An asymptotic expansion for the expected number of real zeros of real random polynomials spanned by OPUC

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Abstract

Let $\{\varphi_i\}_{i=0}^\infty$ be a sequence of orthonormal polynomials on the unit circle with respect to a positive Borel measure μ that is symmetric with respect to conjugation. We study asymptotic behavior of the expected number of real zeros, say $\mathbb{E}_n(\mu)$, of random polynomials

$$P_n(z) := \sum_{i=0}^n \eta_i\varphi_i(z),$$

where $\eta_0,\dots,\eta_n$ are i.i.d. standard Gaussian random variables. When μ is the acrlength measure such polynomials are called Kac polynomials and it was shown by Wilkins that $\mathbb{E}_n(|\mathrm{d}\xi|)$ admits an asymptotic expansion of the form

$$\mathbb{E}_n(|\mathrm{d}\xi|) \sim \frac{2}{\pi}\log(n+1) + \sum_{p=0}^\infty A_p(n+1)^{-p}$$

(Kac himself obtained the leading term of this expansion). In this work we generalize the result of Wilkins to the case where μ is absolutely continuous with respect to arclength measure and its Radon-Nikodym derivative extends to a holomorphic non-vanishing function in some neighborhood of the unit circle. In this case $\mathbb{E}_n(\mu)$ admits an analogous expansion with coefficients the A_p depending on the measure μ for $p \geq 1$ (the leading order term and A_0 remain the same).

Key words: random polynomials, orthogonal polynomials on the unit circle, expected number of real zeros, asymptotic expansion

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1. Introduction and Main Results

In [2], Kac considered random polynomials

$$P_n(z) = \eta_0 + \eta_1 z + \cdots + \eta_n z^n,$$

where η_i are i.i.d. standard real Gaussian random variables. He has shown that $\mathbb{E}_n(\Omega)$, the expected number of zeros of $P_n(z)$ on a measurable set $\Omega \subset \mathbb{R}$, is equal to

$$\mathbb{E}_n(\Omega) = \frac{1}{\pi}\int_\Omega \frac{\sqrt{1-h_{n+1}^2(x)}}{|1-x^2|}\mathrm{d}x, \quad h_{n+1}(x) = \frac{(n+1)x^n(1-x^2)}{1-x^{2n+2}}, \tag{1}$$

from which he proceeded with an estimate

$$\mathbb{E}_n(\mathbb{R}) = \frac{2+o(1)}{\pi}\log(n+1) \quad \text{as} \quad n\to\infty.$$

It was shown by Wilkins [7], after some intermediate results cited in [7], that there exist constants A_p, $p \geq 0$, such that $\mathbb{E}_n(\mathbb{R})$ has an asymptotic expansion of the form

$$\mathbb{E}_n(\mathbb{R}) \sim \frac{2}{\pi}\log(n+1) + \sum_{p=0}^{\infty} A_p(n+1)^{-p}. \tag{2}$$

In another connection, Edelman and Kostlan [1] considered random functions of the form

$$P_n(z) = \eta_0 f_0(z) + \eta_1 f_1(z) + \cdots + \eta_n f_n(z), \tag{3}$$

where η_i are certain real random variables and $f_i(z)$ are arbitrary functions on the complex plane that are real on the real line. Using beautiful and simple geometrical argument they have shown[1] that if $\eta_0, \ldots, \eta_n$ are elements of a multivariate real normal distribution with mean zero and covariance matrix C and the functions $f_i(x)$ are differentiable on the real line, then

$$\mathbb{E}_n(\Omega) = \int_\Omega \rho_n(x)\mathrm{d}x, \quad \rho_n(x) = \frac{1}{\pi}\frac{\partial^2}{\partial s\partial t}\log\left(v(s)^{\mathsf{T}}Cv(t)\right)\Big|_{t=s=x},$$

where $v(x) = \left(f_0(x), \ldots, f_n(x)\right)^{\mathsf{T}}$. If random variables η_i in (3) are again i.i.d. standard real Gaussians, then the above expression for $\rho_n(x)$ specializes to

$$\rho_n(x) = \frac{1}{\pi}\frac{\sqrt{K_{n+1}(x,x)K_{n+1}^{(1,1)}(x,x) - K_{n+1}^{(1,0)}(x,x)^2}}{K_{n+1}(x,x)} \tag{4}$$

(this formula was also independently rederived in [3, Proposition 1.1] and [6, Theo-

[1] In fact, Edelman and Kostlan derive an expression for the real intensity function for any random vector $(\eta_0, \ldots, \eta_n)$ in terms of its joint probability density function and of $v(x)$.

rem 1.2]), where

$$\begin{cases} K_{n+1}(z,w) & := & \sum_{i=0}^n f_i(z)\overline{f_i(w)}, \\ K_{n+1}^{(1,0)}(z,w) & := & \sum_{i=0}^n f_i'(z)\overline{f_i(w)}, \\ K_{n+1}^{(1,1)}(z,w) & := & \sum_{i=0}^n f_i'(z)\overline{f_i'(w)}. \end{cases}$$

In this work we concentrate on a particular subfamily of random functions (3), namely random polynomials of the form

$$P_n(z) = \eta_0\varphi_0(z) + \eta_1\varphi_1(z) + \cdots + \eta_n\varphi_n(z), \tag{5}$$

where η_i are i.i.d. standard real Gaussian random variables and $\varphi_i(z)$ are orthonormal polynomials on the unit circle with real coefficients. That is, for some probability Borel measure μ on the unit circle that is symmetric with respect to conjugation, it holds that

$$\int_{\mathbb{T}} \varphi_i(\xi)\overline{\varphi_j(\xi)}\mathrm{d}\mu(\xi) = \delta_{ij}, \tag{6}$$

where δ_{ij} is the usual Kronecker symbol. In this case it can be easily shown using Christoffel-Darboux formula, see [8, Theorem 1.1], that (4) can be rewritten as

$$\rho_n(x) = \frac{1}{\pi}\frac{\sqrt{1-h_{n+1}^2(x)}}{|1-x^2|}, \quad h_{n+1}(x) := \frac{(1-x^2)b_{n+1}'(x)}{1-b_{n+1}^2(x)}, \quad b_{n+1}(x) := \frac{\varphi_{n+1}(x)}{\varphi_{n+1}^*(x)}, \tag{7}$$

where $\varphi_{n+1}^*(x) := x^{n+1}\varphi_{n+1}(1/x)$ is the reciprocal polynomial (there is no need for conjugation as all the coefficients are real). When μ is the normalized arclength measure on the unit circle, it is elementary to see that $\varphi_m(z) = z^m$ and therefore (7) recovers (1).

Theorem 1. *Let $P_n(z)$ be given by (5)–(6), where μ is absolutely continuous with respect to the arclength measure and $\mu'(\xi)$, the respective Radon-Nikodym derivative, extends to a holomorphic non-vanishing function in some neighborhood of the unit circle. Then $\mathbb{E}_n(\mu)$, the expected number of real zeros of $P_n(z)$, satisfies*

$$\mathbb{E}_n(\mu) = \frac{2}{\pi}\log(n+1) + A_0 + \sum_{p=1}^{N-1} A_p^\mu (n+1)^{-p} + O_N\left((n+1)^{-N}\right)$$

for any integer N and all n large, where $O_N(\cdot)$ depends on N, but is independent of n,

$$A_0 = \frac{2}{\pi}\left(\log 2 + \int_0^1 t^{-1}f(t)\mathrm{d}t + \int_1^\infty t^{-1}(f(t)-1)\mathrm{d}t\right),$$

$f(t) := \sqrt{1-t^2\mathrm{csch}^2 t}$, and A_p^μ, $p \geq 1$, are some constants that do depend on μ.

Clearly, the above result generalizes (2), where $\mathrm{d}\mu(\xi) = |\mathrm{d}\xi|/(2\pi)$.

2. Auxiliary Estimates

In this section we gather some auxiliary estimates of quantities involving orthonormal polynomials $\varphi_m(z)$. First of all, recall [5, Theorem 1.5.2] that monic orthogonal polynomials, say $\Phi_m(z)$, satisfy the recurrence relations

$$\begin{cases} \Phi_{m+1}(z) = z\Phi_m(z) - \alpha_m\Phi_m^*(z), \\ \Phi_{m+1}^*(z) = \Phi_m^*(z) - \alpha_m z\Phi_m(z), \end{cases}$$

where the recurrence coefficients $\{\alpha_m\}$ belong to the interval $(-1,1)$ due to conjugate symmetry of the measure μ. In what follows we denote by $\rho < 1$ the smallest number such that $\mu'(\xi)$ is non-vanishing and holomorphic in the annulus $\{\rho < |z| < 1/\rho\}$.

With a slight abuse of notation we shall denote various constant that depend on μ and possibly additional parameters r, s by the same symbol $C_{\mu,r,s}$ understanding that the actual value of $C_{\mu,r,s}$ might be different for different occurrences, but it never depends on z or n.

Lemma 2. *It holds that*

$$|h_{n+1}(x)| \leq C_\mu(n+1)e^{-\sqrt{n+1}}, \quad |x| \leq 1-(n+1)^{-1/2}.$$

Proof. It was shown in [8, Section 3.3] that

$$|h_{n+1}(x)| \leq C_\mu|(xb_n(x))'|, \quad |x| \leq 1-(n+1)^{-1/2}.$$

It was also shown in [8, Section 3.3] that

$$|(zb_n(z))'| \leq C_\mu(n+1)\left(r^{n-m} + \sum_{i=m}^\infty |\alpha_i|\right), \quad |z| \leq r < 1.$$

It is further known, see [4, Corollary 2], that the recurrence coefficients α_i satisfy

$$|\alpha_i| \leq C_{\mu,\rho-s}s^{i+1} \quad \Rightarrow \quad \sum_{i=m}^\infty |\alpha_i| \leq \frac{C_{\mu,s-\rho}s^m}{1-\rho}, \quad \rho < s < 1,$$

where $C_{\mu,s-\rho}$ also depends on how close s is to ρ. Given a value of the parameter s, take m to be the integer part of $-\sqrt{n+1}/\log s$ and $r = 1 - 1/\sqrt{n+1}$. By combining the above three estimates, we deduce the desired inequality with a constant that depends on μ, $s-\rho$, and s. Optimizing the constant over s finishes the proof of the lemma. □

Denote by $D(z)$ the Szegő function of μ, i.e.,

$$D(z) := \exp\left\{\frac{1}{4\pi}\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\log\mu'(\xi)|d\xi|\right\}, \quad |z| \neq 1.$$

This function is piecewise analytic and non-vanishing. Denote by $D_{int}(z)$ the restriction of $D(z)$ to $|z| < 1$ and by $D_{ext}(z)$ the restriction to $|z| > 1$. It is known that both $D_{int}(z)$

and $D_{ext}(z)$ extend continuously to the unit circle and satisfy there

$$D_{int}(\xi)/D_{ext}(\xi) = \mu'(\xi), \quad |\xi| = 1.$$

Moreover, since $\mu'(\xi)$ extends to a holomorphic and non-vanishing function in the annulus $\rho < |z| < 1/\rho$, $D_{int}(z)$ and $D_{ext}(z)$ extend to holomorphic and non-vanishing functions in $|z| < 1/\rho$ and $|z| > \rho$, respectively. Hence, the scattering function

$$S(z) := D_{int}(z)D_{ext}(z), \quad \rho < |z| < 1/\rho,$$

is well defined and non-vanishing in this annulus. Since the measure μ is conjugate symmetric, it holds that $D(\bar{z}) = \overline{D(z)}$ and $D_{ext}(1/z) = 1/D_{int}(z)$. Thus, $|S(\xi)| = 1$ for $|\xi| = 1$ and $S(1) = 1$. For future use let us record the following straightforward facts.

Lemma 3. *There exist real numbers s_p, $p \geq 1$, such that*

$$\begin{aligned} S(z) &= 1 + \textstyle\sum_{p=1}^{M-1} s_p(1-z)^p + E_M(S;z) \\ S'(z) &= -\textstyle\sum_{p=0}^{M-1}(p+1)s_{p+1}(1-z)^p + E_M(S';z) \\ \log S(z) &= \textstyle\sum_{p=1}^{M-1} c_p(1-z)^p + E_M(\log S;z) \end{aligned}$$

for $|z-1| < T < 1-\rho$ and any integer $M \geq 1$, where the error terms satisfy

$$\left|E_M(F;z)\right| \leq \frac{\|F\|_{|z-1|\leq T}}{1-|1-z|/T}\left(\frac{|1-z|}{T}\right)^M$$

and $c_p = s_p + \sum_{k=2}^{p}\frac{(-1)^{k-1}}{k}\sum_{j_1+\cdots+j_k=p} s_{j_1}\cdots s_{j_k}$. Moreover, $s_2 = s_1(s_1+1)/2$. In particular, $c_1 = s_1$ and $c_2 = s_1/2$.

Proof. Since $c_1 = s_1$ and $c_2 = s_2 - s_1^2/2$, we only need to show that $s_2 = s_1(s_1+1)/2$. It holds that $s_1 = -S'(1)$ and $s_2 = S''(1)/2$. Using the symmetry $1 \equiv S(z)S(1/z)$, one can check that $S''(1) = S'(1)^2 - S'(1)$, from which the desired claim easily follows. □

Set $\tau := D_{ext}(\infty)$. It has been shown in [4, Theorem 1] that

$$\Phi_m(z) = \tau^{-1}z^m D_{ext}(z)\mathcal{E}_m(z) - \frac{\tau\mathcal{I}_m(z)}{D_{int}(z)}, \quad \rho < |z| < 1/\rho, \tag{8}$$

for some recursively defined functions $\mathcal{E}_m(z), \mathcal{I}_m(z)$ holomorphic in the annulus $\rho < |z| < 1/\rho$ that satisfy

$$\left|\mathcal{E}_m(z) - 1\right| \leq \frac{C_{\mu,s}s^{2m}}{1/s - |z|} \quad \text{and} \quad \left|\mathcal{I}_m(z)\right| \leq \frac{C_{\mu,s}s^m}{|z| - s}, \quad \rho < s < |z| < 1/s, \tag{9}$$

for some explicitly defined constant $C_{\mu,s}$, see [4, Equations (34)-(35)]. In particular, it follows from (8) that

$$b_{n+1}(z) = z^{n+1}S(z)H_n(z), \quad H_n(z) := \frac{\mathcal{E}_{n+1}(z) - \tau^2 z^{-(n+1)}S^{-1}(z)\mathcal{I}_{n+1}(z)}{\mathcal{E}_{n+1}(1/z) - \tau^2 z^{n+1}S(z)\mathcal{I}_{n+1}(1/z)}, \tag{10}$$

for $\rho < |z| < 1/\rho$. It can be checked that the conjugate symmetry of μ yields real-valuedness of $H_n(z)$ on the real line. Bounds (9) also imply that $H_n(x)$ is close to 1 near $x = 1$. More precisely, the following lemma holds.

Lemma 4. *It holds for any $\rho < \rho_* < 1$ that*

$$|H_n(x) - 1|, |\log H_n(x)| \leq (1 - x) C_{\mu,\rho_*} e^{-\sqrt{n+1}}, \quad \rho_* \leq x \leq 1.$$

Moreover, it also holds that $|H_n'(x)| \leq C_{\mu,\rho_*} e^{-\sqrt{n+1}}$ on the same interval.

Proof. Define $W_n(z) := \mathcal{E}_{n+1}(z) - 1 - \tau^2 z^{-(n+1)} S^{-1}(z) \mathcal{I}_{n+1}(z)$ and choose $\rho < s < s_* < \rho_* < 1$. Since $S(z)$ is a fixed non-vanishing holomorphic function in the annulus $\rho < |z| < 1/\rho$, it follows from (9) that

$$|W_n(z)| \leq C_{\mu,s,s_*} (s/s_*)^n, \quad s_* \leq |z| \leq 1/s_*.$$

It further follows from the maximum modulus principle that

$$|W_n(z) - W_n(1/z)| \leq |1 - z| C_{\mu,s,s_*} (s/s_*)^n, \quad s_* \leq |z| \leq 1/s_*,$$

where, as agreed before, the actual constants in the last two inequalities are not necessarily the same. Since $|\log(1 + \zeta)| \leq 2|\zeta|$ for $|\zeta| \leq 1/2$, there exists a constant A_{μ,s,s_*} such that

$$|H_n(z) - 1|, |\log H_n(z)| \leq |1 - z| A_{\mu,s,s_*} (s/s_*)^n, \quad s_* \leq |z| \leq 1/s_*.$$

Observe that the constants $A_{\mu,s,s_*} e^{\sqrt{n+1}} (s/s_*)^n$ are uniformly bounded above. Then the first claim of the lemma follows by minimizing these constants over all parameters $s < s_*$ between ρ and ρ_*. Further, it follows from Cauchy's formula that

$$H_n'(z) = \left(\int_{|\zeta|=1/s_*} - \int_{|\zeta|=s_*} \right) \frac{H_n(\zeta) - 1}{(\zeta - z)^2} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}}$$

for $\rho_* \leq |z| \leq 1/\rho_*$ and therefore it holds in this annulus that

$$|H_n'(z)| \leq C_{\mu,s,s_*,\rho_*} (s/s_*)^n.$$

The last claim of the lemma is now deduced in the same manner as the first one. □

3. Proof of Theorem 1

Using (7), it is easy to show that

$$\mathbb{E}_n(\mu) = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1 - h_{n+1}^2(x)}}{1 - x^2} \mathrm{d}x.$$

Furthermore, if we define $\mathrm{d}\sigma(\xi) := \mu'(-\xi)|\mathrm{d}\xi|$, then $\sigma'(\xi) = \mu'(-\xi)$ is still holomorphic and positive on the unit circle. Moreover, $b_n(z;\sigma) = b_n(-z;\mu)$. Therefore,

$$\mathbb{E}_n(\mu) = \widehat{\mathbb{E}}_n(\mu) + \widehat{\mathbb{E}}_n(\sigma), \quad \widehat{\mathbb{E}}_n(\nu) := \frac{2}{\pi}\int_0^1 \frac{\sqrt{1-h_{n+1}^2(x;\nu)}}{1-x^2}\mathrm{d}x, \tag{11}$$

for $\nu \in \{\mu,\sigma\}$. Thus, it is enough to investigate the asymptotic behavior of $\widehat{\mathbb{E}}_n(\mu)$. To this end, let

$$a := (n+1)^{1/2} \quad \text{and} \quad x =: 1 - t/(n+1), \quad 0 \le t \le a. \tag{12}$$

We shall also write

$$1 - h_{n+1}^2(x) =: f^2(t)(1+E_n(t)), \tag{13}$$

for $1-(n+1)^{-1/2} \le x \le 1$, where $f(t)$ was defined in Theorem 1.

Lemma 5. *Given an integer $N \ge 1$, it holds that*

$$\widehat{\mathbb{E}}_n(\mu) = \frac{1}{\pi}\log(n+1) + \frac{1}{2}A_0 + G_n(t) - \frac{1}{2}\sum_{p=1}^{N-1} H_p (n+1)^{-p} + O_N\left((n+1)^{-N}\right)$$

for large n, where $O_N(\cdot)$ is independent of n, but does depend on N,

$$G_n(t) := \frac{1}{\pi}\int_0^a \left(t^{-1} + (2(n+1)-t)^{-1}\right) f(t)\left((1+E_n(t))^{1/2} - 1\right)\mathrm{d}t,$$

and $H_p := \dfrac{1}{2^{p-1}\pi}\displaystyle\int_0^\infty (1-f(t))t^{p-1}\mathrm{d}t$ *for* $p \ge 1$.

Proof. Set $\delta := 1-(n+1)^{-1/2}$. It trivially holds that

$$\widehat{\mathbb{E}}_n(\mu) = \frac{2}{\pi}\int_0^\delta \frac{\mathrm{d}x}{1-x^2} - \frac{2}{\pi}\int_0^\delta \frac{1-\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x + \frac{2}{\pi}\int_\delta^1 \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x.$$

Denote the third integral above by $B_n(t)$. The second integral above is positive and equals to

$$\frac{2}{\pi}\int_0^\delta \frac{h_{n+1}^2(x)}{1+\sqrt{1-h_{n+1}^2(x)}}\frac{\mathrm{d}x}{1-x^2} \le \frac{2}{\pi}\int_0^\delta h_{n+1}^2(x)\frac{\mathrm{d}x}{1-\delta^2} = O\left(a^5 e^{-2a}\right),$$

where we used Lemma 2 for the last estimate. Therefore,

$$\widehat{\mathbb{E}}_n(\mu) = \frac{1}{\pi}\log\left(\frac{1+\delta}{1-\delta}\right) + B_n(t) + o_N\left((n+1)^{-N}\right),$$

where $o_N(\cdot)$ is independent of n, but does depend on N. Substituting $x = 1 - t/(n+1)$

into the expression for $B_n(t)$ and recalling (13), we get that

$$\begin{aligned} B_n(t) &= \frac{1}{\pi}\int_0^a f(t)(1+E_n(t))^{1/2}\frac{2(n+1)}{t(2(n+1)-t)}\,\mathrm{d}t \\ &= \frac{1}{\pi}\left(\log 2 + \log\frac{1}{1+\delta}\right) + \frac{1}{\pi}\int_0^a \frac{f(t)}{t}\,\mathrm{d}t - \frac{1}{\pi}\int_0^a \frac{1-f(t)}{2(n+1)-t}\,\mathrm{d}t + G_n(t). \end{aligned}$$

It was shown in [7, Lemma 8] that

$$\frac{1}{\pi}\int_0^a \frac{1-f(t)}{2(n+1)-t}\,\mathrm{d}t = \frac{1}{2}\sum_{p=1}^{N-1} H_p(n+1)^{-p} + O_N\left((n+1)^{-N}\right),$$

where $O_N(\cdot)$ is independent of n, but does depend on N. Moreover, it holds that

$$\begin{aligned} &\frac{1}{\pi}\log\left(\frac{1+\delta}{1-\delta}\right) + \frac{1}{\pi}\left(\log 2 + \log\frac{1}{1+\delta}\right) + \frac{1}{\pi}\int_0^a \frac{f(t)}{t}\,\mathrm{d}t = \\ &\qquad\qquad = \frac{1}{\pi}\log\frac{a}{1-\delta} + \frac{1}{2}A_0 + \frac{1}{\pi}\int_a^\infty \frac{1-f(t)}{t}\,\mathrm{d}t. \end{aligned}$$

Since $\log a - \log(1-\delta) = \log(n+1)$ and it was shown in [7, Lemma 7] that

$$\frac{1}{\pi}\int_a^\infty \frac{1-f(t)}{t}\,\mathrm{d}t = O\left(ae^{-2a}\right) = o_N\left((n+1)^{-N}\right),$$

where as usual $o_N(\cdot)$ is independent of n, but does depend on N, the claim of the lemma follows. □

We continue by deriving a different representation for the functions $E_n(t)$. To this end, notice that $t^2\mathrm{csch}^2 t = 1 - t^2/3 + O(t^4)$ as $t \to 0$ and therefore $f^2(t) = t^2/3 + O(t^4)$ as $t \to 0$. Hence, the function

$$\chi(t) := \left(\frac{t^2\mathrm{csch}t}{f(t)}\right)^2 \tag{14}$$

is continuous and non-vanishing at zero. Once again, we use notation from (12).

Lemma 6. *Set* $b_{n+1}^2(x) =: e^{-\mu_n(t)-2t}$ *and* $b_{n+1}'(x) =: (n+1)e^{w_n(t)-t}$. *Then it holds that*

$$E_n(t) = t^{-2}\chi(t)\left[1 - \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{e^{2w_n(t)}}{(1+D_n(t))^2}\right], \quad D_n(t) := \frac{1-e^{-\mu_n(t)}}{e^{2t}-1}.$$

Moreover, $\lim_{t\to 0^+} E_n(t)$ *exists and is finite.*

Proof. Since $h_{n+1}(1) = 1$ and $x = 1 - t/(n+1)$, it follows from (13) and the L'Hôpital's rule that

$$\lim_{t\to 0^+} E_n(t) = \frac{6}{(n+1)^2}\lim_{x\to 1^-}\frac{1-h_{n+1}(x)}{(1-x)^2} - 1 = \frac{3}{(n+1)^2}\lim_{x\to 1^-}\frac{h_{n+1}'(x)}{1-x} - 1.$$

Since $h_{n+1}(z)$ is a holomorphic function around 1, the latter limit is finite if and only if $h'_{n+1}(1) = 0$. As Blaschke products $b_{n+1}(z)$ satisfy $b_{n+1}(x)b_{n+1}(1/x) \equiv 1$, it holds that $h_{n+1}(x) = h_{n+1}(1/x)$, which immediately yields the desired equality.

To derive the claimed representation of $E_n(t)$, recall (7) and substitute $x = 1-t/(n+1)$ into (13) to get that

$$\begin{aligned} f^2(t)(1+E_n(t)) &= 1-\left(1-\frac{t}{2(n+1)}\right)^2 \frac{4t^2 e^{2w_n(t)-2t}}{(1-e^{-\mu_n(t)-2t})^2} \\ &= 1-\left(1-\frac{t}{2(n+1)}\right)^2 \frac{t^2 \operatorname{csch}^2 t e^{2w_n(t)}}{(1+D_n(t))^2} \\ &= f^2(t)\left[1+t^{-2}\chi(t)\left(1-\left(1-\frac{t}{2(n+1)}\right)^2 \frac{e^{2w_n(t)}}{(1+D_n(t))^2}\right)\right] \end{aligned}$$

from which the first claim of the lemma easily follows. □

In the next four lemmas we repeatedly use approximation by Taylor polynomials with the Lagrange remainder:

$$F(y) = \sum_{k=0}^{M-1} \frac{F^{(k)}(0)}{k!} y^K + \frac{F^{(M)}(\theta y)}{M!} y^M \tag{15}$$

for some $\theta \in (0,1)$ that dependents on both y and M.

Lemma 7. *Put $\omega(t) := t/(e^{2t}-1)$. Given an integer $N \geq 1$, it holds for all n large that*

$$(1+D_n(t))^{-2} = 1 + \sum_{p=1}^{N-1} \alpha_p(t)(n+1)^{-p} + \alpha_{n,N}(t)(n+1)^{-N},$$

where the functions $\alpha_p(t)$ are independent of n and N and are polynomials of degree p in ω with coefficients that are polynomials in t of degree at most $2p-1$, and the functions $\alpha_{n,N}(t)$ are bounded in absolute value for $0 \leq t \leq a$ by a polynomial of degree $2N-1$ whose coefficients are independent of n. Moreover,

$$\alpha_p(t) = (p+1)s_1^p - p s_1^{p-1}(2s_1+1)t + O(t^2) \quad \text{as} \quad t \to 0.$$

Proof. We start by deriving an asymptotic expansion of $\mu_n(t)$. It follows from Lemma 4 that $\log H_n(x) = tO(a^{-2}e^{-a}) = to_N(1)(n+1)^{-N}$ uniformly for $0 \leq t \leq a$. Fix T in Lemma 3 and let n_T be such that $1 < \sqrt{n_T+1}T$. Then it holds for all $n \geq n_T$ that

$$\log(SH_n)(x) = \sum_{p=1}^{N-1} c_p t^p (n+1)^{-p} + t\hat{c}_N(t)(n+1)^{-N},$$

where $|\hat{c}_N(t)| \leq C_{\mu,T,N} t^{N-1} + o_N(1)$ uniformly for $0 \leq t \leq a$ and $C_{\mu,T,N} \leq C_{\mu,T} T^{-N}$.

Hence, it follows from (10) and [7, Lemma 2] that

$$\begin{aligned}\mu_n(t) &= -2(n+1)\log x - 2t - 2\log(SH_n)(x)\\ &= \sum_{p=1}^{N-1} t^p m_p(t)(n+1)^{-p} + t m_{n,N}(t)(n+1)^{-N},\end{aligned} \tag{16}$$

where

$$m_p(t) := (2(p+1)^{-1}t - 2c_p) \quad \text{and} \quad m_{n,N}(t) := 2\hat{m}_{n,N}(t)t^N/(N+1) - 2\hat{c}_N(t)$$

with $1 \le \hat{m}_{n,N}(t) \le (3/2)^{N+1}$. Assuming that $T < 2/3$, we have that

$$|m_{n,N}(t)| \le C_{\mu,T,N} t^{N-1}(t+1) + o_N(1) \tag{17}$$

uniformly for $0 \le t \le a$ and $C_{\mu,T,N} \le C_{\mu,T} T^{-N}$. Using (16) with $N = 1$, we get that

$$|\mu_n(t)| = \left|\frac{t m_{n,1}(t)}{n+1}\right| \le \frac{|m_{n,1}(t)|}{\sqrt{n+1}} \le C_{\mu,T}, \quad 0 \le t \le a. \tag{18}$$

Recalling the definition of $D_n(t)$ in Lemma 6, we get from (15) that

$$D_n(t) = \omega(t)\frac{1-e^{-\mu_n(t)}}{t} = \omega(t)\left(-\frac{1}{t}\sum_{k=1}^{N-1}\frac{(-1)^k}{k!}\mu_n^k(t) - \frac{1}{t}e^{-\theta_1\mu_n(t)}\frac{(-1)^N}{N!}\mu_n^N(t)\right)$$

for some $\theta_1 \in (0,1)$ that depends on N and $\mu_n(t)$. Plugging (16) into the above formula gives us

$$D_n(t) = \omega(t)\sum_{p=1}^{N-1} t^{p-1} d_p(t)(n+1)^{-p} + \omega(t) d_{n,N}(t)(n+1)^{-N}, \tag{19}$$

where $d_p(t)$ is a polynomial of degree p with coefficients independent of n and N given by

$$d_p(t) := -\sum_{k=1}^{p}\frac{(-1)^k}{k!}\sum_{j_1+\cdots+j_k=p} m_{j_1}(t)\cdots m_{j_k}(t),$$

here, each index $j_i \in \{1,\ldots,p\}$, and $d_{n,N}(t)$ is given by

$$d_{n,N}(t) := -\sum_{k=1}^{N-1}\frac{(-1)^k}{k!}\sum_{j_1+\cdots+j_k\ge N}\frac{1}{t}\frac{m_{n,j_1,N}(t)\cdots m_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} - \frac{(-1)^N}{N!}\frac{(n+1)^N}{e^{\theta_1\mu_n(t)}}\frac{\mu_n^N(t)}{t}$$

with $m_{n,j,N}(t) := t^j m_j(t)$ when $j < N$ and $m_{n,N,N}(t) := t m_{n,N}(t)$. Recall that $t^2/(n+1) \le 1$ on $0 \le t \le a$ since $a = \sqrt{n+1}$. Hence, the first summand above is bounded in absolute value for $0 \le t \le a$ by a polynomial of degree $2N-1$ whose coefficients depend on N but are independent of n. We also get from (18) and (17) that

$$\left|e^{-\theta_1\mu_n(t)}(n+1)^N\mu_n^N(t)/t\right| \le e^{C_{\mu,T}} t^{N-1}|m_{n,1}(t)|^N \le C^*_{\mu,T} t^{N-1}(t+2)^N$$

for $0 \le t \le a$. Further, using (19) with $N = 1$ and (18) gives us

$$|D_n(t)| = \frac{\omega(t)}{e^{\theta_1\mu_n(t)}}\left|\frac{\mu_n(t)}{t}\right| \le \frac{e^{C_{\mu,T}}}{2}\frac{|m_{n,1}(t)|}{n+1} \le \frac{C_{\mu,T}e^{C_{\mu,T}}}{2\sqrt{n+1}}, \quad 0 \le t \le a. \tag{20}$$

Notice also that since $c_1 = s_1$ and $c_2 = s_1/2$ by Lemma 3, we have that

$$d_1(t) = t - 2s_1 \quad \text{and} \quad d_2(t) = -(1/2)t^2 + t(2s_1 + 2/3) - s_1(2s_1 + 1).$$

It follows from (20) that for any $-1 < D < 0$, there exists an integer $n_D \ge n_T$ such that $D \le D_n(t)$ for $0 \le t \le a$ and $n \ge n_D$. Hence, we get from (15) that

$$(1 + D_n(t))^{-2} = 1 + \sum_{k=1}^{N-1}(-1)^k(k+1)D_n^k(t) + \frac{(-1)^N(N+1)D_n^N(t)}{(1+\theta_2 D_n(t))^{N+2}}$$

for all $n \ge n_D$ and some $\theta_2 \in (0, 1)$ that depends on N and $D_n(t)$. Then the statement of the lemma follows with

$$\alpha_p(t) := \sum_{k=1}^{p}(-1)^k(k+1)\omega^k(t)t^{p-k}\sum_{j_1+\cdots+j_k=p} d_{j_1}(t)\cdots d_{j_k}(t)$$

here again, each index $j_i \in \{1, \ldots, p\}$, and

$$\alpha_{n,N}(t) := \sum_{k=1}^{N-1}(-1)^k(k+1)\omega^k(t)\sum_{j_1+\cdots+j_k\ge N}\frac{d_{n,j_1,N}(t)\cdots d_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} + (n+1)^N\frac{(-1)^N(N+1)D_n^N(t)}{(1+\theta_2 D_n(t))^{N+2}}$$

with $d_{n,j,N}(t) := t^{j-1}d_j(t)$ when $j < N$ and $d_{n,N,N}(t) := d_{n,N}(t)$. Reasoning as before lets us conclude that the first summand in the definition of $\alpha_{n,N}(t)$ is bounded in absolute value for $0 \le t \le a$ by a polynomial of degree $2N - 1$ whose coefficients depend on N but are independent of n. Moreover, since

$$\left|\frac{(n+1)^N D_n^N(t)}{(1+\theta_2 D_n(t))^{N+2}}\right| \le \frac{e^{NC_{\mu,T}}|m_{n,1}(t)|^N}{2^N(1-D)^{N+2}} \le \frac{C^*_{\mu,T}e^{NC_{\mu,T}}(t+2)^N}{2^N(1-D)^{N+2}}, \quad 0 \le t \le a,$$

by (20) and (17), the same is true for the second summand as well. Now, notice that

$$\alpha_p(t) = \big(-\omega(t)d_1(t)\big)^{p-2}\Big((p+1)\big(\omega(t)d_1(t)\big)^2 - p(p-1)t\omega(t)d_2(t)\Big) + O(t^2)$$

as $t \to 0$. Since $2\omega(t) = 1 - t + O(t^2)$ as $t \to 0$, the last claim of the lemma follows after a straightforward computation. □

Lemma 8. *Given $N \ge 1$, it holds for all n large that*

$$e^{2w_n(t)} = 1 + \sum_{p=1}^{N-1}\beta_p(t)(n+1)^{-p} + \beta_{n,N}(t)(n+1)^{-N},$$

where $\beta_p(t)$ is a polynomial of degree $2p$ whose coefficients are independent of n and N and the functions $\beta_{n,N}(t)$ are bounded in absolute value when $0 \le t \le a$ by a polynomial of degree $2N$ whose coefficients are independent of n. Moreover, as $t \to 0$, it holds that

$$\begin{cases} \beta_1(t) = -2s_1 + 2(s_1+1)t - t^2, \\ \beta_2(t) = s_1^2 - 4s_1(s_1+1)t + O(t^2), \\ \beta_3(t) = 2s_1^2(s_1+1)t + O(t^2), \\ \beta_p(t) = O(t^2), \quad p \ge 4. \end{cases}$$

Proof. We start by deriving an asymptotic expansion for $w_n(t)$. It follows from the very definition of $w_n(t)$ in Lemma 6, (10), and [7, Lemma 2] that

$$\begin{aligned} w_n(t) &= t + \log\frac{b'_{n+1}(x)}{n+1} = t + n\log x + \log\left((SH_n)(x) + \frac{x(SH_n)'(x)}{n+1}\right) \\ &= \sum_{p=1}^{N-1} t^p\phi_p(t)(n+1)^{-p} + \phi_{n,N}(t)(n+1)^{-N} + \log\left((SH_n)(x) + \frac{x(SH_n)'(x)}{n+1}\right), \end{aligned}$$

where

$$\phi_p(t) := \frac{p+1-pt}{p(p+1)} \quad \text{and} \quad \phi_{n,N}(t) := \left(N^{-1} - \frac{n\hat{m}_{n,N}(t)t}{(N+1)(n+1)}\right)t^N \tag{21}$$

with some $1 \le \hat{m}_{n,N}(t) \le (3/2)^N$. Further, notice that

$$(S^{(i)}H_n)(x) = S^{(i)}(x) + o_N(1)(n+1)^{-N} \quad \text{and} \quad (SH'_n)(x) = o_N(1)(n+1)^{-N}$$

uniformly for $0 \le t \le a$, $i \in \{0,1\}$, by Lemma 4 and since $S(z)$ is a fixed holomorphic function in a neighborhood of 1. Fix T in Lemma 3. Then it holds for all $n \ge n_T$ that

$$(SH_n)(x) = 1 + \sum_{j=1}^{N-1} s_j\frac{t^j}{(n+1)^j} + \hat{s}_N(t)(n+1)^{-N},$$

and

$$(SH_n)'(x) = -\sum_{j=1}^{N-1} js_j\frac{t^{j-1}}{(n+1)^{j-1}} - \hat{f}_N(t)(n+1)^{-N},$$

where $|\hat{s}_N(t)|, |\hat{f}_N(t)| \le C_\mu(t/T)^N + o_N(1)$ uniformly for $0 \le t \le a$. Therefore,

$$L_n(t) := (SH_n)(x) - 1 + \frac{x(SH_n)'(x)}{n+1} = \sum_{j=1}^{N-1} t^{j-1}l_j(t)(n+1)^{-j} + l_{n,N}(t)(n+1)^{-N}, \tag{22}$$

where

$$l_j(t) := \left(s_j(t-j) + (j-1)s_{j-1}\right)$$

and

$$l_{n,N}(t) := (N-1)s_{N-1}t^{N-1} + \hat{s}_N(t) - \left(1 - \frac{t}{n+1}\right)\frac{\hat{f}_N(t)}{n+1}.$$

In particular, it holds that

$$|l_{n,N}(t)| \le 2C_\mu (t/T)^N + (N-1)s_{N-1}t^{N-1} + o_N(1) \tag{23}$$

and therefore

$$|L_n(t)| \le \frac{|l_{n,1}(t)|}{n+1} \le \frac{C_{\mu,T}}{\sqrt{n+1}}, \quad 0 \le t \le a. \tag{24}$$

Hence, given $-1 < L < 0$, there exists an integer $n_L \ge n_T$ such that $L \le L_n(t)$ for $0 \le t \le a$ and $n \ge n_L$. Thus, we get from (15) that

$$\log(1 + L_n(t)) = \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k} L_n^k(t) + \frac{(-1)^{N-1} L_n^N(t)}{N(1 + \theta_3 L_n(t))^N}$$

for some $\theta_3 \in (0,1)$ that depends on N and $L_n(t)$. Therefore, we get from (22) that

$$\log\left((SH_n)(x) + \frac{x(SH_n)'(x)}{n+1}\right) = \sum_{p=1}^{N-1} \psi_p(t)(n+1)^{-p} + \psi_{n,N}(t)(n+1)^{-N},$$

where $\psi_p(t)$ is a polynomial of degree p with coefficients independent of n and N given by

$$\psi_p(t) := \sum_{k=1}^{p} \frac{(-1)^{k-1}}{k} \sum_{j_1+\cdots+j_k=p} t^{p-k} l_{j_1}(t) \cdots l_{j_k}(t), \tag{25}$$

here, each index $j_i \in \{1, \ldots, p\}$, and $\psi_{n,N}(t)$ is given by

$$\psi_{n,N}(t) := \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k} \sum_{j_1+\cdots+j_k \ge N} \frac{l_{n,j_1,N}(t) \cdots l_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} + (n+1)^N \frac{(-1)^{N-1} L_n^N(t)}{N(1+\theta_3 L_n(t))^N}$$

with $l_{n,j,N}(t) := t^{j-1} l_j(t)$ when $j < N$ and $l_{n,N,N}(t) := l_{n,N}(t)$. As in the previous lemma, since $t^2/(n+1) \le 1$ when $0 \le t \le a$, the first summand above is bounded in absolute value by a polynomial of degree N whose coefficients are independent of n. It also follows from (24) and (23) that

$$\frac{(n+1)^N |L_n^N(t)|}{|1+\theta_3 L_n(t)|^N} \le \frac{|l_{n,1}(t)|^N}{(1-L)^N} \le C_{\mu,T} \frac{(t+1)^N}{(1-L)^N}, \quad 0 \le t \le a,$$

for all $n \ge n_L$. Altogether, we have shown that

$$w_n(t) = \sum_{p=1}^{N-1} (t^p \phi_p(t) + \psi_p(t))(n+1)^{-p} + (\phi_{n,N}(t) + \psi_{n,N}(t))(n+1)^{-N} \tag{26}$$

with ϕ_p, ψ_p and $\phi_{n,N}, \psi_{n,N}$ as described above. We also can deduce from (21) and (25) that $t\phi_1(t) + \psi_1(t) = -s_1 + t(s_1 + 1) - t^2/2$ and

$$t^p \phi_p(t) + \psi_p(t) = \frac{(-1)^{p-1}}{p} l_1^p(t) + (-1)^{p-2} t l_1^{p-2}(t) l_2(t) + O(t^2) = -\frac{s_1^p}{p} + O(t^2) \qquad (27)$$

for $p \geq 2$, where we used that $2s_2 = s_1^2 + s_1$, see Lemma 3. Since

$$\left|\psi_{n,1}(t)\right| \leq (n+1)\frac{|L_n(t)|}{1-L} \leq \sqrt{n+1}\frac{C_{\mu,T}}{1-L}, \quad 0 \leq t \leq a,$$

by (24) for $n \geq n_L$, we get from (26), applied with $N = 1$, and (21) that

$$|w_n(t)| = \left|\frac{\phi_{n,1}(t) + \psi_{n,1}(t)}{n+1}\right| \leq C_{\mu,T,L}, \quad 0 \leq t \leq a, \quad n \geq n_L. \qquad (28)$$

Now, using (15) once more, we get

$$e^{2w_n(t)} = 1 + \sum_{k=1}^{N-1} \frac{2^k}{k!} w_n^k(t) + e^{2\theta_4 w_n(t)} \frac{(2)^N}{N!} w_n^N(t)$$

for some $\theta_4 \in (0,1)$ that depends on N and $w_n(t)$. Plugging (26) into the above formula gives us the desired expansion with

$$\beta_p(t) := \sum_{k=1}^{p} \frac{2^k}{k!} \sum_{j_1+\cdots+j_k=p} (t^{j_1}\phi_{j_1}(t) + \psi_{j_1}(t)) \cdots (t^{j_k}\phi_{j_k}(t) + \psi_{j_k}(t)), \qquad (29)$$

which is a polynomial of degree $2p$ with coefficients independent of n and N, and

$$\beta_{n,N}(t) := \sum_{k=1}^{N-1} \frac{2^k}{k!} \sum_{j_1+\cdots+j_k \geq N} \frac{\prod_{i=1}^k (\phi_{n,j_i,N}(t) + \psi_{n,j_i,N}(t))}{(n+1)^{j_1+\cdots+j_k-N}} + e^{2\theta_4 w_n(t)} \frac{2^N}{N!}(n+1)^N w_n^N(t)$$

with $\phi_{n,j,N}(t) := t^j \phi_j(t), \psi_{n,j,N}(t) := \psi_j(t)$ when $j < N$ and $\phi_{n,N,N}(t) := \phi_{n,N}(t), \psi_{n,N,N}(t) := \psi_{n,N}(t)$, which is bounded in absolute value when $0 \leq t \leq a$ by a polynomial of degree $2N$ whose coefficients are independent of n due to (28) and the same reasons as in the similar previous computations. Thus, it only remains to compute the linear approximation to $\beta_p(t)$ at zero. Now, it follows from (27) and (29) that

$$\beta_p(t) = s_1^p \sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{1}{j_1 \cdots j_k} - \left(s_1^{p-1}(s_1+1) \sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{n(j_1,\ldots,j_k)}{j_1 \cdots j_k}\right) t + O(t^2)$$

where $n(j_1,\ldots,j_k)$ is the number of 1's in the partition $\{j_1,\ldots,j_k\}$ of p. To simplify this expression observe that

$$
\begin{aligned}
(1-x)^2 e^{-2yx} = e^{2\log(1-x)-2yx} &= 1 + \sum_{k=1}^{\infty} \frac{(-2)^k}{k!}(yx - \ln(1-x))^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-2)^k}{k!}\left((1+y)x + \sum_{j=2}^{\infty} \frac{x^j}{j}\right)^k \qquad (30)\\
&= 1 + \sum_{p=1}^{\infty}\left(\sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{(1+y)^{n(j_1,\ldots,j_k)}}{j_1\cdots j_k}\right)x^p,
\end{aligned}
$$

where y is a free parameter. By putting $y = 0$ in this expression, we get that

$$
\sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{1}{j_1\cdots j_k} = \begin{cases} -2 & \text{if} \quad p=1, \\ 1 & \text{if} \quad p=2, \\ 0 & \text{if} \quad p\geq 3. \end{cases}
$$

Moreover, by differentiating (30) with respect to y and then putting $y = 0$, we get

$$
\sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{n(j_1,\ldots,j_k)}{j_1\cdots j_k} = \begin{cases} -2 & \text{if} \quad p=1, \\ 4 & \text{if} \quad p=2, \\ -2 & \text{if} \quad p=3, \\ 0 & \text{if} \quad p\geq 4, \end{cases}
$$

which clearly finishes the proof of the last claim of the lemma. □

Lemma 9. *Let $\chi(t)$ be given by (14). For any integer $N \geq 1$, it holds that*

$$
(1+E_n(t))^{1/2} - 1 = \chi(t)\sum_{p=1}^{N-1} u_p(t)(n+1)^{-p} + \chi(t)u_{n,N}(t)(n+1)^{-N},
$$

where $u_p(t)$ is bounded in absolute value[2] *on $0 \leq t < \infty$ by a polynomial of degree $2p-2$ whose coefficients are independent of n and N and the functions $u_{n,N}(t)$ are bounded in absolute value when $0 \leq t \leq a$ by a polynomial of degree $2N-2$ whose coefficients are independent of n.*

Proof. Set

$$
R_n(t) := \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{e^{2w_n(t)}}{(1+D_n(t))^2}.
$$

Lemmas 7 and 8 yield that $R_n(t)$ has the following asymptotic expansion:

$$
R_n(t) = 1 + \sum_{p=1}^{N-1} r_p(t)(n+1)^{-p} + r_{n,N}(t)(n+1)^{-N},
$$

[2] In fact, $u_p(t)$ is a multivariate polynomial in ω, χ, and t.

where

$$r_p(t) := \sum_{j=0}^{p} \beta_j(t)\alpha_{p-j}(t) - \sum_{j=0}^{p-1} t\beta_j(t)\alpha_{p-1-j}(t) + \sum_{j=0}^{p-2} t^2\beta_j(t)\alpha_{p-2-j}(t)/4$$

with $\alpha_0(t) = \beta_0(t) :\equiv 1$, and $r_{n,N}(t)$ given by

$$\sum_{k=N}^{2N+2}\left(\sum_{j=0}^{k}\frac{\beta_{n,j,N}(t)\alpha_{n,k-j,N}(t)}{(n+1)^{k-N}} - \sum_{j=0}^{k-1}\frac{t\beta_{n,j,N}(t)\alpha_{n,k-1-j,N}(t)}{(n+1)^{k-N}} + \sum_{j=0}^{k-2}\frac{t^2\beta_{n,j,N}(t)\alpha_{n,k-2-j,N}(t)/4}{(n+1)^{k-N}}\right)$$

with $\alpha_{n,j,N}(t) := \alpha_j(t), \beta_{n,j,N}(t) := \beta_j(t)$ when $j < N$, $\alpha_{n,N,N}(t) := \alpha_{n,N}(t), \beta_{n,N,N}(t) := \beta_{n,N}(t)$, and $\alpha_{n,j,N}(t) = \beta_{n,j,N}(t) :\equiv 0$ when $j > N$. It also follows from Lemmas 7 and 8 that the functions $r_p(t)$ are independent of n and N and are polynomials in ω of degree p with coefficients that are polynomials in t of degree at most $2p$, while the functions $r_{n,N}(t)$ are bounded in absolute value for $0 \le t \le a$ by a polynomial of degree $2N$ whose coefficients are independent of n. Finally, we get from Lemmas 7 and 8 that

$$\sum_{j=0}^{1}\beta_j(t)\alpha_{1-j}(t) = t + O(t^2) \quad \text{and} \quad \sum_{j=0}^{k}\beta_j(t)\alpha_{k-j}(t) = O(t^2)$$

for all $k \ge 2$. Therefore, it holds that $r_p(t) = O(t^2)$ as $t \to 0$ for all $p \ge 1$.

It follows from Lemma 6 that $E_n(t) = t^{-2}\chi(t)[1 - R_n(t)]$. Hence, plugging the expansion of $R_n(t)$ into this formula gives us

$$E_n(t) = \chi(t)\left[\sum_{p=1}^{N-1} e_p(t)(n+1)^{-p} + e_{n,N}(t)(n+1)^{-N}\right],$$

where $e_p(t) := -t^{-2}r_p(t)$ for any p and $e_{n,N}(t) := -t^{-2}r_{n,N}(t)$ for any n, N. It follows from the properties of $r_p(t)$ that each $e_p(t)$ is a continuous function and is bounded in absolute value on $0 \le t < \infty$ by a polynomial of degree $2p - 2$. Also, since $\chi(t)$ is a continuous function as well and $\lim_{t\to 0^+} E_n(t)$ exists and is finite according to Lemma 6, so must $\lim_{t\to 0^+} e_{n,N}(t)$ for all n, N. Then it follows from properties of $r_{n,N}(t)$ that $e_{n,N}(t)$ is bounded in absolute value when $0 \le t \le a$ by a polynomial of degree $2N - 2$ whose coefficients are independent of n.

From what precedes, we get that

$$|E_n(t)| \le \frac{\chi(t)|e_{n,1}(t)|}{n+1} \le \frac{C_{\mu,T}}{n+1}, \quad 0 \le t \le a.$$

Hence, for any $-1 < E < 0$ there exists an integer n_E such that $E \le E_n(t)$ for all $0 \le t \le a$ and $n \ge n_E$. Thus, by applying (15) one more time, we get that

$$(1 + E_n(t))^{1/2} - 1 = \sum_{k=1}^{N-1}\binom{1/2}{k}E_n^k(t) + \binom{1/2}{N}\frac{E_n^N(t)}{(1 + \theta_5 E_n(t))^{N-1/2}}$$

for some $\theta_5 \in (0,1)$ that depends on N and $E_n(t)$. Therefore, the claim of the lemma follows with

$$u_p(t) := \sum_{k=1}^{p} \binom{1/2}{k} \chi^{k-1}(t) \sum_{j_1+\cdots+j_k=p} e_{j_1}(t)\cdots e_{j_k}(t),$$

which is bounded in absolute value on $0 \le t < \infty$ by a polynomial of degree $2p-2$ whose coefficients are independent of n and N, and

$$u_{n,N}(t) := \sum_{k=1}^{N-1} \binom{1/2}{k} \chi^{k-1}(t) \sum_{j_1+\cdots+j_k \ge N} \frac{e_{n,j_1,N}(t)\cdots e_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} + \binom{1/2}{N} \frac{(n+1)^N E_n^N(t)}{(1+\theta_5 E_n(t))^{N-1/2}}$$

where $e_{n,j,N}(t) := e_j(t)$ when $j < N$ and $e_{n,N,N}(t) := e_{n,N}(t)$, which is bounded in absolute value on $0 \le t \le a$ by a polynomial of degree $2N-2$ whose coefficients are independent of n due to the same reasoning as in two previous lemmas. □

Lemma 10. *Given $N \ge 1$, it holds that*

$$\frac{(1+E_n(t))^{1/2}-1}{2(n+1)-t} = \chi(t) \sum_{p=2}^{N-1} v_p(t)(n+1)^{-p} + \chi(t) v_{n,N}(t)(n+1)^{-N},$$

where $v_p(t)$ is bounded in absolute value on $0 \le t < \infty$ by a polynomial of degree $2p-4$ whose coefficients are independent of n and N and the functions $v_{n,N}(t)$ is bounded in absolute value when $0 \le t \le a$ by a polynomial of degree $2N-4$ whose coefficients are independent of n.

Proof. Since $0 \le t \le a = \sqrt{n+1}$, we get from (15) that

$$\frac{1}{2(n+1)-t} = \sum_{p=1}^{N-1} z_p(t)(n+1)^{-p} + z_{n,N}(t)(n+1)^{-N},$$

where

$$z_p(t) := 2^{-p} t^{p-1} \quad \text{and} \quad z_{n,N}(t) := \frac{2^{-N} t^{N-1}}{(1-\theta_6 t/2(n+1))^{N+1}}$$

for some $\theta_6 \in (0,1)$ that depends on N and t. Therefore, the claim of the lemma follows from Lemma 9 with

$$v_p(t) := \sum_{j=1}^{p-1} z_j(t) u_{p-j}(t) \quad \text{and} \quad v_{n,N}(t) := \sum_{k=N}^{2N} \sum_{j_1+j_2=k} \frac{z_{n,j_1,N}(t) v_{n,j_2,N}(t)}{(n+1)^{k-N}}$$

where $j_1, j_2 \in \{1, \dots, N\}$, $z_{n,j,N}(t) := z_j(t)$, $u_{n,j,N}(t) := u_j(t)$ for $j < N$, and $z_{n,n,N}(t) := z_{n,N}(t)$, $u_{n,N,N}(t) := u_{n,N}(t)$. □

With the notation introduced in Lemmas 5, 9, and 10, the following lemma holds.

Lemma 11. *Given $N \geq 1$, it holds that*

$$G_n(t) = I_1^\mu (n+1)^{-1} + \sum_{p=2}^{N-1} (I_p^\mu + J_p^\mu)(n+1)^{-p} + O_N\left((n+1)^{-N}\right)$$

for all n large, where

$$I_p^\mu := \frac{1}{\pi}\int_0^\infty t^{-1} f(t)\chi(t) u_p(t)\mathrm{d}t \quad \text{and} \quad J_p^\mu := \frac{1}{\pi}\int_0^\infty f(t)\chi(t) v_p(t)\mathrm{d}t$$

(observe that $t^{-1}f(t)$ is a continuous and bounded function on $0 \leq t < \infty$, $\chi(t)$ decreases exponentially at infinity, and the functions $u_p(t), v_p(t)$ are bounded by polynomials).

Proof. By the very definition of $G_n(t)$ in Lemma 5 we have that $G_n(t) = I_n(t) + J_n(t)$, where

$$I_n(t) := \frac{1}{\pi}\int_0^a t^{-1} f(t)\left((1+E_n(t))^{1/2} - 1\right)\mathrm{d}t$$

and

$$J_n(t) := \frac{1}{\pi}\int_0^a f(t)\frac{(1+E_n(t))^{1/2} - 1}{2(n+1) - t}\mathrm{d}t.$$

Using Lemma 9, we can rewrite the first integral above as

$$I_n(t) = \sum_{p=1}^{N-1} I_p^\mu (n+1)^{-p} - S_n(t) + T_n(t),$$

where

$$S_n(t) := \frac{1}{\pi}\sum_{p=1}^{N-1}(n+1)^{-p}\int_a^\infty t^{-1} f(t)\chi(t) u_p(t)\mathrm{d}t$$

and

$$T_n(t) := \frac{1}{\pi}(n+1)^{-N}\int_0^a t^{-1} f(t)\chi(t) u_{n,N}(t)\mathrm{d}t.$$

Since $u_p(t) = O(t^{2p-2})$, $f(t) = O(1)$, and $\chi(t) = O(t^4 e^{-2t})$ as $t \to \infty$, it holds that

$$S_n(t) = \sum_{p=1}^{N-1}(n+1)^{-p}\int_a^\infty O(t^{2p+1}e^{-2t})\mathrm{d}t = \sum_{p=1}^{N-1}(n+1)^{-p} O(a^{2p+1}e^{-2a}) =$$
$$= O_N\left(a e^{-2a}\right) = o_N\left((n+1)^{-N}\right).$$

Moreover, since $u_{n,N}(t)$ is bounded by a polynomial of degree $2N-2$ for $0 \leq t \leq a$, we have that $T_n(t) = O_N((n+1)^{-N})$.

Similarly, we get from Lemma 10 that

$$J_n(t) = \sum_{p=2}^{N-1} J_p^\mu (n+1)^{-p} - U_n(t) + V_n(t),$$

where

$$U_n(t) := \frac{1}{\pi}\sum_{p=2}^{N-1}(n+1)^{-p}\int_a^\infty f(t)\chi(t)v_p(t)\mathrm{d}t$$

and

$$V_n(t) := \frac{1}{\pi}(n+1)^{-N}\int_0^a f(t)\chi(t)v_{n,N}(t)\mathrm{d}t.$$

An argument as above argument shows that $U_n(t) = O_N(e^{-2a}) = o_N((n+1)^{-N})$ and $V_n(t) = O_N((n+1)^{-N})$ for large n, which finishes the proof of the lemma. □

Lemma 12. *The claim of Theorem 1 holds.*

Proof. It follows from Lemmas 5 and 11 that given an integer $N \geq 1$, it holds that

$$\widehat{\mathbb{E}}_n(\mu) = \frac{1}{\pi}\log(n+1) + \frac{1}{2}A_0 + \sum_{p=1}^{N-1}(I_p^\mu + J_p^\mu - H_p/2)(n+1)^{-p} + O_N\left((n+1)^{-N}\right),$$

where we set $J_1^\mu := 0$. The claim of Theorem 1 now follows from (11) by taking $A_p^\mu := I_p^\mu + I_p^\sigma + J_p^\mu + J_p^\sigma - H_p$. □

Acknowledgments

The work of the first author is done towards completion of her Ph.D. degree at Indiana University-Purdue University Indianapolis under the direction of the second author. The research of the second author was supported in part by a grant from the Simons Foundation, CGM-354538.

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