\right).$$

Proof. Similarly to (32), there exist constants r_p^* such that

$$\frac{2\rho}{r(x)} = 1 + \sum_{p=1}^{N-1} \frac{r_p^* t^p}{(n+1)^p} + \frac{\tilde{r}_N^*(t) t^N}{(n+1)^N}, \tag{43}$$

for any $N \geq 1$, where $|\tilde{r}_N^*(t)|$ is bounded above on I_n by a constant that depends only on N. Then

$$\mathscr{I}_1 := \frac{2\rho}{\pi} \int_0^\rho \frac{f(t/\rho)}{tr(x)} \mathrm{d}t = \frac{1}{\pi} \int_0^1 \frac{f(t)}{t} \mathrm{d}t + \sum_{p=1}^{N-1} \frac{L_p}{(n+1)^p} + \mathscr{O}_N\left((n+1)^{-N}\right),$$

where $L_p := (r_p^* \rho^p/\pi) \int_0^1 f(t) t^{p-1} \mathrm{d}t$ and $\mathscr{O}_N(\cdot)$ does not depend on n. Furthermore, it holds that

$$\mathscr{I}_2 := \frac{2\rho}{\pi} \int_\rho^{\sqrt{n+1}} \frac{\mathrm{d}t}{tr(x)} = \frac{2\rho}{\pi} \int_{-1+\rho/(n+1)}^{-1+1/\sqrt{n+1}} \frac{dx}{(1+x)r(x)}. \tag{44}$$

It can be easily verified by differentiation that an antiderivative of $2\rho/((1+x)r(x))$ is $\log(1+x) + \mathscr{L}(x)$. Again, similarly to (32), there exist constants l_p such that

$$\mathscr{L}(x) = \sum_{p=1}^{N-1} \frac{l_p t^p}{(n+1)^p} + \frac{\tilde{l}_N(t) t^N}{(n+1)^N},$$

for any $N \geq 1$, where $|\tilde{l}_N(t)|$ is bounded above on I_n by a constant that depends only on N. Therefore, it holds that

$$\mathscr{I}_2 = \frac{1}{2\pi} \log(n+1) - \frac{1}{\pi} \log \rho + \frac{1}{\pi} \mathscr{L}\left(-1 + \frac{1}{\sqrt{n+1}}\right) - \sum_{p=1}^{N-1} \frac{l_p \rho^p/\pi}{(n+1)^p} + \mathscr{O}_N\left((n+1)^{-N}\right),$$

where, again, $\mathscr{O}_N(\cdot)$ does not depend on n. Next, we have from (43) that

$$\mathscr{I}_3 := \frac{2\rho}{\pi} \int_\rho^{\sqrt{n+1}} \frac{f(t/\rho)-1}{tr(x)} \mathrm{d}t = \frac{1}{\pi} \int_1^{\sqrt{n+1}/\rho} \frac{f(t)-1}{t} \mathrm{d}t +$$
$$\sum_{p=1}^{N-1} \frac{r_p^* \rho^p/\pi}{(n+1)^p} \int_1^{\sqrt{n+1}/\rho} (f(t)-1) t^{p-1} \mathrm{d}t + \frac{\rho^N/\pi}{(n+1)^N} \int_1^{\sqrt{n+1}/\rho} (f(t)-1) \tilde{r}_N^*(\rho t) t^{N-1} \mathrm{d}t$$

for any $N \geq 1$. Notice that

$$0 < 1 - f(t) < t^2 \operatorname{csch}^2(t) < 8t^2 e^{-2t}, \quad t \geq 1. \tag{45}$$

Therefore, it holds that

$$0 < \int_{\sqrt{n+1}/\rho}^\infty (1-f(t)) t^{p-1} \mathrm{d}t \leq C_p (n+1)^{(p+1)/2} e^{-(2/\rho)\sqrt{n+1}} = o_N\left((n+1)^{-N}\right) \tag{46}$$

for any $p \geq 0$ and $N \geq 1$ and some constant C_p that depends only on p, where $o_N(\cdot)$ does not depend on n. Moreover, since $|\tilde{r}_N^*(t)|$ is bounded above on I_n by a constant that depends only on N, we have that

$$\left| \int_1^{\sqrt{n+1}/\rho} (f(t)-1) \tilde{r}_N^*(\rho t) t^{N-1} \mathrm{d}t \right| \leq C_N^* \int_1^\infty (1-f(t)) t^{N-1} \mathrm{d}t = C_N^{**}.$$

Thus, we can conclude that

$$\mathscr{I}_3 = \frac{1}{\pi} \int_1^\infty \frac{f(t)-1}{t} \mathrm{d}t + \sum_{p=1}^{N-1} \frac{M_p}{(n+1)^p} + \mathscr{O}_N\left((n+1)^{-N}\right),$$

where $M_p := (r_p^* \rho^p/\pi) \int_1^\infty (f(t)-1) t^{p-1} \mathrm{d}t$ and $\mathscr{O}_N(\cdot)$ does not depend on n. Since the integral in the statement of the lemma is equal to $\mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3$, the desired claim now follows from the definition of A_0 in (5), where $O_p = L_p - l_p \rho^p/\pi + M_p$. □

Lemma 8. *There exist constants T_p such that*

$$\frac{2}{\pi}\int_{-1}^{-1+1/\sqrt{n+1}}\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x=\frac{1}{2\pi}\log(n+1)+\frac{A_0}{2}-\frac{1}{\pi}\log(2\rho)+\frac{1}{\pi}\mathscr{L}\left(-1+\frac{1}{\sqrt{n+1}}\right)+\sum_{p=1}^{N-1}\frac{T_p}{(n+1)^p}+\mathscr{O}_N\left((n+1)^{-N}\right),$$

for any $N\geq 1$, where $\mathscr{O}_N(\cdot)$ does not depend on n.

Proof. Recall (42). It follows from (43) and (27)–(28) that

$$\frac{2\rho}{r(x)}\left(\sum_{p=1}^{N-1}\frac{K_p(t)}{(n+1)^p}+\frac{\widetilde{K}_N(t)}{(n+1)^N}\right)=\sum_{p=1}^{N-1}\frac{S_p(t)}{(n+1)^p}+\frac{\widetilde{S}_N(t)}{(n+1)^N}$$

for any $N\geq 2$, where $|S_p(t)|$ is bounded above by a polynomial of degree $2p$ independent of n,N while $|\widetilde{S}_N(t)|$ is bounded above on I_n by a polynomial of degree $2N$ whose coefficients depend on N but not on n. Similarly to (45), it holds that $\gamma(s)<3se^{-s}$ for $s\geq\log 2$. Because $f(s)\to 1$ as $s\to\infty$, it holds as in (46) that

$$0<\int_{\sqrt{n+1}/\rho}^{\infty}|\rho S_p(\rho t)|\gamma(t)f(t)\mathrm{d}t\leq C_p(n+1)^{p+1/2}e^{-\sqrt{n+1}/\rho}=o_N\left((n+1)^{-N}\right)$$

for any $p\geq 1$ and $N\geq 1$ and some constant C_p that depends only on p, where $o_N(\cdot)$ does not depend on n. Moreover, a similar estimate takes place if $S_p(t)$ is replaced by $\widetilde{S}_N(t)$. The claim of the lemma now follows by making a substitution $x=-1+t/(n+1)$ to get

$$\frac{2}{\pi}\int_{-1}^{-1+1/\sqrt{n+1}}\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x=\frac{2}{\pi}\int_{0}^{\sqrt{n+1}}\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x}\frac{\mathrm{d}t}{t}$$

and then using Lemmas 6 and 7, where $T_p=O_p+(\rho/\pi)\int_0^\infty f(t)\gamma(t)S_p(\rho t)\mathrm{d}t$ (since $T_1/(n+1)=\mathscr{O}_N((n+1)^{-1})$, the claim indeed holds for all $N\geq 1$). □

Lemma 9. *It holds that*

$$\frac{2}{\pi}\int_{-1+1/\sqrt{n+1}}^{1-\delta_\alpha^{n+1}}\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x=\frac{1}{2\pi}\log(n+1)+\frac{1}{\pi}\log\left(\frac{4\rho}{|\alpha|}\right)-\frac{1}{\pi}\mathscr{L}\left(-1+\frac{1}{\sqrt{n+1}}\right)+o_N\left((n+1)^{-N}\right)$$

for any integer $N\geq 1$, where $o_N(\cdot)$ is independent of n, but does depend on N.

Proof. Since $|h_{n+1}(x)|,|h(x)|\leq 1$ when $x\in[-1,1]$, it holds that

$$\begin{aligned}\left|\frac{\sqrt{1-h^2(x)}-\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\right| &= \frac{|h_{n+1}^2(x)-h^2(x)|}{(1-x^2)\left(\sqrt{1-h^2(x)}+\sqrt{1-h_{n+1}^2(x)}\right)}\\ &\leq \frac{2|h_{n+1}(x)-h(x)|}{(1-x^2)\sqrt{1-h^2(x)}}=\frac{2}{\rho}\frac{r(x)}{(1+x)}\frac{|h_{n+1}(x)-h(x)|}{(1-x)^2}.\end{aligned}$$

Since $r(x)\leq 2$, $x\in[-1,1]$, we obtain from (15) that

$$\left|\frac{\sqrt{1-h^2(x)}-\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\right|\leq C(n+1)^{3/2}e^{-\sqrt{n+1}/\rho}$$

for $-1+1/\sqrt{n+1} \le x \le 1-\delta_\alpha^{n+1}$ and some constant C. Therefore, it holds that

$$\left| \frac{2}{\pi}\int_{-1+1/\sqrt{n+1}}^{1-\delta_\alpha^{n+1}} \frac{\sqrt{1-h^2(x)}-\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x \right| = o_N\left((n+1)^{-N}\right)$$

for any $N \ge 1$, where $o_N(\cdot)$ is independent of n, but does depend on N. Furthermore, since $r(x) \ge 2|\alpha|\rho$ for $x \in [-1,1]$, it holds that

$$\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1} \frac{\sqrt{1-h^2(x)}}{1-x^2}\mathrm{d}x = \frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1} \frac{\rho\,\mathrm{d}x}{(1+x)r(x)} \le \frac{\delta_\alpha^{n+1}}{|\alpha|\pi} = o_N\left((n+1)^{-N}\right)$$

for any $N \ge 1$ by the very definition of δ_α, where, again, $o_N(\cdot)$ is independent of n, but does depend on N. The observation made after (44) allows us now to conclude that

$$\frac{2}{\pi}\int_{-1+1/\sqrt{n+1}}^{1} \frac{\rho\,\mathrm{d}x}{(1+x)r(x)} = \frac{1}{2\pi}\log(n+1) + \frac{1}{\pi}\log\left(\frac{4\rho}{|\alpha|}\right) - \frac{1}{\pi}\mathscr{L}\left(-1+\frac{1}{\sqrt{n+1}}\right),$$

which finishes the proof of the lemma. □

Lemma 10. *When $\alpha > 0$, it holds that*

$$\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x = 1 + o_N\left((n+1)^{-N}\right)$$

for any $N \ge 1$, where $o_N(\cdot)$ is independent of n, but does depend on N.

Proof. It follows from (20) and (24) that

$$h_{n+1}(x) = h(x) - h(x)\frac{\varepsilon^{n+1}(x)X_{n+1}(x)}{(1-x)^2+\varepsilon^{n+1}(x)Y_{n+1}(x)},$$

where

$$X_{n+1}(x) := \frac{R(x)}{\alpha\rho^2}\left((n+1)r(x)\frac{(1-x)^2}{x} + 2\alpha R(x)\left(1-\varepsilon^{n+1}(x)\right)\right)$$

and

$$Y_{n+1}(x) := \frac{R(x)}{\rho^2}\left(2\alpha(x+1) - R(x)\varepsilon^{n+1}(x)\right).$$

Therefore, we can write

$$1-h_{n+1}^2(x) = \rho^2\frac{(1-x)^2}{r^2(x)} + h^2(x)\frac{(1-x)^2\varepsilon^{n+1}(x)X_{n+1}(x)}{((1-x)^2+\varepsilon^{n+1}(x)Y_{n+1}(x))^2} \times \left(2-\varepsilon^{n+1}(x)\frac{X_{n+1}(x)-2Y_{n+1}(x)}{(1-x)^2}\right).$$

We have that

$$\frac{X_{n+1}(x)-2Y_{n+1}(x)}{(1-x)^2} = \frac{R(x)}{\rho^2(1-x)^2}\left(2S(x)+(n+1)\frac{(1-x)^2r(x)}{\alpha x}\right) = 2+(n+1)\frac{r(x)R(x)}{\alpha\rho^2 x},$$

where we used (24) once more. Hence, it holds that

$$\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2} = \frac{\varepsilon^{(n+1)/2}(x)V_{n+1}(x)}{(1-x)^2+\varepsilon^{n+1}(x)Y_{n+1}(x)}, \tag{47}$$

where

$$V_{n+1}(x) := -\frac{2}{\rho}\frac{h(x)R(x)}{1+x}\sqrt{\left(1-\varepsilon^{n+1}(x)\left(1+(n+1)\frac{r(x)R(x)}{2\alpha\rho^2 x}\right)\right) \times \overline{\times\left(1-\varepsilon^{n+1}(x)+(n+1)\frac{(1-x)^2r(x)}{2\alpha xR(x)}\right) + \left(\rho^2\frac{(1-x)^2+\varepsilon^{n+1}(x)Y_{n+1}(x)}{2\varepsilon^{(n+1)/2}(x)h(x)r(x)R(x)}\right)^2}}$$

(observe that $-h(x)>0$). Recall that $\delta_\alpha=\varepsilon^{1/3}(1)$. In particular, we get from (21) that

$$\frac{(1-x)^2}{\varepsilon^{(n+1)/2}(x)}\leq\varepsilon^{\frac{n+1}{6}}(1)\left(\frac{\varepsilon(1)}{\varepsilon(1-\delta_\alpha^{n+1})}\right)^{\frac{n+1}{2}}=(1+o(1))\varepsilon^{\frac{n+1}{6}}(1)=o_N\left((n+1)^{-N}\right)\tag{48}$$

for $1-\delta_\alpha^{n+1}\leq x\leq 1$. Since

$$Y_{n+1}(x)=(2\alpha/\rho^2)(x+1)R(x)+o_N\left((n+1)^{-N}\right)\tag{49}$$

on any fixed small enough neighborhood of 1, it holds that

$$V_{n+1}(x)=-\frac{2}{\rho}\frac{h(x)R(x)}{1+x}+o_N\left((n+1)^{-N}\right)=\frac{4\alpha}{\rho}+o_N\left((n+1)^{-N}\right)\tag{50}$$

uniformly for $1-\delta_\alpha^{n+1}\leq x\leq 1$. Let

$$Z_{n+1}(x):=\sqrt{Y_{n+1}(x)}-\frac{x-1}{2}\left((n+1)\frac{1-x}{x}\frac{\sqrt{Y_{n+1}(x)}}{r(x)}+\frac{Y_{n+1}'(x)}{\sqrt{Y_{n+1}(x)}}\right).$$

It follows from the definition of $Z_{n+1}(x)$ and an estimate similar to (48) that

$$\begin{aligned}\int_{1-\delta_\alpha^{n+1}}^1\frac{\varepsilon^{(n+1)/2}(x)Z_{n+1}(x)\,\mathrm{d}x}{(1-x)^2+\varepsilon^{n+1}(x)Y_{n+1}(x)}&=\arctan\left(\frac{x-1}{\sqrt{\varepsilon^{n+1}(x)Y_{n+1}(x)}}\right)\Bigg|_{1-\delta_\alpha^{n+1}}^1\\&=\frac{\pi}{2}-\arctan\left(\mathscr{O}(1)\varepsilon^{\frac{n+1}{6}}(1)\right)=\frac{\pi}{2}+o_N\left((n+1)^{-N}\right).\end{aligned}$$

Furthermore, we get from (49), the definition of $Z_{n+1}(x)$, and (50) that

$$Z_{n+1}(x)=\frac{4\alpha}{\rho}+o_N\left((n+1)^{-N}\right)=V_{n+1}(x)+o_N\left((n+1)^{-N}\right)$$

uniformly for $1-\delta_\alpha^{n+1}\leq x\leq 1$. Therefore, (47) yields that

$$\begin{aligned}\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^1\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x&=\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^1\frac{\varepsilon^{(n+1)/2}(x)\left(Z_{n+1}(x)+o_N\left((n+1)^{-N}\right)\right)}{(1-x)^2+\varepsilon^{n+1}(x)Y_{n+1}(x)}\mathrm{d}x\\&=1+o_N\left((n+1)^{-N}\right),\end{aligned}$$

where we used positivity of the integrand for the last estimate. ☐

Proof of Theorem 2. The claim follows from formula (16) and Lemmas 8–10. ☐

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