Asymptotics of Polynomials Orthogonal on a Cross with a Jacobi-type Weight

Ahmad Barhoumi and Maxim L. Yattselev

Abstract. We investigate asymptotic behavior of polynomials $Q_n(z)$ satisfying non-Hermitian orthogonality relations

$$\int_\Delta s^k Q_n(s)\rho(s)\mathrm{d}s = 0, \quad k \in \{0,\dots,n-1\},$$

where $\Delta := [-a,a]\cup[-\mathrm{i}b,\mathrm{i}b]$, $a,b>0$, and $\rho(s)$ is a Jacobi-type weight. The primary motivation for this work is study of the convergence properties of the Padé approximants to functions of the form

$$f(z) = (z-a)^{\alpha_1}(z-\mathrm{i}b)^{\alpha_2}(z+a)^{\alpha_3}(z+\mathrm{i}b)^{\alpha_4},$$

where the exponents $\alpha_i \notin \mathbb{Z}$ add up to an integer.

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1. Introduction

Let $a,b>0$ be fixed. Set

$$\Delta := [-a,a]\cup[-\mathrm{i}b,\mathrm{i}b] \quad \text{and} \quad \Delta^\circ := \Delta\setminus\{0,a_1,a_2,a_3,a_4\}, \tag{1.1}$$

where we put $a_1=-a_3=a$ and $a_2=-a_4=\mathrm{i}b$. Denote by Δ_i, $i\in\{1,2,3,4\}$, the segment joining the origin and a_i, which we orient towards the origin. In this work we are interested in strong asymptotics of polynomials $Q_n(z)$, $\deg(Q_n)\leq n$, satisfying orthogonality relations

$$\int_\Delta s^k Q_n(s)\rho(s)\mathrm{d}s = 0, \quad k\in\{0,\dots,n-1\}, \tag{1.2}$$

where Δ inherits its orientation from the segments Δ_i and $\rho(s)$ is a certain weight function on Δ. Orthogonality relations (1.2) are non-Hermitian and therefore there are no a priori reasons to assume that $\deg(Q_n)=n$. In what follows, we shall understand that $Q_n(z)$ stands for the monic polynomial of

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minimal degree satisfying (1.2). The weight functions we are interested in are holomorphic perturbations of the power functions. More precisely, we define the following nested sequence of classes of weights.

Definition. Let ℓ be a positive integer or infinity. We shall say that a function $\rho(s)$ on Δ belongs to the class $\mathcal{W}_\ell$ if

(i) $\rho_i(s) := \rho_{|\Delta_i}(s)$ factors as a product $\rho_i(s) = \rho_i^*(s)(s-a_i)^{\alpha_i}$, where the function $\rho_i^*(z)$ is non-vanishing and holomorphic in some neighborhood of Δ_i, $\alpha_i > -1$, and $(z-a_i)^{\alpha_i}$ is a branch holomorphic across $\Delta \setminus \{a_i\}$, $i \in \{1,2,3,4\}$;
(ii) the ratio $(\rho_1\rho_3)(z)/(\rho_2\rho_4)(z)$ is constant in some neighborhood of the origin;
(iii) it holds that $\rho_1(0)+\rho_2(0)+\rho_3(0)+\rho_4(0)=0$;
(iv) the quantities $\rho_i^{(l)}(0)/\rho_i(0)$, $0 \leq l < \ell$, do not depend on $i \in \{1,2,3,4\}$.

Observe that conditions (ii) and (iii) say that one of the functions $\rho_i(z)$ is fully determined by the other three. In particular, it must hold that

$$\rho_4(z) = -(\rho_1+\rho_2+\rho_3)(0)(\rho_2/\rho_1\rho_3)(0)(\rho_1\rho_3/\rho_2)(z).$$

Notice also that $\mathcal{W}_{\ell_1} \subset \mathcal{W}_{\ell_2}$ whenever $\ell_2 < \ell_1$ and that $\rho(s) \in \mathcal{W}_\infty$ if and only if there exists a function $F(z)$, holomorphic in some neighborhood of $\Delta \setminus \{a_1,a_2,a_3,a_4\}$, such that $\rho_i(s) = c_i F_{|\Delta_i}(s)$ for some constants c_i that add up to zero.

Holomorphy of the weights $\rho_i(z)$ allows one to deform Δ in (1.2) to any cross-like contour consisting of four arcs connecting the points a_i to the origin (some central point if the weight add up to zero in a neighborhood of the origin). Hence, the following question arises: *which contour do we expect to attract the zeros of the polynomials $Q_n(z)$ as $n \to \infty$?* This fundamental question in the theory of non-Hermitian orthogonal polynomials was answered by Herbert Stahl in [11, 12, 13]. It turns out that the attracting contour is essentially characterized by having the smallest logarithmic capacity among all continua containing $\{a_1,a_2,a_3,a_4\}$. It is also known from the works [8, 10] that this contour must consist of the orthogonal critical trajectories of the quadratic differential

$$\frac{(z-b_1)(z-b_2)\mathrm{d}z^2}{(z^2-a^2)(z^2+b^2)} \tag{1.3}$$

for some uniquely determined constants b_1, b_2. It can be readily verified that Δ is the desired contour and $b_1 = b_2 = 0$.

Strong asymptotics of the polynomials $Q_n(z)$ was considered as part of a study in [14] under much more restrictive assumption $\rho(s) = h(s)/w_+(s)$, where $h(z)$ is a holomorphic and non-vanishing function is some neighborhood of Δ and $w(z)$ is defined in (2.1) further below. It is also worth pointing out that if the points $\{a_1,a_2,a_3,a_4\}$ do not form a cross with two symmetries, then the points b_1, b_2 in (1.3) are distinct and the corresponding minimal capacity contour consists of five arcs: one joining b_1 and b_2, two connecting b_1 to two points in $\{a_1,a_2,a_3,a_4\}$, and two connecting b_2 to the other two points in $\{a_1, a_2, a_3, a_4\}$. Non-Hermitian orthogonal polynomials on such a contour for a class of weights defined similarly to $\mathcal{W}_1$ is a particular example of polynomials studied in [1].

2. Statement of Results

The functions describing the asymptotic behavior of the polynomials $Q_n(z)$ are constructed in three steps, carried out in Sections 2.2-2.4, and naturally defined on a Riemann surface corresponding to Δ that is introduced in Section 2.1. The main results of this work are stated in Sections 2.5 and 2.6.

2.1. Riemann Surface

Let $\Delta = \cup_{i=1}^4 \Delta_i$ be given by (1.1). Set

$$w(z) := \sqrt{(z^2 - a^2)(z^2 + b^2)}, \quad z \in \mathbb{C} \setminus \Delta, \tag{2.1}$$

to be the branch normalized so that $w(z) = z^2 + \mathcal{O}(z)$ as $z \to \infty$. Denote by $\mathfrak{R}$ the Riemann surface of $w(z)$ realized as a two-sheeted ramified cover of $\overline{\mathbb{C}}$ constructed in the following manner. Two copies of $\overline{\mathbb{C}}$ are cut along each arc Δ_i. These copies are glued together along the cuts in such a manner that the right (resp. left) side of the arc Δ_i belonging to the first copy, say $\mathfrak{R}^{(0)}$, is joined with the left (resp. right) side of the same arc Δ_i only belonging to the second copy, $\mathfrak{R}^{(1)}$. We denote by π the canonical projection $\pi : \mathfrak{R} \to \overline{\mathbb{C}}$ and

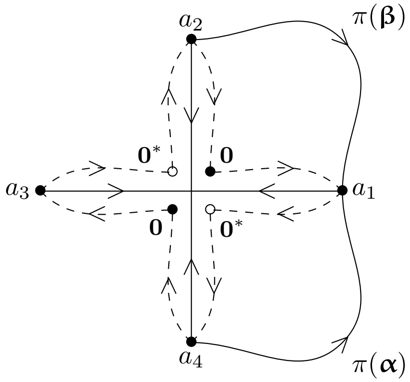

FIGURE 1. The arcs Δ_i together with their orientation (solid lines), a schematic representation of the arcs $\boldsymbol{\Delta}_i = \pi^{-1}(\Delta_i)$ (dashed lines) as viewed from $\mathfrak{R}^{(0)}$, and the chosen homology basis $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ projected down from $\mathfrak{R}^{(0)}$.

define $\boldsymbol{\Delta} := \pi^{-1}(\Delta)$, $\boldsymbol{\Delta}_i := \pi^{-1}(\Delta_i)$, $i \in \{1, 2, 3, 4\}$. Then $\boldsymbol{\Delta}$ is a curve on $\mathfrak{R}$ that intersects itself exactly twice (once at each point on top of the origin), see Figures 1 and 2. We orient $\boldsymbol{\Delta}$ so that $\mathfrak{R}^{(0)}$ remains on the left when $\boldsymbol{\Delta}$ is traversed in the positive direction. We shall denote by $z^{(k)}$, $k \in \{0, 1\}$, the point on $\mathfrak{R}^{(k)}$ with canonical projection z and designate the symbol $\cdot^*$

to stand for the conformal involution that sends $z^{(k)}$ into $z^{(1-k)}$, $k \in \{0,1\}$. We use bold lower case letters such as $\boldsymbol{z}, \boldsymbol{t}, \boldsymbol{s}$ to indicate points on $\mathfrak{R}$ with canonical projections z, t, s. Since $\mathfrak{R}$ is elliptic (genus 1), any homology basis on $\mathfrak{R}$ consists of only two cycles. In what follows, we choose cycles $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to be involution-symmetric and such that $\pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})$ are rectifiable Jordan arcs joining a_1, a_2 and a_4, a_1, respectively, oriented as on Figures 1 and 2.

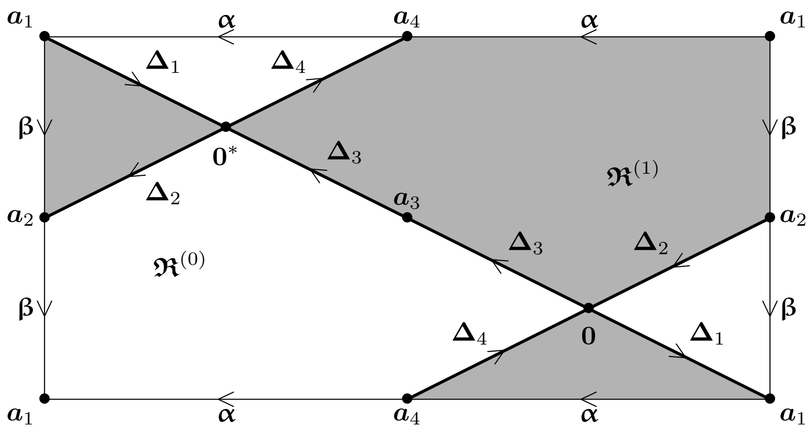

FIGURE 2. Schematic representation of the surface $\mathfrak{R}$ (shaded region represents $\mathfrak{R}^{(1)}$), which topologically is a torus, the arcs $\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \boldsymbol{\Delta}_3, \boldsymbol{\Delta}_4$, and the homology basis $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

2.2. Geometric Term

With a slight abuse of notation, let us set

$$w(\boldsymbol{z}) := (-1)^k w(z), \quad \boldsymbol{z} \in \mathfrak{R}^{(k)} \setminus \boldsymbol{\Delta}, \quad k \in \{0,1\},$$

which we then extend by continuity to $\boldsymbol{\Delta}$. One can readily verify that

$$\Omega(\boldsymbol{z}) := \left(\oint_{\boldsymbol{\alpha}} \frac{\mathrm{d}s}{w(\boldsymbol{s})} \right)^{-1} \frac{\mathrm{d}z}{w(\boldsymbol{z})} \tag{2.2}$$

is the holomorphic differential on $\mathfrak{R}$ normalized to have unit period on $\boldsymbol{\alpha}$. In this case it is known that the constant

$$\mathsf{B} := \oint_{\boldsymbol{\beta}} \Omega \tag{2.3}$$

has positive purely imaginary part. It also readily follows from the properties of the quadratic differential (1.3) that

$$G(\boldsymbol{z}) := \frac{z\mathrm{d}z}{w(\boldsymbol{z})}$$

is a meromorphic differential on $\mathfrak{R}$ having two simple poles at $\infty^{(1)}$ and $\infty^{(0)}$ with respective residues 1 and -1, whose period on any cycle on $\mathfrak{R}$ is purely imaginary. Define

$$\omega := -\frac{1}{2\pi\mathrm{i}}\oint_{\boldsymbol{\beta}} G \quad \text{and} \quad \tau := \frac{1}{2\pi\mathrm{i}}\oint_{\boldsymbol{\alpha}} G, \tag{2.4}$$

which are necessarily real constants. By deforming $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$) into $-\boldsymbol{\Delta}_1-\boldsymbol{\Delta}_4$ (resp. $\boldsymbol{\Delta}_1+\boldsymbol{\Delta}_2$) and using the symmetry $G(\boldsymbol{z}^*)=-G(\boldsymbol{z})$, one gets that

$$\omega = \tau = \frac{1}{4\pi\mathrm{i}}\oint_\Gamma \frac{z\mathrm{d}z}{w(z)} = \frac{1}{2}, \tag{2.5}$$

where Γ is any positively oriented rectifiable Jordan curve encircling Δ. Let

$$\Phi(\boldsymbol{z}) := \exp\left\{\int_{\boldsymbol{a}_3}^{\boldsymbol{z}} G\right\}, \quad \boldsymbol{z}\in\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \mathfrak{R}\setminus\{\boldsymbol{\alpha},\boldsymbol{\beta}\}, \tag{2.6}$$

where the path of integration lies entirely in $\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. The function $\Phi(\boldsymbol{z})$ is holomorphic and non-vanishing on $\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ except for a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$. Furthermore, it possesses continuous traces on both sides of each cycle of the canonical basis that satisfy[1]

$$\Phi_+(\boldsymbol{s}) = -\Phi_-(\boldsymbol{s}), \quad \boldsymbol{s}\in\boldsymbol{\alpha}\cup\boldsymbol{\beta}, \tag{2.7}$$

by (2.4)–(2.5). It is not a difficult computation to verify that $\Phi(\boldsymbol{z})\Phi(\boldsymbol{z}^*)\equiv 1$ and

$$\left|\Phi(\boldsymbol{z})\right| = \exp\left\{(-1)^k g_\Delta(z;\infty)\right\}, \quad \boldsymbol{z}\in\mathfrak{R}^{(k)},$$

$k\in\{0,1\}$, where $g_\Delta(z;\infty)$ is the Green function for $\overline{\mathbb{C}}\setminus\Delta$ with pole at ∞[2]. In fact, the above properties allow us to verify that

$$\Phi^2\left(z^{(k)}\right) = \frac{2}{a^2+b^2}\left(z^2+\frac{b^2-a^2}{2}+(-1)^k w(z)\right), \tag{2.8}$$

$k\in\{0,1\}$. In particular, this implies that the logarithmic capacity of Δ is equal to $\sqrt{a^2+b^2}/2$ since

$$\Phi\left(z^{(0)}\right) = \frac{-2z}{\sqrt{a^2+b^2}} + \mathcal{O}(1) \quad \text{as} \quad z\to\infty \tag{2.9}$$

(the sign in (2.9) is determined by the fact that $\Phi(\boldsymbol{a}_3)=1$ and $\Phi(\boldsymbol{z})$ is non-vanishing on $\pi^{-1}((-\infty,-a))$).

2.3. Szegő Function

Let $\rho(s)\in\mathcal{W}_1$. For each $i\in\{1,2,3,4\}$, fix $\log\rho_i(s)$ to be a branch continuous on $\Delta_i\setminus\{a_i\}$, selected so that

$$\nu := \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^4(-1)^i\log\rho_i(0) \quad \text{satisfies} \quad \mathrm{Re}(\nu)\in\left(-\frac{1}{2},\frac{1}{2}\right]. \tag{2.10}$$

[1] Here and in what follows we state jump relations understanding that they hold outside the points of self-intersection of the considered arcs.

[2] $g_\Delta(z;\infty)$ is equal to zero on Δ, is positive and harmonic in $\mathbb{C}\setminus\Delta$, and satisfies $g(z;\infty)=\log|z|+\mathcal{O}(1)$ as $z\to\infty$.

Further, it can be readily verified that we can set

$$\log w_+(s) = \log|w_+(s)| + (-1)^i\frac{\pi\mathrm{i}}{2}, \quad s\in\Delta_i^\circ := \Delta_i\setminus\{0,a_i\}, \tag{2.11}$$

where $w_+(s)$ is the trace of (2.1) on the positive side of Δ_i° according to the chosen orientation. We also let $\log(\rho_i w_+)(s)$ to stand for $\log\rho_i(s)+\log w_+(s)$ with the just selected branches. Put

$$S_\rho(\boldsymbol z) := \exp\left\{-\frac{1}{4\pi\mathrm{i}}\oint_{\boldsymbol\Delta}\log(\rho w_+)(s)\Omega_{\boldsymbol z,\boldsymbol z^*}(\boldsymbol s)\right\}, \tag{2.12}$$

where $\Omega_{\boldsymbol z,\boldsymbol z^*}(\boldsymbol s)$ is the meromorphic differential with two simple poles at $\boldsymbol z$ and $\boldsymbol z^*$ with respective residues 1 and -1 normalized to have zero period on $\boldsymbol\alpha$. When $\boldsymbol z$ does not lie on top of the point at infinity, it can be readily verified that

$$\Omega_{\boldsymbol z,\boldsymbol z^*}(\boldsymbol s) = \frac{w(\boldsymbol z)}{s-z}\frac{\mathrm{d}s}{w(\boldsymbol s)} - \left(\oint_{\boldsymbol\alpha}\frac{w(\boldsymbol z)}{t-z}\frac{\mathrm{d}t}{w(\boldsymbol t)}\right)\Omega(\boldsymbol s), \tag{2.13}$$

where $\Omega(\boldsymbol s)$ is the holomorphic differential (2.2).

Proposition 2.1. *Let $\rho(s)\in\mathcal W_1$ and $S_\rho(\boldsymbol z)$ be given by (2.12). Then $S_\rho(\boldsymbol z)$ is a holomorphic and non-vanishing function in $\mathfrak R\setminus\{\boldsymbol\Delta\cup\boldsymbol\alpha\}$ with continuous traces on $(\boldsymbol\Delta\cup\boldsymbol\alpha)\setminus\{\boldsymbol a_1,\boldsymbol a_2,\boldsymbol a_3,\boldsymbol a_4,\boldsymbol 0,\boldsymbol 0^*\}$ that satisfy*

$$S_{\rho+}(\boldsymbol s) = S_{\rho-}(\boldsymbol s)\begin{cases}\exp\left\{2\pi\mathrm{i}c_\rho\right\}, & \boldsymbol s\in\boldsymbol\alpha,\\ 1/(\rho w_+)(s), & \boldsymbol s\in\boldsymbol\Delta,\end{cases} \tag{2.14}$$

where $c_\rho := \frac{1}{2\pi\mathrm{i}}\oint_{\boldsymbol\Delta}\log(\rho w_+)\Omega$. It also holds that $S_\rho(\boldsymbol z)S_\rho(\boldsymbol z^*)\equiv 1$ and [3]

$$\left|S_\rho\big(z^{(0)}\big)\right| \sim \begin{cases}|z-a_i|^{-(2\alpha_i+1)/4} & \text{as} \quad z\to a_i,\\ |z|^{(-1)^j\mathrm{Re}(\nu)} & \text{as} \quad \mathcal Q_j\ni z\to 0,\end{cases} \tag{2.15}$$

for $i,j\in\{1,2,3,4\}$, where $\mathcal Q_j$ is the j-th quadrant and ν is given by (2.10).

2.4. Theta Function

Let $\mathsf{Jac}(\mathfrak R) := \mathbb C/\{\mathbb Z+\mathsf B\mathbb Z\}$ be the Jacobi variety of $\mathfrak R$, where $\mathsf B$ is given by (2.3). We shall represented elements of $\mathsf{Jac}(\mathfrak R)$ as equivalence classes $[s] = \{s+l+\mathsf Bm : l,m\in\mathbb Z\}$, where $s\in\mathbb C$. Since $\mathfrak R$ is elliptic, Abel's map

$$\boldsymbol z\in\mathfrak R\mapsto\left[\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega\right]\in\mathsf{Jac}(\mathfrak R)$$

is a holomorphic bijection. Hence, given any $s\in\mathbb C$, there exists a unique $\boldsymbol z_{[s]}\in\mathfrak R$ such that $\left[\int_{\boldsymbol a_3}^{\boldsymbol z_{[s]}}\Omega\right] = [s]$.

Denote by $\theta(\zeta)$ the Riemann theta function associated to $\mathsf B$, i.e.,

$$\theta(\zeta) := \sum_{n\in\mathbb Z}\exp\left\{\pi\mathrm{i}\mathsf Bn^2 + 2\pi\mathrm{i}n\zeta\right\}.$$

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[3]In what follows we write $|g_1(z)|\sim|g_2(z)|$ as $z\to z_0$ if there exists a constant $C>1$ such that $C^{-1}|g_1(z)|\le|g_2(z)|\le C|g_1(z)|$ for all z close to z_0.

As shown by Riemann, $\theta(\zeta)$ is an entire, even function that satisfies

$$\theta(\zeta+l+m\mathsf{B}) = \theta(\zeta)\exp\{-\pi\mathrm{i}m^2\mathsf{B} - 2\pi\mathrm{i}m\zeta\} \tag{2.16}$$

for any integers l,m. Moreover, its zeros are simple and $\theta(\zeta)=0$ if and only if $[\zeta]=[(1+\mathsf{B})/2]$. The constant $(1+\mathsf{B})/2$, known as the Riemann constant, will appear often in our computations. So, we choose to abbreviate the representatives of its "half"-classes by

$$\mathsf{K}_+ := (1+\mathsf{B})/4 \quad \text{and} \quad \mathsf{K}_- := (1-\mathsf{B})/4, \tag{2.17}$$

i.e., $[2\mathsf{K}_+]=[2\mathsf{K}_-]$. The symmetries of $\Omega(\boldsymbol{z})$ ($\Omega(-\boldsymbol{z})=-\Omega(\boldsymbol{z})=\Omega(\boldsymbol{z}^*)$) yield that

$$\int_{\infty^{(1)}}^{\infty^{(0)}}\Omega = \frac{1}{2}\int_{\boldsymbol{\delta}}\Omega = 2\mathsf{K}_+ \quad \Rightarrow \quad \int_{\boldsymbol{a}_3}^{\infty^{(k)}} = (-1)^k\mathsf{K}_+, \tag{2.18}$$

$k\in\{0,1\}$, where $\boldsymbol{\delta}=\pi^{-1}\big((-\infty,-a]\cup[a,\infty)\big)$ is a cycle on $\mathfrak{R}$ oriented from $\infty^{(1)}$ to $\infty^{(0)}$ (on Figure 2, $\boldsymbol{\delta}$ would be represented by the anti-diagonal), which is clearly is homologous to $\boldsymbol{\alpha}+\boldsymbol{\beta}$.

With c_ρ as in Proposition 2.1, define

$$T_k(\boldsymbol{z}) := \exp\left\{\pi\mathrm{i}k\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega\right\}\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - c_\rho - (-1)^k\mathsf{K}_+\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - \mathsf{K}_+\big)} \tag{2.19}$$

for $k\in\{0,1\}$ and $\boldsymbol{z}\in\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, where the path of integration lies entirely within $\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. $T_k(\boldsymbol{z})$ is a meromorphic function that is finite and non-vanishing except for a simple pole at $\infty^{(1)}$, see (2.18), and a simple zero at $\boldsymbol{z}_k := \boldsymbol{z}_{[c_\rho-(-1)^k\mathsf{K}_+]}$, where $\boldsymbol{z}_k\in\mathfrak{R}$ is uniquely characterized by

$$\int_{\boldsymbol{a}_3}^{\boldsymbol{z}_k}\Omega = c_\rho - (-1)^k\mathsf{K}_+ + l_k + m_k\mathsf{B}, \tag{2.20}$$

$k\in\{0,1\}$, for some $l_0,m_0,l_1,m_1\in\mathbb{Z}$. Furthermore, it follows from the normalization in (2.2), the definition of B in (2.3), and (2.16) that

$$T_{k+}(\boldsymbol{s}) = T_{k-}(\boldsymbol{s})\begin{cases}\exp\{2\pi\mathrm{i}(k/2-c_\rho)\}, & \boldsymbol{s}\in\boldsymbol{\alpha},\\ \exp\{\pi\mathrm{i}k\}, & \boldsymbol{s}\in\boldsymbol{\beta}.\end{cases} \tag{2.21}$$

2.5. Asymptotics

Given $\rho(s)\in\mathcal{W}_1$, let c_ρ be as in Proposition 2.1. Set

$$\{0,1\}\ni\imath(n) := n \mod 2, \quad n\in\mathbb{Z},$$

to be the parity function. Then it follows from (2.7), (2.14), and (2.21) that the function

$$\Psi_n(\boldsymbol{z}) := \big(\Phi^n S_\rho T_{\imath(n)}\big)(\boldsymbol{z}), \quad \boldsymbol{z}\in\mathfrak{R}\setminus\boldsymbol{\Delta}, \tag{2.22}$$

is meromorphic in $\mathfrak{R}\setminus\boldsymbol{\Delta}$ with a pole of order n at $\infty^{(0)}$, a zero of multiplicity $n-1$ at $\infty^{(1)}$, a simple zero at $\boldsymbol{z}_{\imath(n)}$, and otherwise non-vanishing and finite, whose traces on $\boldsymbol{\Delta}$ satisfy

$$\Psi_{n+}(\boldsymbol{s}) = \Psi_{n-}(\boldsymbol{s})/(\rho w_+)(s), \quad \boldsymbol{s}\in\boldsymbol{\Delta}, \tag{2.23}$$

and whose behavior around the ramification points of $\mathfrak{R}$ as well as $\mathbf{0}^*,\mathbf{0}$ is governed by (2.15).

In what follows, we have to restrict the subsequence of indices n. To this end, set $\sigma_k:=(-1)^{l_k+m_k+k}$, $k\in\{0,1\}$, see (2.20) . Put $A_{\rho,n}:=0$ when $\mathrm{Re}(\nu)=0$, and otherwise set

$$A_{\rho,n}:=\sigma_{\imath(n)}A_{\rho,n}'\Phi(\boldsymbol z_{\imath(n)})\Phi^{2(n-1)}(\boldsymbol s), \tag{2.24}$$

where $\boldsymbol s=\mathbf{0}$ when $\mathrm{Re}(\nu)>0$ and $\boldsymbol s=\mathbf{0}^*$ when $\mathrm{Re}(\nu)<0$, and

$$A_{\rho,n}':=A_\rho e^{\pi\mathrm{i}\varsigma_\nu(c_\rho+1/4)}\frac{\sqrt{a^2+b^2}}{2}\frac{\Gamma(1-\varsigma_\nu\nu)}{\sqrt{2\pi}}\times \\ \times\left[\lim_{z\to0,\arg(z)=5\pi/4}|z|^{2\nu}S_\rho^2\big(z^{(0)}\big)\right]^{\varsigma_\nu}\left(\frac{ab}{2n}\right)^{1/2-\varsigma_\nu\nu},$$

where $\varsigma_\nu:=1$ when $\mathrm{Re}(\nu)>0$ and $\varsigma_\nu:=-1$ when $\mathrm{Re}(\nu)<0$, and

$$A_\rho:=e^{\pi\mathrm{i}\nu}\rho_3(0)\frac{(\rho_2+\rho_3)(0)}{\rho_2(0)} \quad \text{or} \quad A_\rho:=\frac{1}{(ab)^2}\frac{(\rho_3+\rho_4)(0)}{(\rho_3\rho_4)(0)}$$

depending on whether $\mathrm{Re}(\nu)>0$ or $\mathrm{Re}(\nu)<0$. Observe also that a calculus level computation tells us that

$$\Phi(\mathbf{0})=\overline{\Phi(\mathbf{0}^*)}=\exp\left\{\mathrm{i}\arctan\left(\frac{a}{b}\right)\right\}. \tag{2.25}$$

Proposition 2.2. *For every $\varepsilon\in(0,1/2)$, let*

$$\mathbb{N}_{\rho,\varepsilon}:=\left\{n\in\mathbb{N}:\boldsymbol z_{\imath(n)}\neq\infty^{(0)} \text{ and } |1-A_{\rho,n}|\geq\varepsilon\right\}.$$

It holds that

$$\mathbb{N}_{\rho,\varepsilon}=\mathbb{N}_\rho:=\begin{cases} 2\mathbb{N} & \text{when} \quad [c_\rho]=[0], \\ \mathbb{N}\setminus2\mathbb{N} & \text{when} \quad [c_\rho]=[(1+\mathsf{B})/2]. \end{cases}$$

Otherwise, $\mathbb{N}_{\rho,\varepsilon}=\mathbb{N}_\rho:=\mathbb{N}$ when $\mathrm{Re}(\nu)\in(-1/2,1/2)$ and it is an infinite subsequence when $\mathrm{Re}(\nu)=1/2$. In particular, dependence on ε is significant only when $\mathrm{Re}(\nu)=1/2$ and $[c_\rho]\notin\{[0],[(1+\mathsf{B})/2]\}$.

Indeed, it readily follows from (2.20) and (2.18) that

$$[c_\rho]=[k(1+\mathsf{B})/2] \quad\Leftrightarrow\quad \boldsymbol z_1=\infty^{(k)} \quad\Leftrightarrow\quad \boldsymbol z_0=\infty^{(1-k)}$$

for $k\in\{0,1\}$ (in which case $\Phi(\boldsymbol z_{\imath(n)})=\Phi\big(\infty^{(1)}\big)=0=A_{\rho,n}$). On the other hand, because Abel's map is a bijection, we also get that $|\pi(\boldsymbol z_1)|<\infty\Leftrightarrow|\pi(\boldsymbol z_0)|<\infty$. Since $A_{\rho,n}\to0$ as $n\to\infty$ when $\mathrm{Re}(\nu)\in(-1/2,1/2)$, this proves the first three claims of the proposition. As $\arctan(a/b)\in(0,\pi/2)$, the last claim follows from the arithmetic properties of numbers.

Theorem 2.3. *Let $\rho(s)\in\mathcal{W}_\ell$, where ℓ is a positive integer or infinity and $\Psi_n(\boldsymbol z)$ be given by (2.22). Assume in addition that*

$$\ell(3-2|\mathrm{Re}(\nu)|)>2|\mathrm{Re}(\nu)|(3+2|\mathrm{Re}(\nu)|).$$

Given $\varepsilon > 0$, it holds for all $n \in \mathbb{N}_{\rho,\varepsilon}$ large enough that

$$Q_n(z) = \gamma_n\big(1 + \upsilon_{n1}(z)\big)\Psi_n\big(z^{(0)}\big) + \gamma_n\upsilon_{n2}(z)\Psi_{n-1}\big(z^{(0)}\big) \tag{2.26}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \Delta$, where $\gamma_n := \lim_{z\to\infty} z^n\Psi_n^{-1}\big(z^{(0)}\big)$ is the normalizing constant, $\upsilon_{ni}(\infty) = 0$,

$$\upsilon_{ni}(z) = z^{-1}L_{ni} + \mathcal{O}\left(n^{-d_{\nu,\ell}}\right), \quad L_{ni} = \mathcal{O}\big(n^{|\mathrm{Re}(\nu)|-1/2}\big), \tag{2.27}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \{0\}$ (uniformly in $\overline{\mathbb{C}}$ when $\ell = \infty$) with

$$d_{\nu,\ell} := \begin{cases} \frac{(\frac12+|\mathrm{Re}(\nu)|)(\ell-2|\mathrm{Re}(\nu)|)}{\ell+1+2|\mathrm{Re}(\nu)|}, & \ell \geq \frac{4|\mathrm{Re}(\nu)|(1+|\mathrm{Re}(\nu)|)}{1-2|\mathrm{Re}(\nu)|}, \\ \frac{\ell(3-2|\mathrm{Re}(\nu)|)-2|\mathrm{Re}(\nu)|(3+2|\mathrm{Re}(\nu)|)}{2(\ell+3+2|\mathrm{Re}(\nu)|)}, & \text{otherwise}, \end{cases} \tag{2.28}$$

($d_{\nu,\infty} = 1/2 + |\mathrm{Re}(\nu)|$), and

$$L_{ni} = (-1)^{\imath(n)}\frac{A_{\rho,n}}{1-A_{\rho,n}}\left(-\frac{\Phi T_{\imath(n)}}{T_{\imath(n-1)}}\right)^{i-1}(\boldsymbol{s})\frac{(T_0/T_1)(\boldsymbol{s})}{(T_0/T_1)'(\boldsymbol{s})} \tag{2.29}$$

for $i \in \{1,2\}$, where $\boldsymbol{s} = \boldsymbol{0}$ when $\mathrm{Re}(\nu) > 0$ and $\boldsymbol{s} = \boldsymbol{0}^*$ when $\mathrm{Re}(\nu) < 0$ when $|\pi(\boldsymbol{z}_k)| < \infty$ (when $|\pi(\boldsymbol{z}_k)| = \infty$, the expression for L_{ni} is even more cumbersome and therefore is omitted here). In particular, the polynomials $Q_n(z)$ have degree n for all $n \in \mathbb{N}_{\rho,\varepsilon}$ large enough.

The condition on the index ℓ amounts to saying that it can be any when $|\mathrm{Re}(\nu)| \in [0, -1+\sqrt{7}/2)$, it must be at least 2, when $|\mathrm{Re}(\nu)| \in [-1+\sqrt{7}/2, 1/2)$, and it must be at least 3 when $|\mathrm{Re}(\nu)| = 1/2$.

Notice that the behavior of the orthogonal polynomials is qualitatively different for $\mathrm{Re}(\nu) \in (-1/2, 1/2)$ and $\mathrm{Re}(\nu) = 1/2$ as the first summand in (2.27) is decaying with n in the former case, but not in the latter.

2.6. Padé Approximation

For an integrable weight $\rho(s)$ on Δ define

$$\widehat{\rho}(z) := \frac{1}{2\pi \mathrm{i}}\int_\Delta \frac{\rho(s)\mathrm{d}s}{s-z}, \quad z \in \overline{\mathbb{C}} \setminus \Delta. \tag{2.30}$$

In particular, it can be readily verified that the functions

$$\sum_{i=1}^4 C_i \log(z-a_i) \quad \text{and} \quad \prod_{i=1}^4 (z-a_i)^{\alpha_i},$$

where the constants C_i add up to zero and the exponents $-1 < \alpha_i \notin \mathbb{Z}$ add up to an integer, possess branches holomorphic off Δ that can be represented by (2.30) for certain weight functions in $\mathcal{W}_\infty$ (the second function can represented by (2.30) up to an addition of a polynomial).

Given $\widehat{\rho}(z)$ as in (2.30), it follows from the orthogonality relations (1.2) that there exists a polynomial $P_n(z)$ of degree at most $n-1$ such that

$$R_n(z) := \big(Q_n\widehat{\rho}\big)(z) - P_n(z) = \mathcal{O}\big(z^{-n-1}\big) \quad \text{as} \quad z \to \infty. \tag{2.31}$$

The rational function $[n/n]_{\widehat{\rho}}(z) := P_n(z)/Q_n(z)$ is called the n-th diagonal Padé approximant to $\widehat{\rho}(z)$.

Theorem 2.4. *Let $\widehat\rho(z)$ be given by* (2.30) *and $R_n(z)$ be defined by* (2.31). *In the setting of Theorem 2.3, it holds for all $n\in\mathbb N_{\rho,\varepsilon}$ large enough that*

$$(wR_n)(z)=\gamma_n\big(1+\upsilon_{n1}(z)\big)\Psi_n\big(z^{(1)}\big)+\gamma_n\upsilon_{n2}(z)\Psi_{n-1}\big(z^{(1)}\big) \tag{2.32}$$

locally uniformly in $\overline{\mathbb C}\setminus\Delta$, where $\upsilon_{ni}(z)$ are the same as in Theorem 2.3.

3. Examples

In this section, we illustrate Theorem 2.3 by three examples. In them, we shall not compute $S_\rho(\boldsymbol z)$ and c_ρ via their integral representations, see (2.12), but rather construct a candidate $\widehat S_\rho(\boldsymbol z)$ with the desired jump over $\boldsymbol\Delta$ and the singular behavior as in (2.15). This construction will also determine a candidate constant $\widehat c_\rho$. It is simple to argue that

$$S_\rho(\boldsymbol z)=\widehat S_\rho(\boldsymbol z)\exp\left\{2\pi\mathrm i m\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega\right\},\quad c_\rho=\widehat c_\rho-m\mathsf B,$$

for some integer m. Using $\widehat c_\rho$ in (2.19), we then construct $\widehat T_{\imath(n)}(\boldsymbol z)$ for which it holds that

$$T_{\imath(n)}(\boldsymbol z)=\widehat T_{\imath(n)}(\boldsymbol z)\exp\left\{-2\pi\mathrm i m\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega-\pi\mathrm i m^2\mathsf B+2\pi\mathrm i(-1)^{\imath(n)}\mathsf K_+\right\}$$

with the same integer m. This means that

$$\big(S_\rho T_{\imath(n)}\big)(\boldsymbol z)/\big(S_\rho T_{\imath(n)}\big)\big(\infty^{(0)}\big)=\big(\widehat S_\rho\widehat T_{\imath(n)}\big)(\boldsymbol z)/\big(\widehat S_\rho\widehat T_{\imath(n)}\big)\big(\infty^{(0)}\big)$$

and therefore (2.26) and (2.32) remain valid with $S_\rho(\boldsymbol z)$, $T_{\imath(n)}(\boldsymbol z)$ replaced by $\widehat S_\rho(\boldsymbol z)$, $\widehat T_{\imath(n)}(\boldsymbol z)$. Furthermore, the value of $A_{\rho,n}$ in (2.24) will not change either as the limit in the definition of $A'_{\rho,n}$ will be augmented by $e^{\pi\mathrm i m(1-\mathsf B)}$, see (4.1), that will be offset by the change in c_ρ and σ_k ($\widehat\sigma_k=(-1)^m\sigma_k$). Thus, with a slight abuse of notation, we shall keep on writing $S_\rho(\boldsymbol z)$, $T_{\imath(n)}(\boldsymbol z)$ below.

3.1. Chebyshëv-type case

Let $2\widehat\rho(z)=1/w(z)$, in which case it holds that

$$\rho(s)=1/w_+(s),\quad s\in\Delta,$$

where $\widehat\rho(z)$ and $w(z)$ were defined in (2.30) and (2.1), respectively, and the implication follows from Plemelj-Sokhotski formulae and Privalov's theorem. Using analytic continuations of $w(z)$ one can easily see that $\rho(s)\in\mathcal W_\infty$ and $\nu=0$. Since $(\rho w_+)(s)\equiv1$, we get that $S_\rho(\boldsymbol z)\equiv1$ and necessarily $c_\rho=0$. Thus, $\mathbb N_{\rho,\varepsilon}=2\mathbb N$ and $\boldsymbol z_0=\infty^{(1)}$ ($\boldsymbol z_1=\infty^{(0)}$). Moreover, we get that $T_0(\boldsymbol z)\equiv1$ and $T_1(\boldsymbol z)=1/\Phi(\boldsymbol z)$, see (4.2). Hence, it follows from (2.8) and (2.26) that

$$Q_{2n}(z)=\frac{1+o(1)}{2^n}\left(z^2+\frac{b^2-a^2}{2}+w(z)\right)^n,$$

where it holds that $o(1)$ is geometrically small on closed subsets of $\overline{\mathbb C}\setminus\Delta$ (see [14] for the error rate in this case). To show that the above result is in a way best possible, assume that $a=b=1$. Recall that the n-th monic Chebyshëv polynomial of the first kind is defined by

$$2^nT_n(z)=\left(z+\sqrt{z^2-1}\right)^n+\left(z-\sqrt{z^2-1}\right)^n$$

and is orthogonal to x^j, $j\in\{0,\dots,n-1\}$, on $(-1,1)$ with respect to the weight $1/\sqrt{1-x^2}$. Hence,

$$\mathrm{i}\int_\Delta s^kT_n\big(s^2\big)\rho(s)\mathrm{d}s=\\ \left(\int_0^1-\int_{-1}^0\right)\frac{x^kT_n\big(x^2\big)\mathrm{d}x}{\sqrt{1-x^4}}-\mathrm{i}^{k+1}\left(\int_0^1-\int_{-1}^0\right)\frac{x^kT_n\big(-x^2\big)\mathrm{d}x}{\sqrt{1-x^4}}.$$

Clearly, the above expression is zero for all even k. Assume now that $k=2j+1$, $j\in\{0,\dots,n-1\}$. Then we can continue the above chain of equalities by

$$\int_0^1\frac{x^jT_n(x)\mathrm{d}x}{\sqrt{1-x^2}}-(-1)^{j+1}\int_0^1\frac{x^jT_n(-x)\mathrm{d}x}{\sqrt{1-x^2}}=\int_{-1}^1\frac{x^jT_n(x)\mathrm{d}x}{\sqrt{1-x^2}}=0,$$

where the last equality follows from the orthogonality properties of the Chebyshëv polynomials. Thus, it holds that

$$Q_{2n+1}(z)=Q_{2n}(z)=T_n\big(z^2\big)$$

in this case, which justifies the exclusion of odd indices from $\mathbb{N}_\rho=\mathbb{N}_{\rho,\varepsilon}$ as for such indices polynomials can and do degenerate.

3.2. Legendre-type case

Let $\widehat{\rho}(z)=\frac{1}{2\pi\mathrm{i}}\big(\log(z^2-1)-\log(z^2+1)\big)$, in which case it holds that

$$\rho(s)=(-1)^i,\quad s\in\Delta_i,$$

$i\in\{1,2,3,4\}$, where the justification for the implication is the same as before. As in the previous case, it holds that $\nu=0$. Let $\sqrt{w}(z)$ be the branch holomorphic in $\mathbb{C}\setminus\Delta$ such that $\sqrt{w}(z)=z+\mathcal{O}(1)$ as $z\to\infty$. Further, let

$$\Phi_*(z):=\sqrt{\frac{2}{a^2+b^2}}\left(z^2+\frac{b^2-a^2}{2}+w(z)\right)^{1/2},$$

be the branch holomorphic in $\mathbb{C}\setminus\Delta$ such that $\Phi_*(z)=z+\mathcal{O}(1)$ as $z\to\infty$. It easily follows from (2.7), (2.8), and (2.9) that $\Phi_*(z)$ is an analytic continuation of $-\Phi\big(z^{(0)}\big)$ across $\pi(\boldsymbol{\alpha})\cup\pi(\boldsymbol{\beta})$. It is now straightforward to check that

$$S_\rho\big(z^{(0)}\big)=e^{-\pi\mathrm{i}/4}\Phi_*(z)/\sqrt{w}(z)$$

and thus $c_\rho=0$. Hence, as in the previous subsection, $\mathbb{N}_{\rho,\varepsilon}=2\mathbb{N}$ and $T_0(\boldsymbol{z})\equiv 1$ while $T_1(\boldsymbol{z})=1/\Phi(\boldsymbol{z})$. Therefore, we again deduce from (2.8) and (2.26) that

$$Q_{2n}(z)=\frac{1+\mathcal{O}(n^{-1/2})}{2^{n+1/2}\sqrt{w}(z)}\left(z^2+\frac{b^2-a^2}{2}+w(z)\right)^{n+1/2},$$

uniformly on closed subsets of $\overline{\mathbb{C}}\setminus\Delta$. Again, to show that the above result is best possible, assume that $a=b=1$. Then we can check exactly as in the previous subsection that

$$Q_{2n+1}(z)=Q_{2n}(z)=L_n\big(z^2\big),$$

where $L_n(x)$ is the n-th monic Legendre polynomial, that is, degree n polynomial orthogonal to x^j, $j\in\{0,\ldots,n-1\}$, on $(-1,1)$ with respect to a constant weight.

3.3. Jacobi-$1/4$ case

Let $\sqrt{2}\widehat\rho(z)=1/\sqrt{w}(z)$, in which case it holds that

$$\rho(s)=-\mathrm{i}^{4-i}/|\sqrt{w}(s)|,\quad s\in\Delta_i,\quad i\in\{1,2,3,4\},$$

where $\sqrt{w}(z)$ is the branch defined in the previous subsection. Observe that

$$(\rho w_+)(s)=\mathrm{i}^{i-1}|\sqrt{w}(s)|,\quad s\in\Delta_i,$$

and that $\nu=1/2$. In particular, the constant A_ρ appearing in the definition of $A_{\rho,n}$ in (2.24) is equal to $A_\rho=\sqrt{2}e^{-\pi\mathrm{i}/4}/\sqrt{ab}$.

To construct a Szegő function of $\rho(s)$, let

$$\Theta^2(\boldsymbol z):=\frac{\theta\big(\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega+\mathsf{K}_-\big)}{\theta\big(\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega-\mathsf{K}_-\big)}\frac{\theta\big(\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega-\mathsf{K}_+\big)}{\theta\big(\int_{\boldsymbol a_3}^{\boldsymbol z}\Omega+\mathsf{K}_+\big)},\quad z\in\boldsymbol{\mathfrak{R}}_{\boldsymbol\alpha,\boldsymbol\beta},$$

where the path of integration lies entirely in $\boldsymbol{\mathfrak{R}}_{\boldsymbol\alpha,\boldsymbol\beta}$. It follows from (2.18) and (4.1) further below that $\Theta^2(\boldsymbol z)$ is a meromorphic function in $\boldsymbol{\mathfrak{R}}_{\boldsymbol\alpha,\boldsymbol\beta}$ with two simple poles, namely, $\infty^{(0)},\boldsymbol 0$, and two simple zeros $\infty^{(1)},\boldsymbol 0^*$. Moreover, $\Theta^2(\boldsymbol z)$ is continuous across $\boldsymbol\beta$ and satisfies $\Theta^2_+(\boldsymbol s)=\Theta^2_-(\boldsymbol s)e^{-2\pi\mathrm{i}\mathsf{B}}$ on $\boldsymbol\alpha$ by (2.16) and $\Theta^2(\boldsymbol z)\Theta^2(\boldsymbol z^*)\equiv1$ by the symmetries of $\theta(\zeta)$ and $\Omega(\boldsymbol z)$. Since each individual fraction in the definition of $\Theta^2(\boldsymbol z)$ is injective, we can define a branch $\Theta(\boldsymbol z)$ such that

$$\Theta_+(\boldsymbol s)=\Theta_-(\boldsymbol s)\begin{cases} e^{-\pi\mathrm{i}\mathsf{B}}, & \boldsymbol s\in\boldsymbol\alpha,\\ -1, & \boldsymbol s\in\boldsymbol\Delta_3\cup\pi^{-1}((-\infty,-a]),\end{cases}$$

and $\Theta(\boldsymbol z)\Theta(\boldsymbol z^*)\equiv1$. Further, let $w^{1/4}(z)$ be the branch holomorphic in $\mathbb{C}\setminus\big(\Delta\cup(-\infty,a)\big)$ that is positive for $z>a$. Now, one can verify that $c_\rho=-\mathsf{B}/2$ and

$$S_\rho\big(z^{(k)}\big)=\Theta\big(z^{(k)}\big)w^{\frac{2k-1}{4}}(z),\quad k\in\{0,1\}.$$

Let us now compute $A'_{\rho,n}$ appearing in (2.24). Since $\sqrt{w}(z)\to e^{-3\pi\mathrm{i}/4}\sqrt{ab}$ as $\mathcal{Q}_3\ni z\to0$, we get that

$$\begin{aligned}\lim_{z\to0,\arg(z)=5\pi/4}|z|S^2_\rho\big(z^{(0)}\big) &= \frac{e^{-\pi\mathrm{i}/2}}{\sqrt{ab}}\lim_{\mathcal{Q}_3\ni z\to0}z\Theta^2\big(z^{(0)}\big)\\ &= e^{\pi\mathrm{i}\mathsf{B}/2}\frac{2\sqrt{ab}}{\sqrt{a^2+b^2}}\Phi(\boldsymbol 0),\end{aligned}$$

where the second equality follows from (4.1), (4.5), (4.9), and (4.10) further below. Therefore, it holds that $A'_{\rho,n}=\Phi(\boldsymbol 0)$. It is easy to see from (4.1) that

$\boldsymbol{z}_0 = \mathbf{0}$, $l_0 = 0, m_0 = 1$, and $\boldsymbol{z}_1 = \mathbf{0}^*$, $l_1 = m_1 = 0$. Therefore, $\sigma_{\imath(n)} = -1$ and the condition defining $\mathbb{N}_{\rho,\varepsilon}$ in Proposition 2.2 specializes to

$$\left|1 + \exp\left\{2\mathrm{i}(n - \imath(n)) \arctan(a/b)\right\}\right| > \varepsilon$$

by (2.25) and since $\Phi(\boldsymbol{z}_1)\Phi(\boldsymbol{z}_0) = 1$, see (4.4) further below. As $T_0(\mathbf{0}) = 0$ and respectively $L_{n1} = 0$, we then get that $Q_n(z)$, $n \in \mathbb{N}_{\rho,\varepsilon}$, is equal to

$$\gamma_n\big(S_\rho \Phi^n\big)\big(z^{(0)}\big) \left\{ \begin{array}{ll} \big(T_0\big(z^{(0)}\big) + \mathcal{O}_\varepsilon(n^{-1})\big), & n \in 2\mathbb{N}, \\ \big(T_1\big(z^{(0)}\big) + z^{-1} L_{n2}(T_0/\Phi)\big(z^{(0)}\big) + \mathcal{O}_\varepsilon(n^{-1})\big), & n \notin 2\mathbb{N}, \end{array} \right.$$

uniformly on closed subsets of $\overline{\mathbb{C}} \setminus \Delta$, where

$$L_{n2} = \frac{-1}{(T_0/T_1)'(\mathbf{0})} \frac{\Phi^{2n-1}(\mathbf{0})}{1 + \Phi^{2(n-1)}(\mathbf{0})}$$

for all odd n. When $a = b$, we further get that $L_{n2} = -e^{\pi\mathrm{i}/4}/[2(T_0/T_1)'(\mathbf{0})]$ for $n \in \mathbb{N}_{\rho,\varepsilon}$ and

$$\mathbb{N}_{\rho,\varepsilon} = \{n = 4k, 4k+1 : \ k \in \mathbb{N}\}.$$

Assume further that $a = b = 1$ and let $P_{n,1}(x)$ be the n-th degree monic polynomial orthogonal on $[0,1]$ to x^j, $j \in \{0, \ldots, n-1\}$, with respect to the weight function $x^{-3/4}(1-x)^{-1/4}$. Then

$$\int_\Delta s^k P_{n,1}\big(s^4\big)\rho(s)\mathrm{d}s = \big(1 + \mathrm{i}^k\big) \int_{-1}^{1} y^k P_{n,1}\big(y^4\big) \frac{\mathrm{d}y}{(1-y^4)^{1/4}},$$

which is equal to zero for all k odd by symmetry and for all $k = 4j+2$ due to the factor $1+\mathrm{i}^k$. When $k = 4j$, $j \in \{0, \ldots, n-1\}$, we can further continue the above equality by

$$4\int_0^1 y^{4j} P_{n,1}\big(y^4\big) \frac{\mathrm{d}y}{(1-y^4)^{1/4}} = \int_0^1 x^j P_{n,1}(x) \frac{\mathrm{d}x}{x^{3/4}(1-x)^{1/4}} = 0,$$

where the last equality now holds by the very choice of $P_{n,1}(z)$. Hence, it holds that

$$Q_{4n}(z) = P_{n,1}\big(z^4\big) \quad \text{and} \quad Q_{4n+1}(z) = Q_{4n+2}(z) = Q_{4n+3}(z) = zP_{n,2}\big(z^4\big),$$

where the second set of relations can be shown similarly with $P_{n,2}(x)$ being the n-th degree monic polynomial orthogonal on $[0,1]$ to x^j, $j \in \{0, \ldots, n-1\}$, with respect to the weight function $x^{1/4}(1-x)^{-1/4}$. That is, the restriction to the sequence of indices $\{n = 4k, 4k+1 : \ k \in \mathbb{N}\}$ is not superfluous and the main term of the asymptotics of the polynomials does depend on the parity of n.

4. Auxiliary Identities

In this section we state a number of identities, some of which we have already used and some of which we shall use later.

Lemma 4.1. *Recall* (2.17). *It holds that*

$$\int_{\boldsymbol{a}_3}^{\mathbf{0}} \Omega = -\mathsf{K}_- \quad \textit{and} \quad \int_{\boldsymbol{a}_3}^{\mathbf{0}^*} \Omega = \mathsf{K}_-, \tag{4.1}$$

where the path of integration lies entirely in $\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Proof. Exactly as in the case of (2.18), the symmetries of $\Omega(\boldsymbol{z})$ imply that

$$-\int_{\boldsymbol{a}_3}^{\mathbf{0}} \Omega = \int_{\boldsymbol{a}_3}^{\mathbf{0}^*} \Omega = \frac{1}{2}\int_{\boldsymbol{\Delta}_3} \Omega = \frac{1}{4}\int_{\boldsymbol{\Delta}_3-\boldsymbol{\Delta}_1} \Omega.$$

The claim now follows from the fact that $\boldsymbol{\Delta}_3-\boldsymbol{\Delta}_1$ is homologous to $\boldsymbol{\alpha}-\boldsymbol{\beta}$. □

Lemma 4.2. *It holds that*

$$\Phi(\boldsymbol{z}) = \exp\left\{-\pi\mathrm{i}\int_{\boldsymbol{a}_3}^{\boldsymbol{z}} \Omega\right\} \frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}} \Omega - \mathsf{K}_+\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}} \Omega + \mathsf{K}_+\big)}. \tag{4.2}$$

Proof. It follows from (2.18) and (2.16) that the right hand side of (4.2) is a meromorphic functions with a simple pole at $\infty^{(0)}$, a simple zero at $\infty^{(1)}$, and otherwise non-vanishing and finite that satisfies (2.7). As only holomorphic functions on $\mathfrak{R}$ are constants, the normalization at $\boldsymbol{a}_3$ yields (4.2). □

Lemma 4.3. *Let* l_0, l_1, m_0, m_1 *be given by* (2.20). *Then it holds that*

$$\begin{cases} \Phi(\boldsymbol{z}_0) & = (-1)^{l_0+m_0} e^{-\pi\mathrm{i}(c_\rho-\mathsf{K}_+)}\theta(c_\rho+2\mathsf{K}_-)/\theta(c_\rho), \\ \Phi(\boldsymbol{z}_1) & = (-1)^{l_1+m_1} e^{-\pi\mathrm{i}(c_\rho+\mathsf{K}_+)}\theta(c_\rho)/\theta(c_\rho+2\mathsf{K}_+). \end{cases} \tag{4.3}$$

In particular, when $|\pi(\boldsymbol{z}_k)|<\infty$, *it holds that*

$$\Phi(\boldsymbol{z}_0)\Phi(\boldsymbol{z}_1) = -(-1)^{l_0-l_1+m_0-m_1}. \tag{4.4}$$

Moreover, we have that

$$\Phi\big(\mathbf{0}\big) = e^{\pi\mathrm{i}\mathsf{K}_-}\theta(1/2)/\theta(\mathsf{B}/2). \tag{4.5}$$

Proof. Since $-2\mathsf{K}_+ = 2\mathsf{K}_- - 1$, we get from (4.2) that

$$\Phi(\boldsymbol{z}_0) = e^{\pi\mathrm{i}(\mathsf{K}_+-c_\rho-l_0-m_0\mathsf{B})}\frac{\theta(c_\rho+2\mathsf{K}_-+m_0\mathsf{B})}{\theta(c_\rho+m_0\mathsf{B})}.$$

The first relation in (4.3) now follows from (2.16). Similarly, we have that

$$\Phi(\boldsymbol{z}_1) = e^{\pi\mathrm{i}(-\mathsf{K}_+-c_\rho-l_1-m_1\mathsf{B})}\frac{\theta(c_\rho+m_1\mathsf{B})}{\theta(c_\rho+2\mathsf{K}_++m_1\mathsf{B})},$$

which yields the second relation in (4.3), again by (2.16). To get (4.4), observe that

$$\theta(c_\rho+2\mathsf{K}_-) = \theta(c_\rho+2\mathsf{K}_+-\mathsf{B}) = -e^{2\pi\mathrm{i}c_\rho}\theta(c_\rho+2\mathsf{K}_+)$$

by (2.16). Finally, (4.5) follows from (4.2) and (4.1). □

Lemma 4.4. *Let*

$$X_n := \lim_{z\to\infty} z^{-2}\Psi_n\big(z^{(0)}\big)\Psi_{n-1}\big(z^{(1)}\big). \tag{4.6}$$

When $|\pi(\boldsymbol{z}_k)|<\infty$, *it holds that*

$$X_n = \frac{4}{a^2+b^2}\frac{\theta^2(c_\rho)}{\theta^2(0)}\frac{(-1)^{\imath(n)}}{\Phi^{2\imath(n)}(\boldsymbol{z}_1)}. \tag{4.7}$$

Proof. Since $\Phi(\boldsymbol{z})\Phi(\boldsymbol{z}^*)\equiv 1$ and $S_\rho(\boldsymbol{z})S_\rho(\boldsymbol{z}^*)\equiv 1$, the desired limit is equal to

$$\frac{4}{a^2+b^2}T_{\imath(n)}\big(\infty^{(0)}\big)\lim_{z\to\infty}\Phi\big(z^{(1)}\big)T_{\imath(n-1)}\big(z^{(1)}\big),$$

where we also used (2.9). Since $-2\mathsf{K}_+ = 2\mathsf{K}_- - 1$, it follows from (2.19) and (2.18) that

$$T_{\imath(n)}\big(\infty^{(0)}\big) = e^{\pi\mathrm{i}\imath(n)\mathsf{K}_+}\frac{\theta\big(c_\rho+2\imath(n)\mathsf{K}_-\big)}{\theta(0)}.$$

We further deduce from (2.19) and (4.2) that

$$\big(\Phi T_{\imath(n-1)}\big)(\boldsymbol{z}) = \exp\left\{-\pi\mathrm{i}\imath(n)\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega\right\}\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - c_\rho + (-1)^{\imath(n)}\mathsf{K}_+\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_+\big)}.$$

Therefore, it follows from (2.18) that

$$\big(\Phi T_{\imath(n-1)}\big)\big(\infty^{(1)}\big) = e^{\pi\mathrm{i}\imath(n)\mathsf{K}_+}\frac{\theta\big(c_\rho+2\imath(n)\mathsf{K}_+\big)}{\theta(0)}.$$

Hence, we get from (4.3) that

$$X_n = \frac{4}{a^2+b^2}\frac{\theta^2(c_\rho)}{\theta^2(0)}\left((-1)^{l_0-l_1+m_0-m_1}\frac{\Phi(\boldsymbol{z}_0)}{\Phi(\boldsymbol{z}_1)}\right)^{\imath(n)}.$$

The claim of the lemma now follows from (4.4). □

Lemma 4.5. *It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(e^{\pi\mathrm{i}\zeta}\frac{\theta(\zeta+\mathsf{K}_+)}{\theta(\zeta-\mathsf{K}_+)}\right) = \mathrm{i}\pi\theta^2(0)e^{\pi\mathrm{i}\zeta}\frac{\theta(\zeta-\mathsf{K}_-)\theta(\zeta+\mathsf{K}_-)}{\theta^2(\zeta-\mathsf{K}_+)}. \tag{4.8}$$

Proof. See [4, Eq. (20.7.25)] (observe that $\theta(\zeta)=\theta_3(\pi\zeta|\mathsf{B})$ in the notation of [4, Chapter 20]). □

Lemma 4.6. *It holds that*

$$z = -\frac{\sqrt{a^2+b^2}}{2}\frac{e^{-\pi\mathrm{i}\mathsf{K}_+}\theta^2(0)}{\theta(1/2)\theta(\mathsf{B}/2)}\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega-\mathsf{K}_-\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_-\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega-\mathsf{K}_+\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_+\big)}. \tag{4.9}$$

Proof. It follows from (2.16), (2.18), and (4.1) that

$$z = C\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega-\mathsf{K}_-\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_-\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega-\mathsf{K}_+\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_+\big)}$$

for some normalizing constant C. It further follows from (2.9), (4.2), and (2.18) that

$$-\frac{\sqrt{a^2+b^2}}{2} = \lim_{z\to\infty} z\Phi^{-1}\big(z^{(0)}\big) = Ce^{\pi\mathrm{i}\mathsf{K}_+}\frac{\theta(1/2)\theta(\mathsf{B}/2)}{\theta^2(0)},$$

which yields the desired result. □

Lemma 4.7. *It holds that*

$$e^{\pi\mathrm{i}\mathsf{B}/2}\frac{\theta^2(1/2)\theta^2(\mathsf{B}/2)}{\theta^4(0)} = \frac{a^2+b^2}{4ab}. \tag{4.10}$$

Proof. To prove (4.10), evaluate (4.9) at $\boldsymbol{a}_3$ to get

$$\frac{\theta(1/2)\theta(\mathsf{B}/2)}{\theta^2(0)} = \frac{\sqrt{a^2+b^2}}{2a}e^{-\pi\mathrm{i}\mathsf{K}_+}\frac{\theta^2(\mathsf{K}_-)}{\theta^2(\mathsf{K}_+)}.$$

Since $\boldsymbol{\Delta}_3-\boldsymbol{\Delta}_1$ is homologous to $\boldsymbol{\alpha}-\boldsymbol{\beta}$, one can easily deduce from Figure 1 that it also holds that

$$\int_{\boldsymbol{a}_3}^{\boldsymbol{a}_2}\Omega = \left(\int_{\boldsymbol{a}_3}^{\boldsymbol{0}^*}+\int_{\boldsymbol{0}^*}^{\boldsymbol{a}_1}+\int_{\boldsymbol{a}_1}^{\boldsymbol{a}_2}\right)\Omega = \frac{1}{2}\int_{\boldsymbol{\Delta}_3-\boldsymbol{\Delta}_1+\boldsymbol{\beta}}\Omega = \frac12,$$

where the initial path of integration (except for $\boldsymbol{a}_2$) belongs to $\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Thus, evaluating (4.9) at $\boldsymbol{a}_2$ gives us

$$\frac{\theta(1/2)\theta(\mathsf{B}/2)}{\theta^2(0)} = -\frac{\sqrt{a^2+b^2}}{2\mathrm{i}b}e^{-\pi\mathrm{i}\mathsf{K}_+}\frac{\theta^2(\mathsf{K}_+)}{\theta^2(\mathsf{K}_-)},$$

where we used (2.16). Multiplying two expressions for $\theta(1/2)\theta(\mathsf{B}/2)/\theta^2(0)$ yields the desired result. □

Lemma 4.8. *It holds that*

$$\oint_{\boldsymbol{\alpha}}\frac{\mathrm{d}s}{w(\boldsymbol{s})} = \frac{2\pi\mathrm{i}}{\sqrt{a^2+b^2}}e^{\pi\mathrm{i}\mathsf{K}_+}\theta(1/2)\theta(\mathsf{B}/2). \tag{4.11}$$

Proof. We can deduce from (4.2), (4.8), and the evenness of the theta function that

$$\Phi'(\boldsymbol{z}) = -\mathrm{i}\pi\theta^2(0)\left(\oint_{\boldsymbol{\alpha}}\frac{\mathrm{d}s}{w(\boldsymbol{s})}\right)^{-1}\frac{\Phi(\boldsymbol{z})}{w(\boldsymbol{z})}\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_-\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega-\mathsf{K}_-\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega+\mathsf{K}_+\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega-\mathsf{K}_+\big)}.$$

Since $\Phi'(\boldsymbol{z}) = z\Phi(\boldsymbol{z})/w(\boldsymbol{z})$ by (2.6), (4.11) follows from (4.9). □

Lemma 4.9. *Let*

$$Y_n := \big(T'_{\imath(n)}T_{\imath(n-1)}/\Phi - T_{\imath(n)}(T_{\imath(n-1)}/\Phi)'\big)(\boldsymbol{0}). \tag{4.12}$$

When $|\pi(\boldsymbol{z}_k)| = \infty$, it holds that $Y_n = 0$, otherwise, we have that

$$Y_n = (-1)^{l_0+m_0+\imath(n)}\frac{2e^{\pi\mathrm{i}c_\rho}}{\sqrt{a^2+b^2}}\frac{\Phi(\boldsymbol{z}_0)}{\Phi^2(\boldsymbol{0})}\frac{\theta^2(c_\rho)}{\theta^2(0)}, \tag{4.13}$$

where the integers l_0, m_0 were defined in (2.20).

Proof. Since $\Phi'(\boldsymbol{z}) = z\Phi(\boldsymbol{z})/w(\boldsymbol{z})$ by (2.6), $\Phi'\big(\boldsymbol{0}\big) = 0$. Therefore,

$$Y_n = \big(T^2_{\imath(n-1)}/\Phi\big)\big(\boldsymbol{0}\big)\big(T_{\imath(n)}/T_{\imath(n-1)}\big)'\big(\boldsymbol{0}\big).$$

Assume that $|\pi(\boldsymbol{z}_k)| < \infty$. Then it follows from (2.19), (4.8), and (4.11) that

$$\left(\frac{T_{\imath(n)}}{T_{\imath(n-1)}}\right)'(\boldsymbol{z}) = -(-1)^{\imath(n)}\frac{\sqrt{a^2+b^2}}{2w(\boldsymbol{z})}\frac{e^{-\pi\mathrm{i}\mathsf{K}_+}\theta^2(0)}{\theta(1/2)\theta(\mathsf{B}/2)}\left(\frac{T_{\imath(n)}}{T_{\imath(n-1)}}\right)(\boldsymbol{z})\times$$
$$\times\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - c_\rho + \mathsf{K}_-\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - c_\rho - \mathsf{K}_-\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - c_\rho + \mathsf{K}_+\big)\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega - c_\rho - \mathsf{K}_+\big)}.$$

We further deduce from (2.19), (4.1), and (4.5) that

$$\big(T_{\imath(n-1)}T_{\imath(n)}\big)\big(\boldsymbol{0}\big) = \frac{1}{\Phi\big(\boldsymbol{0}\big)}\frac{\theta(c_\rho - \mathsf{B}/2)\theta(c_\rho + 1/2)}{\theta(1/2)\theta(\mathsf{B}/2)}.$$

Since $w\big(\boldsymbol{0}\big) = \mathrm{i}ab$, we therefore get from (4.1) that

$$Y_n = \frac{\sqrt{a^2+b^2}}{2ab}\frac{\mathrm{i}(-1)^{\imath(n)}}{\Phi^2\big(\boldsymbol{0}\big)}\frac{e^{-\pi\mathrm{i}\mathsf{K}_+}\theta^4(0)}{\theta^2(1/2)\theta^2(\mathsf{B}/2)}\frac{\theta(c_\rho)\theta(c_\rho+2\mathsf{K}_-)}{\theta^2(0)}.$$

(4.13) now follows from (4.10) and the first formula in (4.3).

Let now $\boldsymbol{z}_0 = \infty^{(1)}$, in which case $[c_\rho] = [0]$. Since $\Phi\big(\infty^{(1)}\big) = 0$, we get that $Y_n = 0$. Finally, let $\boldsymbol{z}_1 = \infty^{(1)}$. Then we have that $-c_\rho = -(-1)^k 2\mathsf{K}_+ + l_k + m_k\mathsf{B}$ and therefore

$$\begin{aligned}\frac{T_1(\boldsymbol{z})}{T_0(\boldsymbol{z})} &= \exp\left\{\pi\mathrm{i}\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega\right\}\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega + m_1\mathsf{B} + 3\mathsf{K}_+\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega + (m_1+1)\mathsf{B} - 3\mathsf{K}_+\big)}\\ &= \exp\left\{\pi\mathrm{i}\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega\right\}\frac{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega + (m_1+1)\mathsf{B} - \mathsf{K}_+\big)}{\theta\big(\int_{\boldsymbol{a}_3}^{\boldsymbol{z}}\Omega + m_1\mathsf{B} + \mathsf{K}_+\big)}\\ &= e^{2\pi\mathrm{i}(2m_1+1)\mathsf{K}_-}\Phi(\boldsymbol{z})\end{aligned}$$

by (2.16) and (4.2). As $\Phi'\big(\boldsymbol{0}\big) = 0$, it also holds that $Y_n = 0$. □

Lemma 4.10. *Let*

$$Z_n := \big(T'_{\imath(n)}T_{\imath(n-1)}/\Phi - T_{\imath(n)}(T_{\imath(n-1)}/\Phi)'\big)\big(\boldsymbol{0}^*\big). \tag{4.14}$$

When $|\pi(\boldsymbol{z}_k)| = \infty$, *it holds that* $Z_n = 0$, *otherwise, we have that*

$$Z_n = (-1)^{l_0+m_0+\imath(n)}\frac{2e^{-\pi\mathrm{i}c_\rho}}{\sqrt{a^2+b^2}}\frac{\Phi(\boldsymbol{z}_0)}{\Phi^2\big(\boldsymbol{0}^*\big)}\frac{\theta^2(c_\rho)}{\theta^2(0)}. \tag{4.15}$$

Proof. The proof is the same as in the previous lemma. □

Lemma 4.11. *Let* σ_0,σ_1 *be as in* (2.24). *When* $|\pi(\boldsymbol{z}_k)| < \infty$, *it holds that*

$$Y_nX_n^{-1} = \sigma_{\imath(n)}e^{\pi\mathrm{i}c_\rho}\frac{\sqrt{a^2+b^2}}{2}\frac{\Phi\big(\boldsymbol{z}_{\imath(n)}\big)}{\Phi^2\big(\boldsymbol{0}\big)} \tag{4.16}$$

and

$$Z_nX_n^{-1} = \sigma_{\imath(n)}e^{-\pi\mathrm{i}c_\rho}\frac{\sqrt{a^2+b^2}}{2}\frac{\Phi\big(\boldsymbol{z}_{\imath(n)}\big)}{\Phi^2\big(\boldsymbol{0}^*\big)}, \tag{4.17}$$

where X_n, Y_n, and Z_n are given by (4.6), (4.12), and (4.14), respectively.

Proof. The claim follows immediately from (4.7), (4.13), (4.15), and (4.4). $\square$

5. Proof of Proposition 2.1

It follows from (2.13) that $\Omega_{\boldsymbol z,\boldsymbol z^*}=-\Omega_{\boldsymbol z^*,\boldsymbol z}$ for all $\boldsymbol z\in\mathfrak R$ such that $\pi(\boldsymbol z)\in\mathbb C$ and therefore $S_\rho(\boldsymbol z)S_\rho(\boldsymbol z^*)\equiv 1$ for such $\boldsymbol z$. Clearly, this relation extends to the points on top of infinity by continuity. It is also immediate from (2.12) and (2.13) that

$$S_\rho\big(z^{(0)}\big)=\exp\left\{-\sum_{i=1}^4\frac{w(z)}{2\pi\mathrm i}\int_{\Delta_i}\frac{\log(\rho_iw_+)(s)}{s-z}\frac{\mathrm ds}{w_{|\Delta_i+}(s)}\right\}\times \\ \times\exp\big\{2\pi\mathrm i(wH)(z)c_\rho\big\},\quad (5.1)$$

where, for emphasis, we write $w_{|\Delta_i+}(s)$ for $w_+(s)$ on $s\in\Delta_i^\circ$ and

$$H(z):=\frac{1}{2\pi\mathrm i}\int_{\pi(\boldsymbol\alpha)}\frac{\mathrm dt}{(t-z)w(t)}.\quad (5.2)$$

Relations (2.14) now easily follow from (5.1), (5.2), and Plemelj-Sokhotski formulae [7, equations (4.9)]. As for the behavior near a_i, note that by [7, equation (8.8)], the function $(wH)(z)$ is bounded as $z\to a_i$. Furthermore, [7, equations (8.8) and (8.35)] yield that

$$-\frac{w(z)}{2\pi\mathrm i}\int_{\Delta_i}\frac{\log(\rho_iw_+)(s)}{s-z}\frac{\mathrm ds}{w_{|\Delta_i+}(s)}=-\frac12\log(z-a_i)^{\alpha_i+1/2}+\mathcal O(1).$$

Since the above integral is the only one with singular contribution around a_i, the validity of the top line in (2.15) follows. As for the behavior near the origin, note that $\lim_{\mathcal Q_j\ni z\to0}w(z)=(-1)^{j-1}\mathrm iab$, where, as before, $\mathcal Q_j$ stands for the j-th quadrant. Recall that each segment Δ_i is oriented towards the origin, see Figure 1. Hence, it follows from [7, equation (8.2)] that

$$\begin{aligned}-\frac{w(z)}{2\pi\mathrm i}\int_{\Delta_i}\frac{\log(\rho_iw_+)(s)}{s-z}\frac{\mathrm ds}{w_{\Delta_i+}(s)}&=-\frac{w(z)}{2\pi\mathrm i}\frac{\log(\rho_iw_+)(0)}{w_{|\Delta_i+}(0)}\log(z)+\mathcal O(1)\\&=\frac{(-1)^{j+i}}{2\pi\mathrm i}\log(\rho_iw_+)(0)\log(z)+\mathcal O(1),\quad z\in\mathcal Q_j.\end{aligned}$$

Thus, summing over i yields

$$-\frac{w(z)}{2\pi\mathrm i}\int_{\Delta}\frac{\log(\rho_iw_+)(s)}{s-z}\frac{\mathrm ds}{w_+(s)}=(-1)^j\nu\log(z)+\mathcal O(1),\quad z\in\mathcal Q_j,$$

where ν was defined in (2.10) and we used (2.11). Since $(wH)(z)$ is holomorphic around the origin, the second line in (2.15) follows.

6. Proofs of Theorems 2.3 and 2.4

6.1. Initial RH problem

Just as was first done by Fokas, Its, and Kitaev [5, 6], we connect the orthogonal polynomials $Q_n(z)$ to a 2×2 matrix Riemann-Hilbert problem. To this end, suppose that the index n is such that

$$\deg Q_n = n \quad \text{and} \quad R_{n-1}(z)\sim z^{-n} \quad \text{as} \quad z\to\infty, \tag{6.1}$$

where $R_n(z)$ is given by (2.31). Furthermore, let

$$\boldsymbol{Y}(z) := \begin{pmatrix} Q_n(z) & R_n(z) \\ k_{n-1}Q_{n-1}(z) & k_{n-1}R_{n-1}(z)\end{pmatrix}, \tag{6.2}$$

where k_{n-1} is a constant such that $k_{n-1}R_{n-1}(z) = z^{-n}(1+o(1))$ near infinity. Then $\boldsymbol{Y}(z)$ solves the following Riemann-Hilbert problem (RHP-$\boldsymbol{Y}$):

(a) $\boldsymbol{Y}(z)$ is analytic in $\mathbb{C}\setminus\Delta$ and $\lim_{z\to\infty}\boldsymbol{Y}(z)z^{-n\sigma_3} = \boldsymbol{I}$ [4].

(b) $\boldsymbol{Y}(z)$ has continuous traces on Δ° that satisfy

$$\boldsymbol{Y}_+(s) = \boldsymbol{Y}_-(s)\begin{pmatrix} 1 & \rho(s) \\ 0 & 1\end{pmatrix}, \quad s\in\Delta^\circ.$$

(c) $\boldsymbol{Y}(z)$ is bounded around the origin and

$$\boldsymbol{Y}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix} & \text{if } \alpha_i > 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-a_i| \\ 1 & \log|z-a_i|\end{pmatrix} & \text{if } \alpha_i = 0, \\ \mathcal{O}\begin{pmatrix} 1 & |z-a_i|^{\alpha_i} \\ 1 & |z-a_i|^{\alpha_i}\end{pmatrix} & \text{if } -1<\alpha_i<0, \end{cases}$$

as $z\to a_i$ for each $i\in\{1,2,3,4\}$.

Indeed, property RHP-$\boldsymbol{Y}$(a) is an immediate consequence of (6.1). The jump relations in RHP-$\boldsymbol{Y}$(b) follow from (2.30), (2.31), and an application of the Plemelj-Sokhotski formulae. Behavior of Cauchy integrals around the contours of integration, see [7, Section 8], and an integral representation

$$R_n(z) = \frac{1}{2\pi\mathrm{i}}\int_\Delta \frac{Q_n(s)\rho(s)}{s-z}\mathrm{d}s, \quad z\in\overline{\mathbb{C}}\setminus\Delta,$$

which easily follows from Cauchy integral formula and (2.30), yield RHP-$\boldsymbol{Y}$(c) (to deduce boundedness around the origin one needs to utilize the third condition in the definition of the class $\mathcal{W}_1$).

On the other hand, it also can shown that if a solution of RHP-$\boldsymbol{Y}$ exists, then it must be of the form (6.2) with the diagonal entries satisfying (6.1) (see, for example, [1, Lemma 1]).

In what follows we prove solvability of RHP-$\boldsymbol{Y}$ for all $n\in\mathbb{N}_{\rho,\varepsilon}$ large enough via the matrix steepest descent method developed by Deift and Zhou [3].

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[4] Hereafter, we set $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ and $\boldsymbol{I}$ to be the identity matrix.

6.2. Opening of the Lenses

Let $\delta_0 > 0$ be small enough so that all the functions $\rho_i(z)$ are holomorphic in some neighborhood of $\{|z| \leq \delta_0\}$. Define $\tilde\Delta_i$ and $\tilde\Delta_i^\circ$ to be the closed and open segments connecting the origin and $\delta_0 e^{(2i-1)\pi\mathrm{i}/4}$, $i \in \{1,2,3,4\}$, that are oriented towards the origin. Further, let Γ_{i-}, Γ_{i+} be open smooth

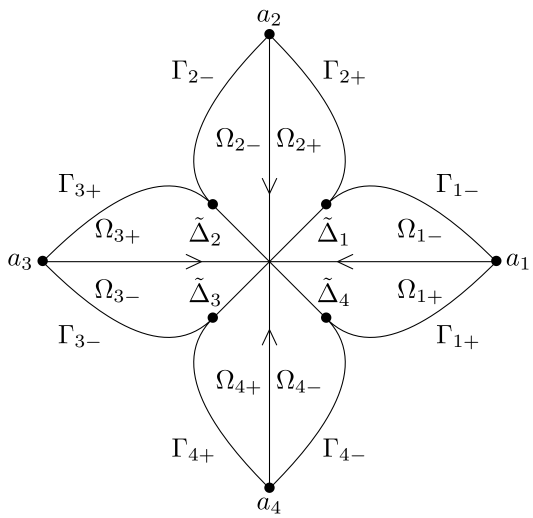

FIGURE 3. The arcs Δ_i, $\tilde\Delta_i$ and $\Gamma_{i\pm}$, and domains $\Omega_{i\pm}$.

arcs that lie within the domain of holomorphy of $\rho_i(z)$ and connect a_i to $\delta_0 e^{(2i-1)\pi\mathrm{i}/4}, \delta_0 e^{(2i-3)\pi\mathrm{i}/4}$, respectively. We orient $\Gamma_{i\pm}$ away from a_i and assume that no open arcs $\Delta_i^\circ, \tilde\Delta_i^\circ, \Gamma_{i\pm}$ intersect, see Figure 3. We denote by $\Omega_{i\pm}$ the domain partially bounded by Δ_i and $\Gamma_{i\pm}$. Let

$$\boldsymbol{X}(z) := \boldsymbol{Y}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp 1/\rho_i(z) & 1 \end{pmatrix}, & z \in \Omega_{i\pm}, \\ \boldsymbol{I}, & z \notin \overline{\Omega}_{i+} \cup \overline{\Omega}_{i-}. \end{cases} \tag{6.3}$$

Then $\boldsymbol{X}(z)$ satisfies the following Riemann-Hilbert problem (RHP-$\boldsymbol{X}$):

(a) $\boldsymbol{X}(z)$ is analytic in $\mathbb{C} \setminus \cup_i(\Delta_i \cup \tilde\Delta_i \cup \Gamma_{i\pm})$ and $\lim_{z\to\infty} \boldsymbol{X}(z) z^{-n\sigma_3} = \boldsymbol{I}$;

(b) $\boldsymbol{X}(z)$ has continuous traces on each Δ_i°, $\tilde\Delta_i^\circ$, and $\Gamma_{i\pm}$ that satisfy

$$\boldsymbol{X}_+(s) = \boldsymbol{X}_-(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1/\rho_i(s) & 1 \end{pmatrix}, & s \in \Gamma_{i+} \cup \Gamma_{i-}, \\ \begin{pmatrix} 0 & \rho_i(s) \\ -1/\rho_i(s) & 0 \end{pmatrix}, & s \in \Delta_i^\circ, \\ \begin{pmatrix} 1 & 0 \\ \dfrac{1}{\rho_i(s)} + \dfrac{1}{\rho_{i+1}(s)} & 1 \end{pmatrix}, & s \in \tilde\Delta_i^\circ, \end{cases}$$

where $i \in \{1,2,3,4\}$ and $\rho_5 := \rho_1$.

(c) $\boldsymbol{X}(z)$ is bounded around the origin and behaves like

$$\boldsymbol{X}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha_i > 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-a_i| \\ 1 & \log|z-a_i| \end{pmatrix} & \text{if } \alpha_i = 0, \\ \mathcal{O}\begin{pmatrix} 1 & |z-a_i|^{\alpha_i} \\ 1 & |z-a_i|^{\alpha_i} \end{pmatrix} & \text{if } -1 < \alpha_i < 0, \end{cases}$$

as $z \to a_i$ from outside the lens while from inside the lens,

$$\boldsymbol{X}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z-a_i|^{-\alpha_i} & 1 \\ |z-a_i|^{-\alpha_i} & 1 \end{pmatrix} & \text{if } \alpha_i > 0, \\ \mathcal{O}\begin{pmatrix} 1 & \log|z-a_i| \\ 1 & \log|z-a_i| \end{pmatrix} & \text{if } \alpha_i = 0, \\ \mathcal{O}\begin{pmatrix} 1 & |z-a_i|^{\alpha_i} \\ 1 & |z-a_i|^{\alpha_i} \end{pmatrix} & \text{if } -1 < \alpha_i < 0. \end{cases}$$

The following observation can be easily checked: RHP-$\boldsymbol{X}$ is solvable if and only if RHP-$\boldsymbol{Y}$ is solvable. When solutions of RHP-$\boldsymbol{X}$ and RHP-$\boldsymbol{Y}$ exist, they are unique and connected by (6.3).

6.3. Global Parametrix

Let $\Psi_n(\boldsymbol{z})$ be given by (2.22). For each $n \in \mathbb{N}_{\rho,\varepsilon}$, define

$$\boldsymbol{N}(z) := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma^*_{n-1} \end{pmatrix} \begin{pmatrix} \Psi_n\big(z^{(0)}\big) & \Psi_n\big(z^{(1)}\big)/w(z) \\ \Psi_{n-1}\big(z^{(0)}\big) & \Psi_{n-1}\big(z^{(1)}\big)/w(z) \end{pmatrix}, \tag{6.4}$$

where the constants γ_n and γ^*_{n-1} are defined by the relations

$$\lim_{z\to\infty} \gamma_n z^{-n} \Psi_n\big(z^{(0)}\big) = 1 \quad \text{and} \quad \lim_{z\to\infty} \gamma^*_{n-1} z^n \Psi_{n-1}\big(z^{(1)}\big)/w(z) = 1. \tag{6.5}$$

Such constants do exist, see the explanation after Proposition 2.2. The product $\gamma_n\gamma^*_{n-1}$ assumes only two necessarily finite and non-zero values depending on the parity of n (when $|\pi(\boldsymbol{z}_k)| < \infty$, it is equal to X_n^{-1}, see (4.6)). The matrix $\boldsymbol{N}(z)$ solves the following Riemann-Hilbert problem (RHP-$\boldsymbol{N}$):

(a) $\boldsymbol{N}(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z\to\infty} \boldsymbol{N}(z) z^{-n\sigma_3} = \boldsymbol{I}$;

(b) $\boldsymbol{N}(z)$ has continuous traces on Δ° that satisfy

$$\boldsymbol{N}_+(s) = \boldsymbol{N}_-(s) \begin{pmatrix} 0 & \rho(s) \\ -1/\rho(s) & 0 \end{pmatrix}, \quad s \in \Delta^\circ;$$

(c) $\boldsymbol{N}(z)$ satisfies

$$\boldsymbol{N}(z) = \mathcal{O}\begin{pmatrix} |z-a_i|^{-(2\alpha_i+1)/4} & |z-a_i|^{(2\alpha_i-1)/4} \\ |z-a_i|^{-(2\alpha_i+1)/4} & |z-a_i|^{(2\alpha_i-1)/4} \end{pmatrix} \quad \text{as} \quad z \to a_i,$$

$i \in \{1,2,3,4\}$, and

$$\boldsymbol{N}(z) = \mathcal{O}\begin{pmatrix} |z|^{(-1)^j\mathrm{Re}(\nu)} & |z|^{(-1)^{j+1}\mathrm{Re}(\nu)} \\ |z|^{(-1)^j\mathrm{Re}(\nu)} & |z|^{(-1)^{j+1}\mathrm{Re}(\nu)} \end{pmatrix} \quad \text{as} \quad z \to 0,$$

where $j \in \{1,2,3,4\}$ is the number of the quadrant from which $z \to 0$ an ν is given by (2.10).

Indeed, RHP-$\boldsymbol{N}$(a) holds by construction, while RHP-$\boldsymbol{N}$(b,c) follow from (2.23) and (2.15), respectively (notice that the actual rate of behavior in RHP-$\boldsymbol{N}$(c) can be different if the considered point happens to coincide with $z_{\imath(n)}$ or $z_{\imath(n-1)}$). Notice also that $\det(\boldsymbol{N}(z)) \equiv 1$ since this is an entire function (it clearly has no jumps and it can have at most square root singularities at the points a_i) that converges to 1 at infinity.

For later calculations it will be convenient to set

$$\boldsymbol{M}^\star(z) := \begin{pmatrix} (S_\rho T_{\imath(n)})(z^{(0)}) & (S_\rho T_{\imath(n)})(z^{(1)})/w(z) \\ (S_\rho T_{\imath(n-1)}/\Phi)(z^{(0)}) & (S_\rho T_{\imath(n-1)}/\Phi)(z^{(1)})/w(z) \end{pmatrix}, \tag{6.6}$$

and $\boldsymbol{M}(z) := (\boldsymbol{I} + \boldsymbol{L}_\nu/z)\,\boldsymbol{M}^\star(z)$, where $\boldsymbol{L}_\nu$ is a certain constant matrix with zero trace and determinant defined further below in (6.26). Observe that $\boldsymbol{N}(z) = \boldsymbol{C}\boldsymbol{M}^\star(z)\boldsymbol{D}(z)$, where

$$\boldsymbol{C} := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_{n-1}^* \end{pmatrix} \quad \text{and} \quad \boldsymbol{D}(z) := \Phi^{n\sigma_3}\big(z^{(0)}\big). \tag{6.7}$$

When $\mathrm{Re}(\nu) \in (-1/2, 1/2)$, it is possible to take $\boldsymbol{L}_\nu$ to be the zero matrix, but this would worsen the error rates in (2.26) and (2.32). When $\mathrm{Re}(\nu) = 1/2$, our analysis necessitates introduction of $\boldsymbol{L}_\nu$. Notice that neither the normalization of $\boldsymbol{M}(z)$ at infinity not its determinate do not depend on $\boldsymbol{L}_\nu$. In fact, it holds that $\det(\boldsymbol{M}(z)) = \det(\boldsymbol{M}^\star(z)) = (\gamma_n\gamma_{n-1}^*)^{-1}$.

6.4. Local Parametrix around a_i

Let U_i be a disk around a_i of small enough radius so that $\rho_i(z)$ is holomorphic around $\overline{U}_i$, $i \in \{1,2,3,4\}$. In this section we construct solution of RHP-$\boldsymbol{X}$ locally in each U_i. More precisely, we seeking a solution of the following local Riemann-Hilbert problem (RHP-$\boldsymbol{P}_{a_i}$):

(a,b,c) $\boldsymbol{P}_{a_i}(z)$ satisfies RHP-$\boldsymbol{X}$(a,b,c) within U_i;

(d) $\boldsymbol{P}_{a_i}(s) = \boldsymbol{M}(s)\big(\boldsymbol{I} + \boldsymbol{\mathcal{O}}(1/n)\big)\boldsymbol{D}(s)$ uniformly for $s \in \partial U_i$.

We shall only construct a solution of RHP-$\boldsymbol{P}_{a_1}$ as other constructions are almost identical.

6.4.1. Model Problem.

Below, we always assume that the real line as well as its subintervals are oriented from left to right. Further, we set

$$I_\pm := \big\{z : \ \arg(\zeta) = \pm 2\pi/3\big\},$$

where the rays $I_\pm$ are oriented towards the origin. Given $\alpha > -1$, let $\boldsymbol{\Psi}_\alpha(\zeta)$ be a matrix-valued function such that

(a) $\boldsymbol{\Psi}_\alpha(\zeta)$ is analytic in $\mathbb{C} \setminus \big(I_+ \cup I_- \cup (-\infty, 0]\big)$;

(b) $\boldsymbol{\Psi}_\alpha(\zeta)$ has continuous traces on $I_+ \cup I_- \cup (-\infty, 0)$ that satisfy

$$\boldsymbol{\Psi}_{\alpha+} = \boldsymbol{\Psi}_{\alpha-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm \pi \mathrm{i} \alpha} & 1 \end{pmatrix} & \text{on} \quad I_\pm; \end{cases}$$

(c) as $\zeta \to 0$ it holds that

$$\boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} \log|\zeta| & \log|\zeta| \\ \log|\zeta| & \log|\zeta| \end{pmatrix}$$

when $\alpha < 0$ and $\alpha = 0$, respectively, and

$$\boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix}$$

when $\alpha > 0$, for $|\arg(\zeta)| < 2\pi/3$ and $2\pi/3 < |\arg(\zeta)| < \pi$, respectively;

(d) it holds uniformly in $\mathbb{C} \setminus \big(I_+ \cup I_- \cup (-\infty, 0]\big)$ that

$$\boldsymbol{\Psi}_\alpha(\zeta) = \boldsymbol{S}(\zeta)\left(\boldsymbol{I} + \boldsymbol{\mathcal{O}}\left(\zeta^{-1/2}\right)\right)\exp\left\{2\zeta^{1/2}\sigma_3\right\},$$

where $\boldsymbol{S}(\zeta) := \dfrac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$ and we take the principal branch of $\zeta^{1/4}$.

Explicit construction of this matrix can be found in [9] (it uses modified Bessel and Hankel functions). Observe that

$$\boldsymbol{S}_+(\zeta) = \boldsymbol{S}_-(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.8}$$

since the principal branch of $\zeta^{1/4}$ satisfies $\zeta_+^{1/4} = \mathrm{i}\zeta_-^{1/4}$. Also notice that the matrix $\sigma_3\boldsymbol{\Psi}_\alpha(\zeta)\sigma_3$ satisfies RHP-$\boldsymbol{\Psi}_\alpha$ only with the reversed orientation of $(-\infty, 0]$ and $I_\pm$.

6.4.2. Conformal Map. Since $w(z)$ has a square root singularity and a_1 and satisfies $w_+(s) = -w_-(s)$, $s \in \Delta$, the function

$$\zeta_{a_1}(z) := \left(\frac{1}{2}\int_{a_1}^z \frac{s\mathrm{d}s}{w(s)}\right)^2, \quad z \in U_1, \tag{6.9}$$

is holomorphic in U_1 with a simple zero at a_1. Thus, the radius of U_1 can be made small enough so that $\zeta_{a_1}(z)$ is conformal on $\overline{U}_1$. Observe that $s\mathrm{d}s/w_\pm(s)$ is purely imaginary on Δ_1° and therefore $\zeta_{a_1}(z)$ maps $\Delta_1 \cap U_1$ into the negative reals. It is also rather obvious that $\zeta_{a_1}(z)$ maps the interval $(a_1, \infty) \cap U_1$ into the positive reals. As we have had some freedom in choosing the arcs $\Gamma_{1\pm}$, we shall choose them within U_1 so that Γ_{1-} is mapped into I_+ and Γ_{1+} is mapped into I_-. Notice that the orientation of the images of $\Delta_1, \Gamma_{1+}, \Gamma_{1-}$ under $\zeta_{a_1}(z)$ are opposite from the ones of $(-\infty, 0], I_-, I_+$.

In what follows, we understand that $\zeta_{a_1}^{1/2}(z)$ stands for the branch given by the expression in the parenthesis in (6.9).

6.4.3. Matrix $\boldsymbol{P}_{a_1}$. According to the definition of the class $\mathcal{W}_1$, it holds that

$$\rho(z) = \rho_*(z)(a_1 - z)^{\alpha_1}, \quad z \in U_1,$$

where $\rho_*(z)$ is non-vanishing and holomorphic in U_1 and $(a_1 - z)^{\alpha_1}$ is the branch holomorphic in $U_1 \setminus [a_1, \infty)$ and positive on Δ_1. Define

$$r_{a_1}(z) := \sqrt{\rho_*(z)}(z - a_1)^{\alpha_1/2}, \quad z \in U_1 \setminus \Delta_1,$$

where $(z - a_1)^{\alpha_1/2}$ is the principle branch. It clearly holds that

$$(z - a_1)^{\alpha_1} = e^{\pm\pi\mathrm{i}\alpha_1}(a_1 - z)^{\alpha_1}, \quad z \in U_1^{\pm},$$

where $U_1^{\pm} := U_1 \cap \{\pm\mathrm{Im}(z) > 0\}$. Then

$$\begin{cases} r_{a_1+}(s)r_{a_1-}(s) = \rho(s), & s \in \Delta_1 \cap U_1, \\ r_{a_1}^2(z) = \rho(z)e^{\pm\pi\mathrm{i}\alpha_1}, & z \in U_1^{\pm}. \end{cases}$$

The above relations and RHP-$\boldsymbol{\Psi}_\alpha$(a,b,c) imply that

$$\boldsymbol{P}_{a_1}(z) := \boldsymbol{E}_{a_1}(z)\sigma_3\boldsymbol{\Psi}_{\alpha_1}\left(n^2\zeta_{a_1}(z)\right)\sigma_3 r_{a_1}^{-\sigma_3}(z) \tag{6.10}$$

satisfies RHP-$\boldsymbol{P}_{a_1}$(a,b,c) for any holomorphic matrix $\boldsymbol{E}_{a_1}(z)$.

6.4.4. Matrix $\boldsymbol{E}_{a_1}$. Now we choose $\boldsymbol{E}_{a_1}(z)$ so that RHP-$\boldsymbol{P}_{a_1}$(d) is fulfilled. To this end, denote by V_1, V_2, V_3 the sectors within U_1 delimited by $\pi(\boldsymbol{\alpha}) \cup \pi(\boldsymbol{\beta})$, $\pi(\boldsymbol{\beta}) \cup \Delta_1$, and $\Delta_1 \cup \pi(\boldsymbol{\alpha})$, respectively, see Figure 1. Let $\gamma \subset \mathbb{C} \setminus \Delta$ be a path from a_3 to a_1 that does not intersect $\pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})$. Further, let $\boldsymbol{\gamma} := \pi^{-1}(\gamma)$ be a cycle oriented so that $\boldsymbol{\gamma}^{(0)} := \boldsymbol{\gamma} \cap \mathfrak{R}^{(0)}$ proceeds from $\boldsymbol{a}_3$ to $\boldsymbol{a}_1$. Define

$$K_{a_1}(z) := \begin{cases} \exp\left\{\int_{\boldsymbol{\gamma}^{(0)}} G\right\} = \exp\{\pi\mathrm{i}\,(\tau - \omega)\} = 1, & z \in V_1, \\ \exp\left\{\int_{\boldsymbol{\gamma}^{(0)} - \boldsymbol{\alpha}} G\right\} = \exp\{-\pi\mathrm{i}(\tau + \omega)\} = -1, & z \in V_2, \\ \exp\left\{\int_{\boldsymbol{\gamma}^{(0)} - \boldsymbol{\beta}} G\right\} = \exp\{\pi\mathrm{i}(\tau + \omega)\} = -1, & z \in V_3, \end{cases}$$

where we used the symmetry $G(\boldsymbol{z}^*) = -G(\boldsymbol{z})$, the fact that $\boldsymbol{\gamma}$ is homologous to $\boldsymbol{\alpha} + \boldsymbol{\beta}$, see Figure 2, and (2.4)–(2.5). Recalling the definition of $\Phi(\boldsymbol{z})$ in (2.6) (the path of integration must lie in $\mathfrak{R}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$), one can see that

$$\Phi\big(z^{(0)}\big) = K_{a_1}(z)\exp\left\{2\zeta_{a_1}^{1/2}(z)\right\}, \quad z \in V_1 \cup V_2 \cup V_3.$$

Clearly, $|K_{a_1}(z)| = 1$. It now follows from RHP-$\boldsymbol{\Psi}_\alpha$(d) that

$$\boldsymbol{P}_{a_1}(s) = \boldsymbol{E}_{a_1}(s)\sigma_3\boldsymbol{S}\big(n^2\zeta_{a_1}(s)\big)\sigma_3 r_{a_1}^{-\sigma_3}(s)K_{a_1}^{-n\sigma_3}(s)\big(\boldsymbol{I} + \boldsymbol{\mathcal{O}}(1/n)\big)\boldsymbol{D}(s)$$

for $s \in \partial U_1$. Thus, if the matrix

$$\boldsymbol{E}_{a_1}(z) := \boldsymbol{M}(z)K_{a_1}^{n\sigma_3}(z)r_{a_1}^{\sigma_3}(z)\sigma_3\boldsymbol{S}^{-1}\big(n^2\zeta_{a_1}(z)\big)\sigma_3$$

is holomorphic in U_1, RHP-$\boldsymbol{P}_{a_1}$(d) is clearly fulfilled. The fact that it has no jumps on $\Delta_1, \pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})$ follows from RHP-$\boldsymbol{N}$(b), (6.8), (2.7), and the definition of $K_{a_1}(z)$. Thus, it is holomorphic in $U_1 \setminus \{a_1\}$. Since $|r_{a_1}(z)| \sim |z - a_1|^{\alpha_1/2}$, $\boldsymbol{S}^{-1}\big(n^2\zeta_{a_1}(z)\big) \sim |z - a_1|^{\sigma_3/4}$, and $\boldsymbol{M}(z)$ satisfies RHP-$\boldsymbol{N}$(c) around a_1, the desired claim follows.

6.5. Approximate Local Parametrix around the Origin

Let $0 < \delta \leq \delta_0$, see Section 6.2. We can assume that the closure of $U_\delta := \{|z| < \delta\}$ is disjoint from $\pi(\boldsymbol{\alpha}), \pi(\boldsymbol{\beta})$. In this section we construct an approximate solution of RHP-$\boldsymbol{X}$ in U_δ when $\ell < \infty$ and an exact solution of RHP-$\boldsymbol{X}$ in U_δ when $\ell = \infty$.

To this end, let the functions $b_i(z)$, $i \in \{1,2,3,4\}$, be defined in $\overline{U}_{\delta_0}$ by

$$b_1 := \frac{\rho_1 + \rho_2}{\rho_2}, \; b_2 := -\frac{\rho_2 + \rho_3}{\rho_4}, \; b_3 := -\frac{\rho_3 + \rho_4}{\rho_2}, \text{ and } b_4 := \frac{\rho_1 + \rho_4}{\rho_4}, \quad (6.11)$$

which are holomorphic and non-vanishing on $\overline{U}_\delta$. It follows from item (iv) in the definition of class $\mathcal{W}_l$ that

$$\frac{b_i(0)}{b_i(z)} - 1 = \mathcal{O}\big(z^\ell\big) \quad \text{as} \quad z \to 0, \quad i \in \{1,2,3,4\}. \quad (6.12)$$

Notice that $b_i(z) \equiv b_i(0)$ when $\ell = \infty$. Observe also that $b_1(0) = b_3(0)$ and $b_2(0) = b_4(0)$ by item (ii) in the definition of class $\mathcal{W}_l$. We seeking a solution of the following local Riemann-Hilbert problem (RHP-$\boldsymbol{P}_0$):

(a) $\boldsymbol{P}_0(z)$ satisfies RHP-$\boldsymbol{X}$(a) within U_δ;

(b) $\boldsymbol{P}_0(z)$ satisfies RHP-$\boldsymbol{X}$(b) within U_δ, where the jump matrix on each $\tilde{\Delta}_i^\circ$ needs to be replaced by

$$\begin{pmatrix} 1 & 0 \\ \dfrac{b_i(0)}{b_i(s)}\left(\dfrac{1}{\rho_i(s)} + \dfrac{1}{\rho_{i+1}(s)}\right) & 1 \end{pmatrix};$$

(c) $\boldsymbol{P}_0(s) = \boldsymbol{M}(s)\big(\boldsymbol{I} + \boldsymbol{\mathcal{O}}\big((n\delta^2)^{-1/2-|\mathrm{Re}(\nu)|}\big)\big)\boldsymbol{D}(s)$ uniformly for $s \in \partial U_\delta$ and $\delta \leq \delta_0$.

6.5.1. Model Problem. Let $s_1, s_2 \in \mathbb{C}$ be independent parameters and let $\nu \in \mathbb{C}$, $\mathrm{Re}(\nu) \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be given by

$$e^{-2\pi \mathrm{i}\nu} := 1 - s_1 s_2 \quad (6.13)$$

(we slightly abuse the notation here as the parameter ν has already been fixed in (2.10); however, we shall use the construction below with parameters s_1, s_2 such that (6.13) holds with ν from (2.10)). Define constants d_1, d_2 by

$$d_1 := -s_1 \frac{\Gamma(1+\nu)}{\sqrt{2\pi}} \quad \text{and} \quad d_2 := -s_2 e^{\pi \mathrm{i}\nu} \frac{\Gamma(1-\nu)}{\sqrt{2\pi}}, \quad (6.14)$$

where $\Gamma(z)$ is the standard Gamma function. It follows from the well-known Gamma function identities that

$$d_1 d_2 = \mathrm{i}\nu. \quad (6.15)$$

Denote by $D_\mu(\zeta)$ the parabolic cylinder function in Whittaker's notations, see [4, Section 12.2]. It is an entire function with the asymptotic expansion

$$D_\mu(\zeta) \sim e^{-\zeta^2/4} \zeta^\mu \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-2k)} \frac{1}{(2\zeta^2)^k} \quad (6.16)$$

valid uniformly in each $|\arg(\zeta)| \leq 3\pi/4 - \epsilon$, $\epsilon > 0$, see [4, Equation (12.9.1)].

Let the matrix function $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ be given by

$$\begin{pmatrix} D_\nu(2\zeta) & d_1 D_{-\nu-1}(-2\mathrm{i}\zeta) \\ d_2 D_{\nu-1}(2\zeta) & D_{-\nu}(-2\mathrm{i}\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\pi\mathrm{i}\nu/2} \end{pmatrix}, \qquad \arg(\zeta) \in \left(0, \tfrac{\pi}{2}\right),$$

$$\begin{pmatrix} D_\nu(-2\zeta) & d_1 D_{-\nu-1}(-2\mathrm{i}\zeta) \\ -d_2 D_{\nu-1}(-2\zeta) & D_{-\nu}(-2\mathrm{i}\zeta) \end{pmatrix} \begin{pmatrix} e^{\pi\mathrm{i}\nu} & 0 \\ 0 & e^{-\pi\mathrm{i}\nu/2} \end{pmatrix}, \quad \arg(\zeta) \in \left(\tfrac{\pi}{2}, \pi\right),$$

$$\begin{pmatrix} D_\nu(-2\zeta) & -d_1 D_{-\nu-1}(2\mathrm{i}\zeta) \\ -d_2 D_{\nu-1}(-2\zeta) & D_{-\nu}(2\mathrm{i}\zeta) \end{pmatrix} \begin{pmatrix} e^{-\pi\mathrm{i}\nu} & 0 \\ 0 & e^{\pi\mathrm{i}\nu/2} \end{pmatrix}, \quad \arg(\zeta) \in \left(-\tfrac{\pi}{2}, -\pi\right),$$

$$\begin{pmatrix} D_\nu(2\zeta) & -d_1 D_{-\nu-1}(2\mathrm{i}\zeta) \\ d_2 D_{\nu-1}(2\zeta) & D_{-\nu}(2\mathrm{i}\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi\mathrm{i}\nu/2} \end{pmatrix}, \qquad \arg(\zeta) \in \left(0, -\tfrac{\pi}{2}\right).$$

Then, $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ satisfies the following RH problem (RHP-$\boldsymbol{\Psi}_{s_1,s_2}$):

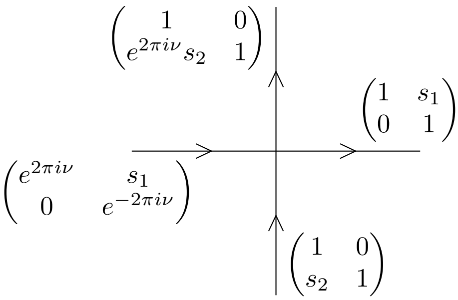

FIGURE 4. Matrices $\boldsymbol{\Psi}_{s_1,s_2-}^{-1}\boldsymbol{\Psi}_{s_1,s_2+}$ on the corresponding rays.

(a) $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ is analytic in $\mathbb{C}\setminus(\mathbb{R}\cup\mathrm{i}\mathbb{R})$;
(b) $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ has continuous traces on $\mathbb{R}\cup\mathrm{i}\mathbb{R}$ outside of the origin that satisfy the jump relations shown in Figure 4;
(c) $\boldsymbol{\Psi}_{s_1,s_2}(\zeta)$ has the following asymptotic expansion as $\zeta\to\infty$:

$$\left(\boldsymbol{I} + \frac{1}{2\zeta}\begin{pmatrix} 0 & \mathrm{i}d_1 \\ d_2 & 0 \end{pmatrix} + \frac{\nu(\nu-1)}{8\zeta^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \boldsymbol{\mathcal{O}}\left(\frac{1}{\zeta^3}\right)\right)(2\zeta)^{\nu\sigma_3}e^{-\zeta^2\sigma_3},$$

which holds uniformly in $\mathbb{C}$.

Indeed, RHP-$\boldsymbol{\Psi}_{s_1,s_2}$(a) follows from the fact that $D_\nu(\zeta)$ is entire, while RHP-$\boldsymbol{\Psi}_{s_1,s_2}$(c) is a consequence of (6.16). The jump relations RHP-$\boldsymbol{\Psi}_{s_1,s_2}$(b) can be verified using the identities $\Gamma(-\nu)\Gamma(1+\nu) = -\pi/\sin(\pi\nu)$, (6.13), and

$$D_\mu(2\xi) = e^{-\mu\pi\mathrm{i}}D_\mu(-2\xi) + \frac{\sqrt{2\pi}}{\Gamma(-\mu)}e^{-(\mu+1)\pi\mathrm{i}/2}D_{-\mu-1}(2\mathrm{i}\xi),$$

suitably applied with parameter values $\mu = -\nu, \nu - 1$ and $\xi = \zeta, -\zeta, \mathrm{i}\zeta$. For later, it will be important for us to make the following observation. Define

$$d_\nu := \begin{cases} d_2, & \mathrm{Re}(\nu) > 0, \\ 0, & \mathrm{Re}(\nu) = 0, \\ \mathrm{i}d_1, & \mathrm{Re}(\nu) < 0 \end{cases} \quad \text{and} \quad \boldsymbol{A}_\nu := \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \mathrm{Re}(\nu) \geq 0, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \mathrm{Re}(\nu) < 0, \end{cases} \tag{6.17}$$

Recall that we set $\varsigma_\nu = 1, 0, -1$ depending on whether $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(\nu) = 0$, or $\mathrm{Re}(\nu) < 0$. Observe that

$$\begin{aligned} &\big(\boldsymbol{I} - (2\zeta)^{-1} d_\nu \boldsymbol{A}_\nu\big) \boldsymbol{\Psi}_{s_1,s_2}(\zeta) \\ &\qquad = (2\zeta)^{\nu\sigma_3} \left(\boldsymbol{I} + (2\zeta)^{-1-2\varsigma_\nu \nu} d_{-\nu} \boldsymbol{A}_{-\nu} + \boldsymbol{\mathcal{O}}\left(\zeta^{-1-|\varsigma_\nu|}\right)\right) e^{-\zeta^2\sigma_3}. \end{aligned} \tag{6.18}$$

6.5.2. Conformal Map. Let, as before, $\mathcal{Q}_j$ stand for the j-th quadrant, $j \in \{1, 2, 3, 4\}$. Set

$$\zeta_0(z) := \left((-1)^{j-1} \int_0^z \frac{s\mathrm{d}s}{w(s)}\right)^{1/2}, \quad z \in U_\delta \cap \mathcal{Q}_j. \tag{6.19}$$

Since $w(z)$ is bounded at 0 and satisfies $w_+(s) = -w_-(s)$, $s \in \Delta$, the branch of the square root can be chosen so that the function $\zeta_0(z)$ is in fact holomorphic in U_δ with a simple zero at the origin. Without loss of generality we can assume that δ is small enough for $\zeta_0(z)$ to be conformal on $\overline{U}_\delta$.

Since the integrand $(-1)^{j-1} s\mathrm{d}s/w(s)$ becomes negative purely imaginary on $\Delta_1 \cup \Delta_3$, the square root in (6.19) can be chosen so that $\arg\big(\zeta_0(z)\big) = -\pi/4$, $z \in \Delta_3^\circ$. As we have had some freedom in selecting the arcs $\tilde{\Delta}_i$, we shall choose them so that $\tilde{\Delta}_3^\circ$ and $\tilde{\Delta}_1^\circ$ are mapped by $\zeta_0(z)$ into positive and negative reals, respectively, while $\tilde{\Delta}_4^\circ$ and $\tilde{\Delta}_2^\circ$ are mapped into positive and negative purely imaginary numbers.

6.5.3. Matrix $\boldsymbol{P}_0$. Define the function $r(z) := r_j(z)$, $z \in \mathcal{Q}_j$, where we let

$$r_1 := \mathrm{i}e^{\pi\mathrm{i}\nu}\sqrt{\rho_1}, \; r_2 := \mathrm{i}e^{-\pi\mathrm{i}\nu}\frac{\rho_2}{\sqrt{\rho_1}}, \; r_3 := -\mathrm{i}e^{-\pi\mathrm{i}\nu}\frac{\rho_4}{\sqrt{\rho_1}}, \; r_4 := -\mathrm{i}e^{-\pi\mathrm{i}\nu}\sqrt{\rho_1} \tag{6.20}$$

for a fixed determination of $\sqrt{\rho_1(z)}$. Furthermore, let

$$\boldsymbol{J}(z) := \begin{cases} e^{2\pi\mathrm{i}\nu\sigma_3}, & \arg z \in \left(-\frac{\pi}{2}, 0\right), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{2\pi\mathrm{i}\nu\sigma_3}, & \arg z \in \left(0, \frac{\pi}{4}\right), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \arg z \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\pi\right), \\ \boldsymbol{I}, & \arg z \in \left(\frac{\pi}{2}, \pi\right). \end{cases} \tag{6.21}$$

Finally, recalling (6.11), put

$$s_1 := b_1(0) = b_3(0) \quad \text{and} \quad s_2 := b_2(0) = b_4(0). \tag{6.22}$$

Notice that since $(\rho_1+\rho_2+\rho_3+\rho_4)(0)=0$, the parameters s_1,s_2 satisfy (6.13) with ν given by (2.10). Then

$$\boldsymbol{P}_0(z) := \boldsymbol{E}_0(z)\boldsymbol{\Psi}_{s_1,s_2}\big(n^{1/2}\zeta_0(z)\big)\boldsymbol{J}^{-1}(z)r^{-\sigma_3}(z) \tag{6.23}$$

satisfies RHP-$\boldsymbol{P}_0$(a,b) for any matrix $\boldsymbol{E}_0(z)$ holomorphic in U_δ. Indeed, RHP-$\boldsymbol{P}_0$(a) is an immediate consequence of RHP-$\boldsymbol{\Psi}_{s_1,s_2}$(a). It further follows from RHP-$\boldsymbol{\Psi}_{s_1,s_2}$(b) that the jumps of $\boldsymbol{P}_0(z)$ are as on Figure 5. To verify RHP-

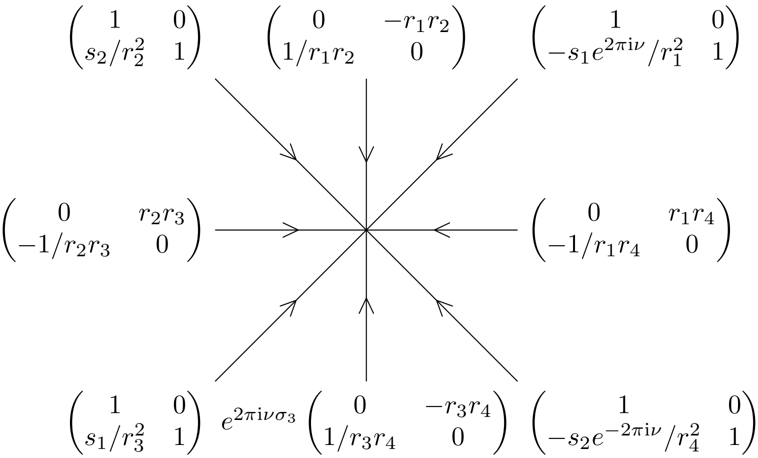

FIGURE 5. The jump matrices of $\boldsymbol{P}_0(z)$.

$\boldsymbol{P}_0$(b), it remains to observe that

$$r_1r_4=\rho_1,\ -r_1r_2=\rho_2,\ r_2r_3=e^{-2\pi i\nu}\rho_2\rho_4/\rho_1=\rho_3,\ -r_3r_4e^{2\pi i\nu}=\rho_4,$$

since $e^{-2\pi i\nu}=(\rho_1\rho_3)/(\rho_2\rho_4)$, and that

$$\begin{aligned}
-e^{2\pi i\nu}\frac{s_1}{r_1^2} &= \frac{b_1(0)}{\rho_1}=\frac{b_1(0)}{b_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right),\\
\frac{s_2}{r_2^2} &= -e^{2\pi i\nu}b_2(0)\frac{\rho_1}{\rho_2^2}=-b_2(0)\frac{\rho_4}{\rho_2\rho_3}=\frac{b_2(0)}{b_2}\left(\frac{1}{\rho_2}+\frac{1}{\rho_3}\right),\\
\frac{s_1}{r_3^2} &= -e^{2\pi i\nu}b_3(0)\frac{\rho_1}{\rho_4^2}=-b_3(0)\frac{\rho_2}{\rho_3\rho_4}=\frac{b_3(0)}{b_3}\left(\frac{1}{\rho_3}+\frac{1}{\rho_4}\right),\\
-e^{-2\pi i\nu}\frac{s_2}{r_4^2} &= \frac{b_4(0)}{\rho_1}=\frac{b_4(0)}{b_4}\left(\frac{1}{\rho_1}+\frac{1}{\rho_4}\right).
\end{aligned}$$

Thus, it remains to choose $\boldsymbol{E}_0(z)$ so that RHP-$\boldsymbol{P}_0$(c) is fulfilled.

6.5.4. Matrix $\boldsymbol{E}_0$. Let $\boldsymbol{\gamma}$ be the part of $\boldsymbol{\Delta}_3$ that proceeds from $\boldsymbol{a}_3$ to $\boldsymbol{0}^*$, see Figures 1 and 2. Define

$$K_0(z) := \begin{cases} \exp\left\{-\int_{\boldsymbol{\gamma}} G\right\} = \Phi(\boldsymbol{0}), & z\in\mathcal{Q}_1\cup\mathcal{Q}_3,\\ \exp\left\{\int_{\boldsymbol{\gamma}} G\right\} = \Phi(\boldsymbol{0}^*), & z\in\mathcal{Q}_2\cup\mathcal{Q}_4.\end{cases} \tag{6.24}$$

(2.25) immediately yields that $|K_0(z)|\equiv 1$. Define

$$\boldsymbol{E}_0^\star(z):=\boldsymbol{M}^\star(z)r^{\sigma_3}(z)K_0^{n\sigma_3}(z)\boldsymbol{J}(z)\zeta_0^{-\nu\sigma_3}(z), \tag{6.25}$$

see (6.6). From RHP-$\boldsymbol{N}$(b), the definition of $\boldsymbol{J}(z)$, and the fact that $\zeta_0(z)$ maps $\tilde{\Delta}_1^\circ$ to the negative reals, it follows that $\boldsymbol{E}_0^\star(z)$ is holomorphic in $U_\delta\setminus\{0\}$. Furthermore, RHP-$\boldsymbol{N}$(c) combined with the fact that $\zeta_0(z)$ possesses a simple zero at $z=0$ imply that $\boldsymbol{E}_0^\star(z)$ is holomorphic in U_δ. Observe also that the absolute values of the entries of $\boldsymbol{E}^\star(\boldsymbol{z})$ depend on only the parity of n.

Put for brevity $\epsilon_{\nu,n}:=(4n)^{\varsigma_\nu\nu-1/2}$, where, as before, ς_ν is equal to $1,0,-1$ depending on whether $\mathrm{Re}(\nu)$ is positive, zero, or negative. Set

$$\boldsymbol{L}_\nu:=\frac{d_\nu\epsilon_{\nu,n}}{\zeta_0'(0)D_n}\boldsymbol{E}^\star(0)\boldsymbol{A}_\nu\boldsymbol{E}^{\star-1}(0), \tag{6.26}$$

where $d_\nu,\mathbf{A}_\nu$ were defined in (6.17) and we assume that

$$0\neq D_n:=1-d_\nu\epsilon_{\nu,n}\big(\zeta_0'(0)\big)^{-1}E_\nu \tag{6.27}$$

with

$$E_\nu:=\begin{cases}\left[\boldsymbol{E}^{\star-1}(0)\boldsymbol{E}^{\star\prime}(0)\right]_{12} & \text{if}\quad \mathrm{Re}(\nu)\geq 0,\\ \left[\boldsymbol{E}^{\star-1}(0)\boldsymbol{E}^{\star\prime}(0)\right]_{21} & \text{if}\quad \mathrm{Re}(\nu)<0.\end{cases}$$

Notice that $\boldsymbol{L}_\nu$ is the zero matrix when $\mathrm{Re}(\nu)=0$ as $d_\nu=0$ by (6.17). Let

$$\boldsymbol{E}_0(z):=(\boldsymbol{I}+\boldsymbol{L}_\nu/z)\boldsymbol{E}_0^\star(z)(4n)^{-\nu\sigma_3/2}\big(\boldsymbol{I}-d_\nu\big(2n^{1/2}\zeta_0(z)\big)^{-1}\boldsymbol{A}_\nu\big). \tag{6.28}$$

Let us show that thus defined matrix $\boldsymbol{E}_0(z)$ is holomorphic at the origin. Indeed, it has at most double pole there. It is quite simple to see that the coefficient next to z^{-2} is equal to

$$-d_\nu\epsilon_{\nu,n}(4n)^{-\varsigma_\nu\nu/2}\big(\zeta_0'(0)\big)^{-1}\boldsymbol{L}_\nu\boldsymbol{E}_0^\star(0)\boldsymbol{A}_\nu,$$

which is equal to the zero matrix since $\boldsymbol{A}_\nu^2$ is equal to the zero matrix. Using this observation we also get the coefficient next to z^{-1} is equal to

$$\boldsymbol{L}_\nu\boldsymbol{E}_0^\star(0)(4n)^{-\nu\sigma_3/2}-d_\nu\epsilon_{\nu,n}(4n)^{-\varsigma_\nu\nu/2}\big(\zeta_0'(0)\big)^{-1}\big(\boldsymbol{E}_0^\star(0)+\boldsymbol{L}_\nu\boldsymbol{E}_0^{\star\prime}(0)\big)\boldsymbol{A}_\nu,$$

which simplifies to

$$\frac{d_\nu\epsilon_{\nu,n}(4n)^{-\varsigma_\nu\nu/2}}{\zeta_0'(0)D_n}\left(1-\frac{d_\nu\epsilon_{\nu,n}}{\zeta_0'(0)}E_\nu-D_n\right)\boldsymbol{E}_0^\star(0)\boldsymbol{A}_\nu$$

that is equal to the zero matrix by the very definition of D_n.

Now, recalling the definition of $\Phi(\boldsymbol{z})$ in (2.6) and of $\zeta_0(z)$ in (6.19), one can see that

$$\exp\left\{-\zeta_0^2(z)\right\}=e^{-\int_\gamma G}\begin{cases}\Phi\big(z^{(1)}\big), & z\in\mathcal{Q}_1\cup\mathcal{Q}_3,\\ \Phi\big(z^{(0)}\big), & z\in\mathcal{Q}_2\cup\mathcal{Q}_4.\end{cases} \tag{6.29}$$

In particular, since $\boldsymbol{D}(z)=\Phi^{n\sigma_3}\big(z^{(0)}\big)$ and $\Phi\big(z^{(0)}\big)\Phi\big(z^{(1)}\big)\equiv 1$, it follows that

$$\exp\left\{-n\zeta_0^2(z)\sigma_3\right\}\boldsymbol{J}^{-1}(z)=\boldsymbol{J}^{-1}(z)K_0^{-n\sigma_3}(z)\boldsymbol{D}(z).$$

For brevity, let $\boldsymbol{H}(z) := r^{\sigma_3}(z)K_0^{n\sigma_3}(z)\boldsymbol{J}(z)$. Then we get from (6.18) and the previous identity that

$$\boldsymbol{E}_0(s)\boldsymbol{\Psi}_{s_1,s_2}\big(n^{1/2}\zeta_0(s)\big)\boldsymbol{J}^{-1}(s)r^{-\sigma_3}(s) = \\ \boldsymbol{M}(s)\boldsymbol{H}(s)\left(\boldsymbol{I}+\boldsymbol{\mathcal{O}}\left(\big(n\zeta_0^2(s)\big)^{-1/2-|\mathrm{Re}(\nu)|}\right)\right)\boldsymbol{H}^{-1}(s)\boldsymbol{D}(s) = \\ \boldsymbol{M}(s)\left(\boldsymbol{I}+\boldsymbol{\mathcal{O}}\left(\big(n\delta^2\big)^{-1/2-|\mathrm{Re}(\nu)|}\right)\right)\boldsymbol{D}(s).$$

It remains to show that (6.27) holds for all $n \in \mathbb{N}_{\rho,\varepsilon}$. It follows from (2.24) that it is enough to show that

$$A_{\rho,n} = d_\nu \epsilon_{\nu,n}\big(\zeta_0'(0)\big)^{-1}E_\nu. \tag{6.30}$$

6.6. Existence of $\boldsymbol{L}_\nu$

Assume that $\mathrm{Re}(\nu) > 0$. It can be readily verified that

$$E_\nu = \gamma_n\gamma_{n-1}^*\big([\boldsymbol{E}_0^{\star\prime}(0)]_{12}[\boldsymbol{E}_0^\star(0)]_{22} - [\boldsymbol{E}_0^{\star\prime}(0)]_{22}[\boldsymbol{E}_0^\star(0)]_{12}\big),$$

where we used the fact that $\det(\boldsymbol{E}_0^\star(z)) = \det(\boldsymbol{M}^\star(z)) = (\gamma_n\gamma_{n-1}^*)^{-1}$. Notice that $d_2 \neq 0$ by (6.15). Using (6.25), (6.21), and (6.24) gives us that $[\boldsymbol{E}_0^\star(z)]_{i2}$ is equal to

$$\zeta_0^\nu(z)\Phi^n\big(\mathbf{0}\big)\begin{cases} e^{-2\pi\mathrm{i}\nu}r_1(z)[\boldsymbol{M}^\star(z)]_{i1}, & \arg(z)\in(0,\pi/4), \\ r_1(z)[\boldsymbol{M}^\star(z)]_{i1}, & \arg(z)\in(\pi/4,\pi/2), \\ [\boldsymbol{M}^\star(z)]_{i2}/r_2(z), & \arg(z)\in(\pi/2,\pi), \\ r_3(z)[\boldsymbol{M}^\star(z)]_{i1}, & \arg(z)\in(\pi,3\pi/2), \\ e^{-2\pi\mathrm{i}\nu}[\boldsymbol{M}^\star(z)]_{i2}/r_4(z), & \arg(z)\in(3\pi/2,2\pi). \end{cases}$$

Define

$$S(z) := \zeta_0^\nu(z)\begin{cases} e^{-2\pi\mathrm{i}\nu}r_1(z)S_\rho\big(z^{(0)}\big), & \arg(z)\in(0,\pi/4), \\ r_1(z)S_\rho\big(z^{(0)}\big), & \arg(z)\in(\pi/4,\pi/2), \\ S_\rho\big(z^{(1)}\big)/(r_2w)(z), & \arg(z)\in(\pi/2,\pi), \\ r_3(z)S_\rho\big(z^{(0)}\big), & \arg(z)\in(\pi,3\pi/2), \\ e^{-2\pi\mathrm{i}\nu}S_\rho\big(z^{(1)}\big)/(r_4w)(z), & \arg(z)\in(3\pi/2,2\pi), \end{cases}$$

which is a holomorphic and non-vanishing function around the origin. Then we obtain from (6.6), (4.6), and (4.12) that

$$E_\nu = S^2(0)\Phi^{2n}\big(\mathbf{0}\big)Y_nX_n^{-1}. \tag{6.31}$$

When $|\pi(\boldsymbol{z}_k)| = \infty$, the first condition in the definition of $\mathbb{N}_{\rho,\varepsilon}$ implies that we are looking only at those indices n for which $\boldsymbol{z}_{\imath(n)} = \infty^{(1)}$. In this case $A_{\rho,n} = 0$ by its very definition in (2.24) and it also follows from Lemma 4.9 that $Y_n = 0$ in this case. Hence, (6.30) does hold in this case.

Let now $|\pi(\boldsymbol{z}_k)| < \infty$ and therefore the first condition in the definition of $\mathbb{N}_{\rho,\varepsilon}$ is void. It follows from (6.19) and (2.1) as well as the fact that $\zeta_0(z)$

maps $\{\arg(z) = 5\pi/4\}$ into positive reals that

$$1/\zeta_0'(0) = e^{5\pi \mathrm{i}/4}\sqrt{2ab}. \tag{6.32}$$

Since $e^{-2\pi\mathrm{i}\nu} = (\rho_1\rho_3)(0)/(\rho_2\rho_4)(0)$ by (2.10), we get from (6.20) that

$$S^2(0) = -\big(\rho_3\rho_4/\rho_2\big)(0)(2ab)^{-\nu} \lim_{z\to 0,\ \arg(z)=5\pi/4} |z|^{2\nu} S_\rho^2\big(z^{(0)}\big). \tag{6.33}$$

Observe also that

$$d_2 = e^{\pi\mathrm{i}\nu}\frac{(\rho_2+\rho_3)(0)}{\rho_4(0)}\frac{\Gamma(1-\nu)}{\sqrt{2\pi}} \tag{6.34}$$

by (6.14), (6.22), and (6.11). Then it follows from (4.16) and the very definitions of $A_{\rho,n}$ in (2.24) that (6.31)–(6.34) yield (6.30). The proof of (6.30) in the case $\mathrm{Re}(\nu) < 0$ is similar.

Assume now that $|\pi(\boldsymbol{z}_k)| < \infty$. Then the quantities Y_n and Z_n in (4.12) and (4.14) are non-zero and equal to

$$W_{\imath(n)}'(\boldsymbol{s})\frac{T_{\imath(n-1)}^2(\boldsymbol{s})}{\Phi(\boldsymbol{s})}, \quad W_{\imath(n)}(\boldsymbol{z}) := \frac{T_{\imath(n)}(\boldsymbol{z})}{T_{\imath(n-1)}(\boldsymbol{z})},$$

where $\boldsymbol{s} = \boldsymbol{0}$ when $\mathrm{Re}(\nu) > 0$ and $\boldsymbol{s} = \boldsymbol{0}^*$ when $\mathrm{Re}(\nu) < 0$. Hence, it follows from (6.26), (6.30), (6.31), and a computation similar to the one carried out at the beginning of this subsection that

$$\boldsymbol{L}_\nu = \frac{A_{\rho,n}}{1-A_{\rho,n}}\frac{1}{W_{\imath(n)}'(\boldsymbol{s})}\begin{pmatrix} W_{\imath(n)}(\boldsymbol{s}) & -\Phi(\boldsymbol{s})W_{\imath(n)}^2(\boldsymbol{s}) \\ 1/\Phi(\boldsymbol{s}) & -W_{\imath(n)}(\boldsymbol{s}) \end{pmatrix}.$$

Moreover, since $W_1(\boldsymbol{z}) = 1/W_0(\boldsymbol{z})$ we can rewrite the first row of $\boldsymbol{L}_\nu$ as

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\boldsymbol{L}_\nu = (-1)^{\imath(n)}\frac{A_{\rho,n}}{1-A_{\rho,n}}\frac{W_0(\boldsymbol{s})}{W_0'(\boldsymbol{s})}\begin{pmatrix} 1 & -\Phi(\boldsymbol{s})W_{\imath(n)}(\boldsymbol{s}) \end{pmatrix}, \tag{6.35}$$

where, as before, $\boldsymbol{s} = \boldsymbol{0}$ when $\mathrm{Re}(\nu) > 0$ and $\boldsymbol{s} = \boldsymbol{0}^*$ when $\mathrm{Re}(\nu) < 0$.

6.7. Final Riemann-Hilbert Problem

In what follows, we assume that $\delta = \delta_n \leq \delta_0$ in Section 6.5 when $\ell < \infty$ and shall specify the exact dependence on n later on in this section. When $\ell = \infty$, we simply take $\delta = \delta_0$. Set $U := \cup_{i=1}^4 U_{a_i}$ and define

$$\Sigma_n := \big(\partial U \cup \partial U_{\delta_n}\big) \cup \Big(\cup_{i=1}^4\big(\Gamma_{i-}\cup\Gamma_{i+}\cup\tilde\Delta_i\big)\setminus\overline{U}\Big),$$

see Figure 6. We are looking for a solution of the following Riemann-Hilbert problem (RHP-$\boldsymbol{Z}$):

(a) $\boldsymbol{Z}(z)$ is analytic in $\overline{\mathbb{C}}\setminus\Sigma_n$ and $\lim_{z\to\infty}\boldsymbol{Z}(z) = \boldsymbol{I}$;

(b) $\boldsymbol{Z}(z)$ has continuous traces outside of non-smooth points of Σ_n that satisfy

$$\boldsymbol{Z}_+ = \boldsymbol{Z}_-\begin{cases} \boldsymbol{P}_{a_i}(\boldsymbol{MD})^{-1}, & \text{on } \partial U_{a_i}, \\ \boldsymbol{P}_0(\boldsymbol{MD})^{-1}, & \text{on } \partial U_\delta, \\ \boldsymbol{MD}\begin{pmatrix} 1 & 0 \\ 1/\rho_i & 1 \end{pmatrix}(\boldsymbol{MD})^{-1}, & \text{on } \big(\Gamma_{i+}^\circ\cup\Gamma_{i-}^\circ\big)\setminus\overline{U}, \end{cases}$$

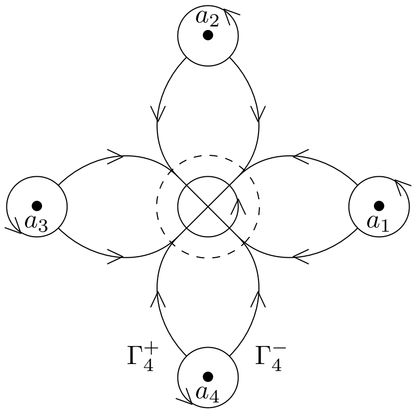

FIGURE 6. Contour Σ_n for RHP-$\boldsymbol Z$ (dashed circle represents $\{|z|=\delta_0\}$).

and

$$\boldsymbol Z_+ = \boldsymbol Z_- \begin{cases} \boldsymbol{MD}\begin{pmatrix} 1 & 0 \\ \frac{\rho_i+\rho_{i+1}}{\rho_i\rho_{i+1}} & 1 \end{pmatrix}(\boldsymbol{MD})^{-1}, & \text{on } \tilde\Delta_i^\circ \setminus \overline{U}_{\delta_n}, \\ \boldsymbol P_{0-}\begin{pmatrix} 1 & 0 \\ \frac{\rho_i+\rho_{i+1}}{\rho_i\rho_{i+1}} & 1 \end{pmatrix}\boldsymbol P_{0+}^{-1}, & \text{on } \tilde\Delta_i^\circ \cap U_{\delta_n} \end{cases}$$

(notice that the second set of jumps is not present when $\ell=\infty$ as $\delta_n=\delta_0$ and $\boldsymbol P_0(z)$ is an exact parametrix).

It follows from RHP-$\boldsymbol P_{a_i}$(d) that the jump of $\boldsymbol Z$ on ∂U_{a_i} can be written as

$$\boldsymbol M(s)\big(\boldsymbol I+\boldsymbol{\mathcal O}(1/n)\big)\boldsymbol M^{-1}(s) = \boldsymbol I + \boldsymbol{\mathcal O}_\varepsilon(1/n)$$

since the matrix $\boldsymbol M(z)$ is invertible (its determinant is equal to the reciprocal of $\gamma_n\gamma_{n-1}^*$), the matrix $\boldsymbol M^\star(z)$ depends only on the parity of n, see (6.6), and the matrix $\boldsymbol L_\nu$ has trace and determinant zero as well as bounded entries for all $n\in\mathbb N_{\rho,\varepsilon}$ and each fixed $\varepsilon>0$, see (6.26). Similarly, we get from RHP-$\boldsymbol P_0$(c) that the jump of $\boldsymbol Z$ on ∂U_{δ_n} can be written as

$$\boldsymbol M(s)\left(\boldsymbol I+\boldsymbol{\mathcal O}\left(\left(n\delta_n^2\right)^{-1/2-|\mathrm{Re}(\nu)|}\right)\right)\boldsymbol M^{-1}(s) \\ = \boldsymbol I + \big(\boldsymbol I+\boldsymbol L_\nu/s\big)\boldsymbol{\mathcal O}\left(\left(n\delta_n^2\right)^{-1/2-|\mathrm{Re}(\nu)|}\right)\big(\boldsymbol I-\boldsymbol L_\nu/s\big),$$

where $\boldsymbol{\mathcal O}(\cdot)$ does not depend on n. Since $\boldsymbol L_\nu = \mathcal O_\varepsilon\big(n^{|\mathrm{Re}(\nu)|-1/2}\big)$ by its very definition in (6.26), we get that the jump of $\boldsymbol Z$ on ∂U_{δ_n} can further written as

$$\boldsymbol I + \boldsymbol{\mathcal O}_\varepsilon\left(\left(n\delta_n^2\right)^{-1/2-|\mathrm{Re}\nu|}\max\left\{1, n^{2|\mathrm{Re}(\nu)|}/(n\delta_n^2)\right\}\right).$$

One can easily check with the help of (6.4) and (6.6) that the jump of $\boldsymbol{Z}$ on $\big(\Gamma_{i+}^\circ\cup\Gamma_{i-}^\circ\big)\setminus\overline{U}$ is equal to

$$\boldsymbol{I}+\frac{\gamma_n\gamma_{n-1}^*}{(w^2\rho_i)(s)}\big(\boldsymbol{I}+\boldsymbol{L}_\nu/s\big)\begin{pmatrix}(\Psi_n\Psi_{n-1})\big(s^{(1)}\big) & -\Psi_n^2\big(s^{(1)}\big)\\ \Psi_{n-1}^2\big(s^{(1)}\big) & -(\Psi_n\Psi_{n-1})\big(s^{(1)}\big)\end{pmatrix}\big(\boldsymbol{I}-\boldsymbol{L}_\nu/s\big) = \boldsymbol{I}+\boldsymbol{\mathcal{O}}_\varepsilon(e^{-cn})$$

for some constant $c>0$ by (2.22) and since the maximum of $|\Phi(s^{(1)})|$ on $\Gamma_{i\pm}\setminus U$ is less than 1. The estimate of the jump of $\boldsymbol{Z}$ on $\tilde\Delta_i^\circ\setminus\overline{U}_{\delta_n}$ is analogous and yields

$$\boldsymbol{I}+\boldsymbol{\mathcal{O}}_\varepsilon\left(e^{-cn\delta_n^2}\max\big\{1,n^{2|\mathrm{Re}(\nu)|}/(n\delta_n^2)\big\}\right)$$

for an adjusted constant $c>0$, where the rate estimate follows from (6.29) as

$$\big|\Phi\big(s^{(1)}\big)\big|=\exp\big\{(-1)^i\mathrm{Re}\big(\zeta_0^2(s)\big)\big\}=\mathcal{O}\big(e^{-c\delta_n^2}\big),\quad s\in\tilde\Delta_i\setminus U_{\delta_n},$$

since $\zeta_0(z)$ is real on $\tilde\Delta_1\cup\tilde\Delta_3$ and is purely imaginary on $\tilde\Delta_2\cup\tilde\Delta_4$.

Finally, it holds on $\tilde\Delta_i^\circ\cap U_{\delta_n}$ that the jump of $\boldsymbol{Z}$ is equal to

$$\boldsymbol{I}+\left(1-\frac{b_i(0)}{b_i(z)}\right)\frac{(\rho_i+\rho_{i+1})(s)}{(\rho_i\rho_{i+1})(s)}\boldsymbol{P}_{0+}(s)\begin{pmatrix}0&0\\1&0\end{pmatrix}\boldsymbol{P}_{0+}^{-1}(s)=$$
$$\boldsymbol{I}+\mathcal{O}(\delta_n^\ell)\boldsymbol{E}_0(s)\begin{pmatrix}[\boldsymbol{\Psi}_+(s)]_{1j}[\boldsymbol{\Psi}_+(s)]_{2j} & -[\boldsymbol{\Psi}_+(s)]_{1j}^2\\ [\boldsymbol{\Psi}_+(s)]_{2j}^2 & -[\boldsymbol{\Psi}_+(s)]_{1j}[\boldsymbol{\Psi}_+(s)]_{2j}\end{pmatrix}\boldsymbol{E}_0^{-1}(s)$$

by (6.12) and (6.23), where $j=1$ for $s\in\tilde\Delta_1\cup\tilde\Delta_3$ and $j=2$ for $s\in\tilde\Delta_2\cup\tilde\Delta_4$, and we set for brevity $\boldsymbol{\Psi}(z):=\boldsymbol{\Psi}_{s_1,s_2}\big(n^{1/2}\zeta_0(z)\big)$ (observe also that $\det(\boldsymbol{\Psi}(z))\equiv1$). It follows from the asymptotic expansion (6.16) that $D_\mu(x)$ is bounded for $x\geq0$. Thus, we deduce from the definition of $\boldsymbol{\Psi}(z)$ that the above jump matrix can be estimated as

$$\boldsymbol{I}+\mathcal{O}(\delta_n^\ell)\boldsymbol{E}_0(s)\boldsymbol{\mathcal{O}}(1)\boldsymbol{E}_0^{-1}(s)=\boldsymbol{I}+\boldsymbol{\mathcal{O}}_\varepsilon\left(n^{|\mathrm{Re}(\nu)|}\delta_n^\ell\right),$$

where the last equality follows from (6.25) and (6.28) as $\boldsymbol{E}_0(z)$ is equal to a bounded matrix that depends only on $\varepsilon_{\nu,n}$ multiplied by $(4n)^{\nu\sigma_3/2}$ on the right.

When $\ell\geq4|\mathrm{Re}(\nu)|(1+|\mathrm{Re}(\nu)|)/(1-2|\mathrm{Re}(\nu)|)$, choose

$$\delta_n=\delta_0\exp\left\{-\frac{1}{2}\frac{1+4|\mathrm{Re}(\nu)|}{\ell+1+2|\mathrm{Re}(\nu)|}\ln n\right\}.$$

Then it holds that $n^{2|\mathrm{Re}(\nu)|}/(n\delta_n^2)=\mathcal{O}(1)$ and

$$n^{|\mathrm{Re}(\nu)|}(\delta_n/\delta_0)^\ell=\big(n(\delta_n/\delta_0)^2\big)^{-|\mathrm{Re}(\nu)|-1/2}=n^{-d_{\nu,\ell}}$$

with $d_{\nu,\ell}$ defined in (2.28). Otherwise, take

$$\delta_n=\delta_0\exp\left\{-\frac{1}{2}\frac{3}{\ell+3+2|\mathrm{Re}(\nu)|}\ln n\right\}.$$

In this case $n^{2|\mathrm{Re}(\nu)|}/(n\delta_n^2)\to\infty$ as $n\to\infty$ and

$$n^{|\mathrm{Re}(\nu)|}(\delta_n/\delta_0)^\ell = n^{2|\mathrm{Re}(\nu)|}\big(n(\delta_n/\delta_0)^2\big)^{-|\mathrm{Re}(\nu)|-3/2} = n^{-d_{\nu,\ell}}.$$

Since $d_{\nu,\ell}<1$, it holds that the jumps of $\boldsymbol Z$ on Σ_n are of order $\boldsymbol I+\boldsymbol{\mathcal O}_\varepsilon(n^{-d_{\nu,\ell}})$, where $\boldsymbol{\mathcal O}_\varepsilon(\cdot)$ does not depend on n. Then, by arguing as in [2, Theorem 7.103 and Corollary 7.108] we obtain that the matrix $\boldsymbol Z$ exists for all $n\in\mathbb N_{\rho,\varepsilon}$ large enough and that

$$\|\boldsymbol Z_\pm-\boldsymbol I\|_{2,\Sigma_n} = \mathcal O_\varepsilon\big(n^{-d_{\nu,\ell}}\big).$$

Since the jumps of $\boldsymbol Z$ on Σ_n are restrictions of holomorphic matrix functions, the standard deformation of the contour technique and the above estimate yield that

$$\boldsymbol Z = \boldsymbol I+\boldsymbol{\mathcal O}_\varepsilon\big(n^{-d_{\nu,\ell}}\big) \tag{6.36}$$

locally uniformly in $\overline{\mathbb C}\setminus\{0\}$.

6.8. Proofs of Theorems 2.3–2.4

Given $\boldsymbol Z(z)$, a solution of RHP-$\boldsymbol Z$, $\boldsymbol P_{a_i}(z)$ and $\boldsymbol P_0(z)$, defined in (6.10) and (6.23), respectively, and $\boldsymbol C(\boldsymbol{MD})(z)$ from (6.6) and (6.7), it can be readily verified that

$$\boldsymbol X(z) := \boldsymbol{CZ}(z)\begin{cases}\boldsymbol P_{a_i}(z), & z\in U_i,\ i\in\{1,2,3,4\},\\ \boldsymbol P_0(z), & z\in U_\delta,\\ (\boldsymbol{MD})(z), & \text{otherwise},\end{cases}$$

solves RHP-$\boldsymbol X$. Given a closed set $K\subset\overline{\mathbb C}\setminus\Delta$, the contour Σ_n can always be adjusted so that K lies in the exterior domain of Σ_n. Then it follows from (6.3) that $\boldsymbol Y(z)=\boldsymbol X(z)$ on K. Formulae (2.26) and (2.32) now follow immediately from (6.2), (6.4), (6.6), and (6.7) since

$$w^{i-1}(z)[(\boldsymbol{ZMD})(z)]_{1i} = (1+\upsilon_{n1}(z))\Psi_n\big(z^{(i-1)}\big)+\upsilon_{n2}(z)\Psi_{n-1}\big(z^{(i-1)}\big),$$

where $\upsilon_{n1}(z),\upsilon_{n2}(z)$ are the first row entries of $\boldsymbol Z(z)(\boldsymbol I+\boldsymbol L_\nu/z)$. Estimates (2.27) are direct consequence of (6.26) and (6.36). Relations (2.29) follow from (6.35).

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Ahmad Barhoumi
Department of Mathematical Sciences
Indiana University-Purdue University Indianapolis
402 North Blackford Street, Indianapolis, IN 46202
e-mail: abarhoum@iupui.edu

Maxim L. Yattselev
Department of Mathematical Sciences
Indiana University-Purdue University Indianapolis
402 North Blackford Street, Indianapolis, IN 46202
e-mail: maxyatts@iupui.edu