TOPOLOGICAL EXPANSION IN THE COMPLEX CUBIC LOG-GAS MODEL. ONE-CUT CASE

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Dedicated to David Ruelle and Yakov Sinai

Abstract. We prove the topological expansion for the cubic log-gas partition function

$$Z_N(t) = \int_\Gamma \cdots \int_\Gamma \prod_{1\le j<k\le N} (z_j - z_k)^2 \prod_{k=1}^{N} e^{-N\left(-\frac{z^3}{3}+tz\right)} \mathrm{d}z_1 \cdots \mathrm{d}z_N,$$

where t is a complex parameter and Γ is an unbounded contour on the complex plane extending from $e^{\pi \mathrm{i}}\infty$ to $e^{\pi \mathrm{i}/3}\infty$. The complex cubic log-gas model exhibits two phase regions on the complex t-plane, with one cut and two cuts, separated by analytic critical arcs of the two types of phase transition: split of a cut and birth of a cut. The common point of the critical arcs is a tricritical point of the Painlevé I type. In the present paper we prove the topological expansion for $\log Z_N(t)$ in the one-cut phase region. The proof is based on the Riemann–Hilbert approach to semiclassical asymptotic expansions for the associated orthogonal polynomials and the theory of S-curves and quadratic differentials.

1. Introduction

The main goal of this work is to analyze the topological expansion in the cubic log-gas model with a general complex coupling constant and semiclassical asymptotics of related orthogonal polynomials. The partition function of the cubic log-gas model is given as

$$Z_N(u) = \int_\Gamma \cdots \int_\Gamma \prod_{1\le j<k\le N} (z_j - z_k)^2 \prod_{j=1}^{N} e^{-N\left(\frac{z_j^2}{2}-uz_j^3\right)} \mathrm{d}z_1 \ldots \mathrm{d}z_N, \tag{1.1}$$

where $u > 0$ is a coupling constant and the contour of integration Γ goes from $e^{\pi \mathrm{i}}\infty$ to $e^{\pi \mathrm{i}/3}\infty$. This work is a continuation of the works of Bleher and Deaño [6, 7].

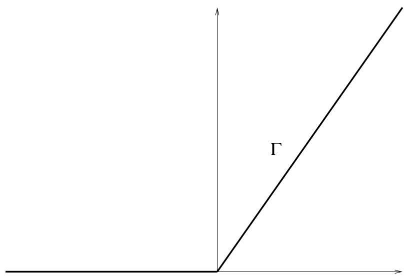

Figure 1. The contour Γ of integration.

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As proven in [6], the free energy of the cubic log-gas model,

$$F_N(u) := \frac{1}{N^2} \ln \frac{Z_N(u)}{Z_N(0)}\,, \tag{1.2}$$

admits an asymptotic expansion as $N \to \infty$ in powers of $\frac{1}{N^2}$,

$$F_N(u) \sim \sum_{g=0}^{\infty} \frac{F^{(2g)}(u)}{N^{2g}}, \tag{1.3}$$

for any u in the interval $0 \le u < u_c$, where

$$u_c = \frac{3^{1/4}}{18} \tag{1.4}$$

is a critical point. In addition, the functions $F^{(2g)}(u)$ admit an analytic continuation to the disk $|u| < u_c$ on the complex plane, and if we expand them in powers of u,

$$F^{(2g)}(u) = \sum_{j=1}^{\infty} \frac{f_{2j}^{(2g)} u^{2j}}{(2j)!}, \tag{1.5}$$

then the coefficient $f_{2j}^{(2g)}$ is a positive integer number that counts the number of 3-valent connected graphs with $2j$ vertices on a Riemann surface of genus g. Asymptotic expansion (1.3) is called the *topological expansion*. For more details on this aspect of the theory, we refer the reader to the classical papers of Bessis, Itzykson and Zuber [4], Brézin, Itzykson, Parisi and Zuber [5], the monograph of Forrester [25, Section 1.6], the works of Mulase [31], Di Francesco [17], Ercolani and McLaughlin [22, 23], and references therein, or the very readable introduction by Zvonkin [39].

As shown in [6], the coefficients $f_{2j}^{(2g)}/(2j)!$ of power series (1.5) behave, when $j \to \infty$, as

$$\frac{f_{2j}^{(2g)}}{(2j)!} = \frac{K_{2g} j^{\frac{5g-7}{2}}}{u_c^{2j}} \left(1 + \mathcal{O}(j^{-1/2})\right), \quad K_{2g} > 0. \tag{1.6}$$

This implies that u_c is the radius of convergence of power series (1.5). In fact, $u = u_c$ is a singular point of the functions (1.5). The topological expansion in a neighborhood of the critical point u_c has been obtain in the work of Bleher and Deaño [7]. This topological expansion is closely related to the Painlevé I equation. The relation to the Painlevé I equation can be already seen in asymptotic formula (1.6). Namely, if we rescale the coefficients K_{2g} in (1.6), by introducing the coefficients

$$C_{2g} = \frac{\Gamma\left(\frac{5g-1}{2}\right) u_c^g K_{2g}}{6 \cdot 3^{1/4}}\,, \tag{1.7}$$

and consider the following generating function:

$$y(t) = \sum_{g=0}^{\infty} C_{2g} t^{\frac{1-5g}{2}}\,, \tag{1.8}$$

then $y(t)$ solves the Painlevé I differential equation,

$$y''(t) = a_0 y^2(t) - a_1 t, \tag{1.9}$$

with $a_0 = 2^{\frac{5}{2}} 3^{\frac{9}{4}}$, $a_1 = 2^{\frac{3}{2}} 3^{-\frac{5}{4}}$ (see [3, 21, 6]).

It is noteworthy that the key ingredient in the proof of topological expansion (1.3) in [6] is the derivation of semiclassical asymptotic formulae for the recurrence coefficients γ_n^2, β_n of the corresponding monic orthogonal polynomials $P_n(z) = z^n + \dots$. The orthogonality condition is stated on the contour Γ:

$$\int_\Gamma P_n(z) z^k w(z) \mathrm{d}z = 0, \quad k = 0, 1, \dots, n-1; \qquad w(z) = e^{-N\left(\frac{z^2}{2} + uz^3\right)} \tag{1.10}$$

Namely, as proven in [6], for any u such that $0 \le u < u_c$, there exists $\varepsilon > 0$ such that as $N, n \to \infty$ with $1 - \varepsilon \le \frac{n}{N} \le 1 + \varepsilon$, the recurrence coefficients γ_n^2 and β_n admit the asymptotic expansions in powers of $\frac{1}{N^2}$:

$$\begin{cases} \gamma_n^2 \sim \sum_{k=0}^{\infty} \frac{1}{N^{2k}} g_{2k}\left(\frac{n}{N}, u\right), \\ \beta_n \sim \sum_{k=0}^{\infty} \frac{1}{N^{2k}} b_{2k}\left(\frac{n}{N} + \frac{1}{2N}, u\right), \end{cases} \tag{1.11}$$

where the functions $g_{2k}(s,u)$, $b_{2k}(s,u)$, $k = 0, 1, \dots$, do not depend on n and N and are analytic in s at $s = 1$.

In the paper [7] this asymptotic expansion is extended to the double scaling asymptotic expansion of the recurrence coefficients at the critical point u_c. In the double scaling regime we set

$$\frac{n}{N} = 1 + vN^{-4/5}. \tag{1.12}$$

where $v \in \mathbb{R}$ is a scaling variable. Then as proven in [7], at $u = u_c$ the recurrence coefficients γ_n^2 and β_n admit the asymptotic expansions in powers of $N^{-2/5}$ as $N \to \infty$:

$$\begin{cases} \gamma_n^2 \sim \gamma_c^2 + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} p_{2k}(v), \\ \beta_n \sim \beta_c + \sum_{k=1}^{\infty} \frac{1}{N^{2k/5}} q_{2k}(\tilde{v}), \end{cases} \tag{1.13}$$

where the functions $p_{2k}(v)$, $q_{2k}(\tilde{v})$ are expressed in terms of the Boutroux tritronquée solution to Painlevé I and $\tilde{v} = v + \frac{N^{-1/5}}{2}$. As shown in [7], expansions (1.11) and (1.13) can be extended for large N to u in overlapping intervals, $[0, u_c - N^{-0.79}]$ for (1.11) and $[u_c - N^{-0.65}, u_c]$ for (1.13), and this can be used to obtain the double scaling asymptotic formula for the partition function.

Namely, let $u - u_c = C\lambda N^{-\frac{4}{5}}$, where $C = 2^{-\frac{12}{5}}3^{-\frac{7}{4}}$ and λ is a complex scaling variable in the double scaling regime. Then for λ outside of a neighborhood of the poles of the Boutroux tritronquée solution to Painlevé I $y(\lambda)$, the partition function $Z_N(u)$ can be written as

$$Z_N(u) = Z_N^{\mathrm{reg}}(u) Z_N^{\mathrm{sing}}(\lambda)\left(1 + \mathcal{O}(N^{-\varepsilon})\right), \quad \varepsilon > 0, \tag{1.14}$$

where the regular factor is

$$Z_N^{\mathrm{reg}}(u) = e^{N^2[a + b(u-u_c) + c(u-u_c)^2] + d}, \tag{1.15}$$

with some explicit constants a, b, c, d, and the singular factor is

$$Z_N^{\mathrm{sing}}(\lambda) = e^{-Y(\lambda)}, \tag{1.16}$$

where $Y(\lambda)$ is a solution of the differential equation

$$Y''(\lambda) = y(\lambda), \tag{1.17}$$

with the boundary condition

$$Y(\lambda) = \frac{2\sqrt{6}}{45}(-\lambda)^{5/2} - \frac{1}{48}\log(-\lambda) + \mathcal{O}((-\lambda)^{-5/2}), \qquad \lambda \to -\infty. \tag{1.18}$$

Asymptotic formula (1.14) is used in [7] to prove the conjecture of David [12, 13] that the poles of $y(\lambda)$ give rise to zeros of $Z_N(u)$.

This work is a continuation of [6, 7]. The main goal of it is to investigate the topological expansion of the cubic log-gas model for complex values of u. Formula (1.1) is not very convenient for this purpose because the contour of integration Γ should be rotated to secure the convergence of the integral. Instead, let us make the change of variables in (1.1),

$$z_j = (3u)^{-1/3}\zeta_j + \frac{1}{6u}, \tag{1.19}$$

where we assume that $u > 0$ and $(3u)^{-1/3} > 0$. Then

$$\frac{z_j^2}{2} - uz_j^3 - \frac{1}{108u^2} = -\frac{\zeta_j^3}{3} + t\zeta_j, \tag{1.20}$$

where

$$t = \frac{1}{4(3u)^{4/3}}, \tag{1.21}$$

and with the help of the Cauchy theorem, formula (1.1) can be reduced to

$$Z_N(u) = C_N \int_\Gamma \cdots \int_\Gamma \prod_{1\le j<k\le N} (\zeta_j - \zeta_k)^2 \prod_{k=1}^N e^{-N\left(-\frac{\zeta_k^3}{3} + t\zeta_k\right)} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_N, \tag{1.22}$$

where C_N is an explicit constant. Now the integral converges for any complex t.

The primary interest of the present study is the asymptotic analysis of the partition function

$$Z_N(t) := \int_\Gamma \cdots \int_\Gamma \prod_{1\le j<k\le N} (z_j - z_k)^2 \prod_{k=1}^N e^{-NV(z;t)} \mathrm{d}z_1 \cdots \mathrm{d}z_N, \tag{1.23}$$

with respect to the parameter $t \in \mathbb{C}$ for the case when

$$V(z;t) = -\frac{z^3}{3} + tz, \quad t \in \mathbb{C}, \tag{1.24}$$

where Γ is an unbounded smooth contour such that for any parametrization $z(s)$, $s \in \mathbb{R}$, of Γ there exists $\epsilon \in (0, \pi/6)$ and $s_0 > 0$ for which

$$\begin{cases} |\mathsf{arg}(z(s)) - \pi/3| \le \pi/6 - \epsilon, & s \ge s_0, \\ |\mathsf{arg}(z(s)) - \pi| \le \pi/6 - \epsilon, & s \le -s_0, \end{cases} \tag{1.25}$$

where $\mathsf{arg}(z(s)) \in [0, 2\pi)$. The above conditions ensure that the partition function $Z_N(t)$ is finite and due to analyticity of the integrand does not depend on a particular Γ satisfying (1.25). Hence, we shall denote by $\mathcal{T}$ the collection of all such contours.

We analyze the partition function via the corresponding monic orthogonal polynomials

$$\int_\Gamma z^k P_n(z;t,N) e^{-NV(z;t)} \mathrm{d}z = 0, \quad k \in \{0, \ldots, n-1\}. \tag{1.26}$$

Due to the non-Hermitian character of the above relations, it might happen that polynomial satisfying (1.26) is non-unique. In this case we understand by $P_n(z;t,N)$ the monic polynomial of the smallest degree (such a polynomial is always unique). One way of connecting $Z_N(t)$ to $P_n(z;t,N)$ is via *three term recurrence relation.* More precisely, it is known that

$$zP_n(z;t,N) = P_{n+1}(z;t,N) + \beta_n(t,N)P_n(z;t,N) + \gamma_n^2(t,N)P_{n-1}(z;t,N), \tag{1.27}$$

granted all the polynomials in (1.27) have prescribed degrees, where

$$\begin{cases} \gamma_n^2(t,N) &=& h_n(t,N)/h_{n-1}(t,N), \\ h_n(t,N) &=& \displaystyle\int_\Gamma P_n^2(z;t,N) e^{-NV(z;t)} \mathrm{d}z. \end{cases} \tag{1.28}$$

Observe that if $P_n(z;t,N) = P_{n+1}(z;t,N)$ with both polynomials having degree n, then $h_n(t,N) = 0$ and $h_{n+1}(t,N) = \infty$. More generally, it holds that $h_n(t,N)$ is a meromorphic function of t and so is $\gamma_n^2(t,N)$. It is further known that the recurrence coefficients $\gamma_N^2(t,N)$ satisfy the *Toda equation*,

$$\frac{\partial^2 F_N(t)}{\partial t^2} = \gamma_N^2(t,N), \qquad F_N(t) = \frac{1}{N^2}\log Z_N(t). \tag{1.29}$$

Another way of connecting $Z_N(t)$ to orthogonal polynomials is through the formula

$$Z_N(t) = N! \prod_{n=0}^{N-1} h_n(t,N),$$

where $h_n(t,N)$ are given in (1.28). However, we shall not elaborate on this approach.

The structure of the paper is as follows:

- In Sections 2 and 3 we describe equilibrium measures and corresponding S-curves for the cubic model under consideration. This leads us to a precise description of the phase diagram of the cubic model on the complex t-plane.

- In Section 4 we present the main results of the paper: the topological expansion in the one-cut phase region and the asymptotic expansion of the orthogonal polynomials and their recurrence coefficients.
- In Section 5 we obtain various results about the detailed structure of the S-curves and critical graphs of the quadratic differential.
- In Section 6 we evaluate the g-function and its asymptotic behavior at singular points.
- In Section 7 we apply the Riemann–Hilbert approach to derive the asymptotic behavior of the orthogonal polynomials and their recurrence coefficients.
- And finally, in Section 8 we prove the topological expansion in the one-cut phase region.

2. Equilibrium Measures and S-Property

It is well understood that the zeros of polynomials satisfying (1.26) asymptotically distribute as a certain *weighted equilibrium measure* on an *S-contour* corresponding to the weight function (1.24). In this section we discuss these notions in greater detail. Our consideration will use the recent works of Huybrechs, Kuijlaars, and Lejon [27] and Kuijlaars and Silva [30]. Let us start with some definitions.

Definition 2.1. *Let V be an entire function. The logarithmic energy in the external field $\mathrm{Re}\, V$ of a measure ν in the complex plane is defined as*

$$E_V(\nu)=\iint \log\frac{1}{|s-t|}\mathrm{d}\nu(s)\mathrm{d}\nu(t)+\int \mathrm{Re}\, V(s)\mathrm{d}\nu(s).$$

The equilibrium energy of a contour Γ in the external field $\mathrm{Re}\, V$ is equal to

$$\mathcal{E}_V(\Gamma)=\inf_{\nu\in\mathcal{M}(\Gamma)} E_V(\nu), \tag{2.1}$$

where $\mathcal{M}(\Gamma)$ denotes the space of Borel probability measures on Γ.

When $\mathrm{Re}\, V(s)-\log|s|\to+\infty$ as $\Gamma\ni s\to\infty$, there exists a unique minimizing measure for (2.1), which is called the *weighted equilibrium measure* of Γ, say μ_Γ, in the external field $\mathrm{Re}\, V$, see [34, Theorem I.1.3] or [27]. We shall use this definition in the case of the cubic polynomial (1.24) and $\Gamma\in\mathcal{T}$. The support of μ_Γ, say J_Γ, is a compact subset of Γ. The equilibrium measure $\mu=\mu_\Gamma$ is characterized by the Euler–Lagrange variational conditions:

$$2U^\mu(z)+\mathrm{Re}\, V(z)\begin{cases}=\ell, & z\in J_\Gamma,\\ \geq\ell, & z\in\Gamma\setminus J_\Gamma,\end{cases} \tag{2.2}$$

where $\ell=\ell_\Gamma$ is a constant, the Lagrange multiplier, and

$$U^\mu(z)=-\int\log|z-s|d\mu(s)$$

is the logarithmic potential of μ, see [34, Theorem I.3.3]. Any $\Gamma\in\mathcal{T}$ can be used to define $Z_N(t)$ in (1.23), nevertheless, it is well understood in the theory of non-Hermitian orthogonal polynomials, starting with the works of Stahl [35, 36, 37] and Gonchar and Rakhmanov [26] that one should use the contour whose equilibrium measure has support *symmetric* (with the *S-property*) in the external field $\mathrm{Re}\, V$. We make this idea precise in the following definition.

Definition 2.2. *The support J_Γ has the S-property in the external field $\mathrm{Re}\, V$, if it consists of a finite number of open analytic arcs and their endpoints, and on each arc it holds that*

$$\frac{\partial}{\partial n_+}\big(2U^{\mu_\Gamma}+\mathrm{Re}\, V\big)=\frac{\partial}{\partial n_-}\big(2U^{\mu_\Gamma}+\mathrm{Re}\, V\big), \tag{2.3}$$

where $\frac{\partial}{\partial n_+}$ and $\frac{\partial}{\partial n_-}$ are the normal derivatives from the $(+)$- and $(-)$-side of Γ. We shall say that a curve $\Gamma\in\mathcal{T}$ is an S-curve in the field $\mathrm{Re}\, V$, if J_Γ has the S-property in this field.

It is also understood that geometrically J_Γ is comprised of *critical trajectories* of quadratic differentials. Recall that if Q is a meromorphic function, a *trajectory* (resp. *orthogonal trajectory*) of a quadratic differential $-Q(z)\mathrm{d}z^2$ is a maximal regular arc on which

$$-Q(z(s))\big(z'(s)\big)^2>0\quad\big(\text{resp.}\quad -Q(z(s))\big(z'(s)\big)^2<0\big)$$

for any local uniformizing parameter. A trajectory is called *critical* if it is incident with a *finite critical point* (zero or a simple pole of $-Q(z)\mathrm{d}z^2$) and it is called *short* if it is incident only with finite critical points. We designate the expression *critical (orthogonal) graph of* $-Q(z)\mathrm{d}z^2$ for the totality of the critical (orthogonal) trajectories $-Q(z)\mathrm{d}z^2$.

The following theorem is a specialization to $V(z;t)$ of [30, Theorem 2.3].

Theorem 2.3. *Let $V(z;t)$ be given by (1.24).*

(1) *There exists a contour $\Gamma_t \in \mathcal{T}$ such that*

$$\mathcal{E}_V(\Gamma_t) = \sup_{\Gamma\in\mathcal{T}} \mathcal{E}_V(\Gamma). \tag{2.4}$$

(2) *The equilibrium measure $\mu_t := \mu_{\Gamma_t}$ is the same for every Γ_t satisfying (2.4). The support J_t of μ_t has the S-property in the external field $\mathsf{Re}\, V(z;t)$.*

(3) *The function*

$$Q(z;t) = \left(\frac{V'(z;t)}{2} - \int \frac{\mathrm{d}\mu_t(s)}{z-s}\right)^2, \quad z \in \mathbb{C}\setminus J_t, \tag{2.5}$$

is a polynomial of degree 4.

(4) *The support J_t consists of short critical trajectories of the quadratic differential $-Q(z;t)\mathrm{d}z^2$ that connect simple zeros of $Q(z;t)$ and the equation*

$$\mathrm{d}\mu_t(z) = -\frac{1}{\pi i} Q_+^{1/2}(z;t)\mathrm{d}z, \quad z\in J_t, \tag{2.6}$$

holds on each such critical trajectory, where $Q^{1/2}(z;t) = \frac{1}{2}z^2 + \mathcal{O}(z)$ as $z\to\infty$.

Much information on the structure of the critical graphs of a quadratic differential can be found in the excellent monographs [28, 33, 38]. Since $\mathsf{deg}\, Q(z) = 4$, J_{Γ_t} consists of one or two arcs, corresponding (respectively) to the cases where $Q(z)$ has two simple zeros and one double zero, and the case where it has four simple zeros. In this paper we study the case of a single arc and investigate the two-cut case in a later publication. In the next section we discuss which values of t correspond to the one-cut case and describe the geometry of the critical graphs in more detail.

3. Structure of Γ_t

The structure of Γ_t and its dependence on t has been heuristically described in [1, 2]. Our goal here is to provide rigorous mathematical justifications for this description, when J_t consists of a single arc. Clearly, in this case $Q(z;t)$ should be of the form

$$Q(z;t) = \frac{1}{4}(z-a(t))(z-b(t))(z-c(t))^2. \tag{3.1}$$

It follows from (2.5) in conjunction with (1.24) that

$$Q(z;t) = \left(\frac{-z^2+t}{2} - \frac{1}{z} + \mathcal{O}(z^{-2})\right)^2 = \frac{(z^2-t)^2}{4} + z + C. \tag{3.2}$$

Thus, by equating the coefficients in (3.1) and (3.2), we obtain a system of equations

$$\begin{cases} a+b+2c = 0, \\ ab + c^2 + 2(a+b)c = -2t\,, \\ 2abc + (a+b)c^2 = -4. \end{cases} \tag{3.3}$$

By setting $x := (a+b)/2$ and eliminating the product ab from the second and third relations in (3.3), we get that

$$x^3 - tx - 1 = 0. \tag{3.4}$$

To study the solutions of (3.4), denote by $\mathcal{C}$ the critical graph of an auxiliary quadratic differential

$$-(1+1/s)^3\mathrm{d}s^2, \tag{3.5}$$

see Figure 2(a). We show in Section 5 that $\mathcal{C}$ consists of 5 critical trajectories emanating from -1 at the angles $2\pi k/5$, $k\in\{0,\dots,4\}$, one of them being $(-1,0)$, other two forming a loop crossing the real line approximately at 0.635, and the last two approaching infinity along the imaginary axis without changing the half-plane (upper or lower). Given $\mathcal{C}$, define

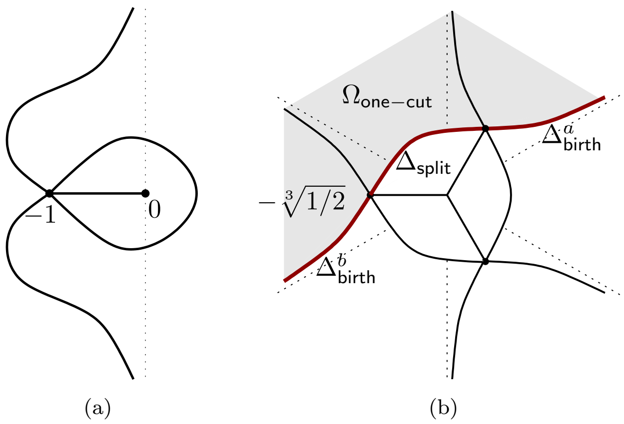

FIGURE 2. schematic representation of (a) the critical graph $\mathcal{C}$; (b) the set Δ (solid lines) and the domain $\Omega_{\mathsf{one-cut}}$ (shaded region).

$$\Delta := \big\{x : \ 2x^3 \in \mathcal{C}\big\}.$$

Further, put $\Omega_{\mathsf{one-cut}}$ to be the shaded region on Figure 2(b) and set

$$\partial\Omega_{\mathsf{one-cut}} = \Delta_{\mathsf{birth}}^b \cup \big\{-2^{-1/3}\big\} \cup \Delta_{\mathsf{split}} \cup \big\{e^{\pi \mathrm{i}/3}2^{-1/3}\big\} \cup \Delta_{\mathsf{birth}}^a,$$

where Δ_{split} connects $-2^{-1/3}$ and $e^{\pi\mathrm{i}/3}2^{-1/3}$, $\Delta_{\mathsf{birth}}^b$ extends to infinity in the direction of the angle $7\pi/6$ while $\Delta_{\mathsf{birth}}^a$ extends to infinity in the direction of the angle $\pi/6$. Finally, let $t(x) := (x^3-1)/x$ and set

$$\begin{cases} t_{\mathsf{cr}} := 3\cdot 2^{-2/3} = t\big(-2^{-1/3}\big), \\ O_{\mathsf{one-cut}} := t(\Omega_{\mathsf{one-cut}}), \\ C_{\mathsf{split}} := t\big(\Delta_{\mathsf{split}}\big), \quad C_{\mathsf{birth}}^b := t\big(\Delta_{\mathsf{birth}}^b\big), \quad C_{\mathsf{birth}}^a := t\big(\Delta_{\mathsf{birth}}^a\big), \\ S := (t_{\mathsf{cr}},\infty), \quad e^{2\pi\mathrm{i}/3}S := \big\{z : \ e^{-2\pi\mathrm{i}/3}z \in S\big\}, \end{cases} \tag{3.6}$$

see Figure 3. The function $t(x)$ is holomorphic in $\Omega_{\mathsf{one-cut}}$ with non-vanishing derivative

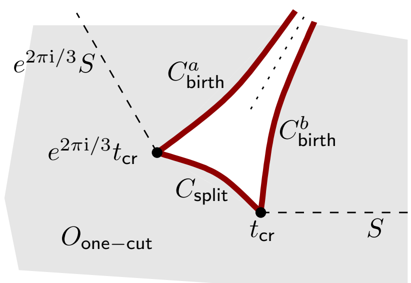

FIGURE 3. Domain $O_{\mathsf{one-cut}}$ (shaded region); $\partial O_{\mathsf{one-cut}}$ consisting of the open bounded arc C_{split}, two open semi-unbounded arcs C_{birth}^a and C_{birth}^b, and two points t_{cr} and $e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}$; the semi-unbounded open horizontal rays S and $e^{2\pi\mathrm{i}/3}S$ (dashed lines).

there. It maps $\Omega_{\mathsf{one-cut}}$ onto $O_{\mathsf{one-cut}}$ in a one-to-one fashion. Hence, the inverse map $x(t)$ exists and is holomorphic. Altogether, the following proposition holds.

Proposition 3.1. *There exists a holomorphic branch $x(t)$ of (3.4) that maps $O_{\mathsf{one-cut}}$ conformally onto $\Omega_{\mathsf{one-cut}}$. The function $x(t)$ possesses analytic continuations across each of* *the arcs C_{split}, C^a_{birth}, and C^b_{birth}. The functions*

$$\begin{cases} a(t) & := & x(t) - \mathrm{i}\sqrt{2}/\sqrt{x}(t), \\ b(t) & := & x(t) + \mathrm{i}\sqrt{2}/\sqrt{x}(t), \\ c(t) & := & -x(t), \end{cases} \tag{3.7}$$

are holomorphic in $O_{\mathsf{one-cut}}$, where $\sqrt{x}(t)$ is the branch holomorphic in $O_{\mathsf{one-cut}}$ satisfying $\sqrt{x}(0) = e^{\pi\mathrm{i}/3}$. It is a matter of a routine verification to check that they also satisfy (3.3).

Below, we adapt the following convention: $\Gamma(z_1, z_2)$ (resp. $\Gamma[z_1, z_2]$) stands for the trajectory or orthogonal trajectory (resp. the closure of) connecting z_1 and z_2, oriented from z_1 to z_2, and $\Gamma\big(z, e^{\mathrm{i}\theta}\infty\big)$ (resp. $\Gamma\big(e^{\mathrm{i}\theta}\infty, z\big)$) stands for the orthogonal trajectory ending at z, approaching infinity at the angle θ, and oriented away from z (resp. oriented towards z).[1]

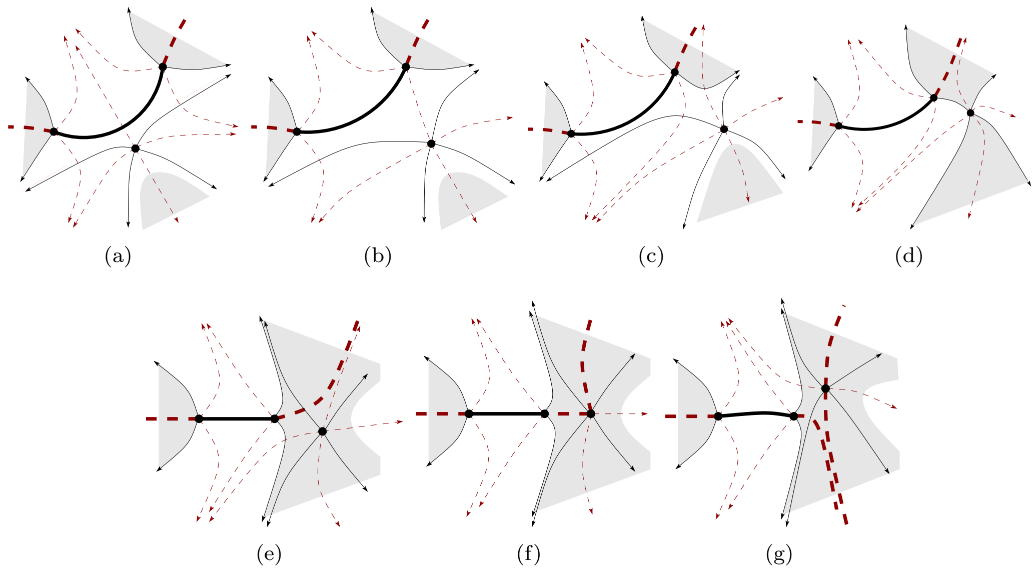

FIGURE 4. Schematic representation of the critical (solid) and critical orthogonal (dashed) graphs of $-Q(z;t)\mathrm{d}z^2$ when $t \in O_{\mathsf{one-cut}}$. The bold curves represent the preferred S-curve Γ_t. Shaded region is the set where $\mathsf{Re}\left(\int_b^z Q^{1/2}(z;t)\mathrm{d}z\right) < 0$.

Theorem 3.2. *Let μ_t and $Q(z;t)$ be as in Theorem 2.3, $J_t = \mathsf{supp}(\mu_t)$. When $t \in \overline{O}_{\mathsf{one-cut}}$, the polynomial $Q(z;t)$ is of the form* (3.1) *with $a(t)$, $b(t)$, and $c(t)$ as in Proposition 3.1 and the set J_t consists of a single arc. Moreover,*

- (I) *if $t \in O_{\mathsf{one-cut}}$, then $J_t = \Gamma[a,b]$ and an S-curve $\Gamma_t \in \mathcal{T}$ can be chosen as*
  - (a) *$\Gamma\big(e^{\pi\mathrm{i}}\infty, a\big) \cup J_t \cup \Gamma\big(b, e^{\pi\mathrm{i}/3}\infty\big)$ when t belongs to the connected component bounded by $S \cup C_{\mathsf{crit}} \cup e^{2\pi\mathrm{i}/3}S$, see Figure 4(a–e);*
  - (b) *$\Gamma\big(e^{\pi\mathrm{i}}\infty, a\big) \cup J_t \cup \Gamma(b,c) \cup \Gamma\big(c, e^{\pi\mathrm{i}/3}\infty\big)$ when $t \in S$, see Figure 4(f);*
  - (c) *$\Gamma\big(e^{\pi\mathrm{i}}\infty, c\big) \cup \Gamma(c,a) \cup J_t \cup \Gamma\big(b, e^{\pi\mathrm{i}/3}\infty\big)$ when $t \in e^{2\pi\mathrm{i}/3}S$;*
  - (d) *$\Gamma\big(e^{\pi\mathrm{i}}\infty, a\big) \cup J_t \cup \Gamma\big(b, e^{-\pi\mathrm{i}/3}\infty\big) \cup \Gamma\big(e^{-\pi\mathrm{i}/3}\infty, c\big) \cup \Gamma\big(c, e^{\pi\mathrm{i}/3}\infty\big)$ when t belongs to the connected component bounded by $S \cup C^b_{\mathsf{birth}}$, see Figure 4(g);*
  - (e) *$\Gamma\big(e^{\pi\mathrm{i}}\infty, c\big) \cup \Gamma\big(c, e^{-\pi\mathrm{i}/3}\big) \cup \Gamma\big(e^{-\pi\mathrm{i}/3}\infty, a\big) \cup J_t \cup \Gamma\big(b, e^{\pi\mathrm{i}/3}\infty\big)$ when t belongs to the connected component bounded by $e^{2\pi\mathrm{i}/3}S \cup C^a_{\mathsf{birth}}$.*
- (II) *if $t = t_{\mathsf{cr}}$ (resp. $t = e^{2\pi i/3}t_{\mathsf{cr}}$), then $J_t = \Gamma[a,b]$, c coincides with b (resp. a), and an S-curve $\Gamma_t \in \mathcal{T}$ can be chosen as in Case I(a), see Figure 5(a).*
- (III) *if $t \in C_{\mathsf{split}}$, then $J_t = \Gamma[a,c] \cup \Gamma[c,b]$ and an S-curve $\Gamma_t \in \mathcal{T}$ can be chosen as in Case I(a), see Figure 5(b).*

[1]This notation is unambiguous as the corresponding trajectories are unique for polynomial differentials as follows from Teichmüller's lemma, see (5.1) further below.

(IV) *if $t \in C^b_{\mathsf{birth}}$ (resp. $t \in C^a_{\mathsf{birth}}$), then $J_t = \Gamma[a,b]$ and an S-curve $\Gamma_t \in \mathcal{T}$ can be chosen as in Case I(d) (resp. Case I(e)), see Figure 5(c).*

We prove Theorem 3.2 in Section 5.

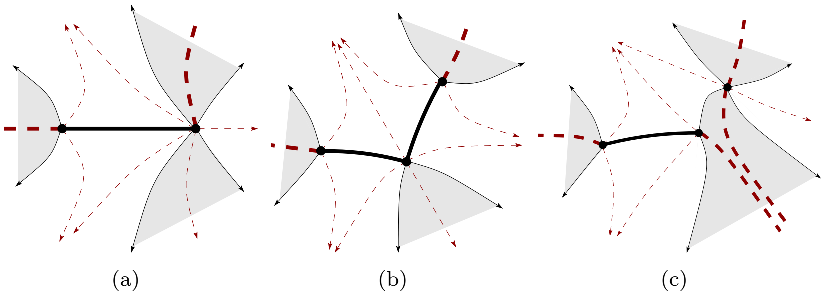

FIGURE 5. This is a continuation of Figure 4 for the case $t \in \partial O_{\mathsf{one-cut}}$.

Remark 3.3. *Even though we shall not dwell on this point here, the reason for the nomenclature introduced above is the following. When $t \notin \overline{O}_{\mathsf{one-cut}}$, the double zero $c(t)$ splits into two simple zeros, say $c_1(t)$ and $c_2(t)$. When t crosses C_{split}, the trajectory connecting a and b will split into two, one connecting a and c_1 and another connecting c_2 and b. When t crosses $C_{\mathsf{birth}} := C^a_{\mathsf{birth}} \cup C^b_{\mathsf{birth}}$, a critical trajectory connecting c_1 and c_2 will appear while a and b will remain being connected by a trajectory.*

Let us elaborate on all the configurations appearing on Figures 4 and 5. To this end,

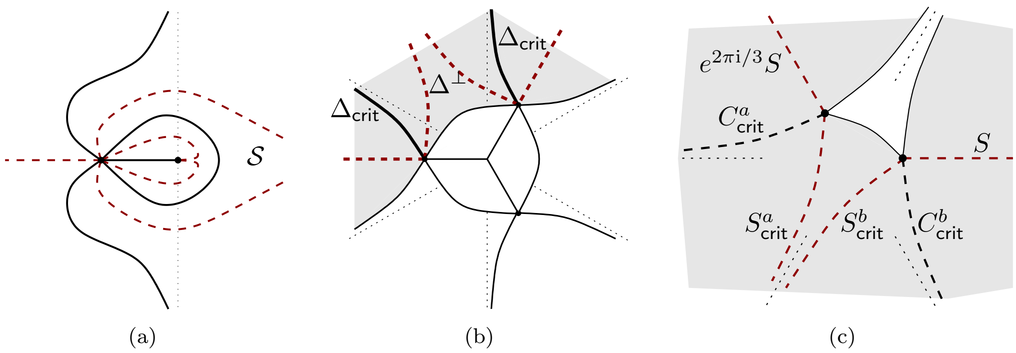

FIGURE 6. Schematic representations of (a) the set $\mathcal{S}$ (dashed lines); (b) the sets $\Delta^{\perp}$ (dashed lines) and Δ_{crit} (solid curves within the shaded region); (c) the images of $\Delta^{\perp}$ and Δ_{crit} under $t(x)$.

we need to introduce the totality of the orthogonal trajectories of the differential (3.5) emanating out of -1, say $\mathcal{S}$, see Figure 6(a), the sets

$$\Delta_{\mathsf{crit}} := \Delta \cap \Omega_{\mathsf{one-cut}}, \quad \Delta^{\perp} := \big\{x : 2x^3 \in \mathcal{S}\big\} \cap \Omega_{\mathsf{one-cut}},$$

see Figure 6(b), as well as the sets

$$t(\Delta_{\mathsf{crit}}) =: C^a_{\mathsf{crit}} \cup C^b_{\mathsf{crit}}, \quad t(\Delta^{\perp}) =: S^a_{\mathsf{crit}} \cup S^b_{\mathsf{crit}} \cup S \cup e^{2\pi\mathrm{i}/3}S,$$

where C^b_{crit}, S^b_{crit} and C^a_{crit}, S^a_{crit} are incident with t_{cr} and $e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}$, respectively, see Figure 6(c).

Theorem 3.4. *The critical and critical orthogonal graphs of $-Q(z;t)\mathrm{d}z^2$ have the structure as on*

- *Figure 5(a,b,c) when $t = t_{\mathsf{cr}}$, $t \in C_{\mathsf{split}}$, and $t \in C^b_{\mathsf{birth}}$, respectively;*
- *Figure 4(f,d,b) when $t \in S$, $t \in C^b_{\mathsf{crit}}$, and $t \in S^b_{\mathsf{crit}}$, respectively;*
- *Figure 4(g) when t belongs to the component of $O_{\mathsf{one-cut}}$ bounded by $S \cup C^b_{\mathsf{birth}}$;*

- *Figure 4(e) when t belongs to the component of $O_{\mathsf{one-cut}}$ bounded by $S \cup C^b_{\mathsf{crit}}$; in fact, C^b_{crit} is the reflection of C^b_{birth} across the real line and the structure is the same as on Figure 4(g) reflected across the real line as well;*
- *Figure 4(c) when t belongs to the component of $O_{\mathsf{one-cut}}$ bounded by $C^b_{\mathsf{crit}} \cup S^b_{\mathsf{crit}}$;*
- *Figure 4(a) when t belongs to the component of $O_{\mathsf{one-cut}}$ bounded by $S^b_{\mathsf{crit}} \cup S^a_{\mathsf{crit}}$.*

In the rest of the cases one needs to pick the reflection of t across the line $L_{\pi/3}$, where

$$L_\theta := \big\{ r e^{\mathrm{i}\theta} : \ r \in (-\infty, \infty) \big\}, \tag{3.8}$$

and then reflect the corresponding graph structures across the line $L_{2\pi/3}$. This symmetry is displayed on Figures 4(a) and 5(a).

We prove Theorem 3.4 in Section 5 as well.

4. Main Results

In what follows, we always assume that $t \in \overline{O}_{\mathsf{one-cut}}$ while $\beta_n(t,N)$ and $\gamma_n(t,N)$ are recurrence coefficients (1.27) of the polynomials $P_n(z;t,N)$ satisfying orthogonality relations (1.26) with $V(z;t)$ as in (1.24) on a contour $\Gamma = \Gamma_t$ as in Theorem 3.2.

Henceforth, we use interval notation preceded by Γ_t to denote subarcs of Γ_t. For example, $\Gamma_t(u,v]$ stands for the subarc of Γ_t connecting u and v, not containing u and containing v, and u precedes v according to the orientation of Γ_t.

Below, we prove existence of various expansions that depend on the parameter t. To indicate the way of dependence, we introduce the following two notions.

Definition 4.1. *Given an increasing sequence $\alpha(i) \to \infty$ as $i \to \infty$, we say that an expansion*

$$A_N(t) \sim \sum_{i=0}^{\infty} A^{(i)}(t) N^{-\alpha(i)}$$

holds t-locally uniformly if for any $T \subset \overline{O}_{\mathsf{one-cut}}$ such that $T \cap O_{\mathsf{one-cut}}$, $T \cap C_{\mathsf{split}}$, $T \cap C_{\mathsf{birth}}$ are compact, there exist constants $C_I(T) < \infty$ for which

$$|A_N(t) - \sum_{i=0}^{I-1} A^{(i)}(t) N^{-\alpha(i)}| \leq C_I(T) N^{-\alpha(I)}, \quad t \in T.$$

Moreover, we say that an expansion is strongly t-locally uniform if $C_I(T) < \infty$ exists as long as $T \cap C_{\mathsf{split}}$ is compact, $T \cap C_{\mathsf{birth}}$ is closed, and there exists $\epsilon(T) > 0$ for which

$$\mathsf{Re}\left(\int_{b(t)}^{c(t)} Q^{1/2}(z;t) \mathrm{d}z \right) \leq -\epsilon(T) \tag{4.1}$$

for all $t \in T \cap O_{\mathsf{one-cut}}$ large with $\mathsf{arg}(t) \in (0, \pi/2)$.

To understand (4.1) geometrically, notice that its left-hand side is equal to 0 when $t \in \partial O_{\mathsf{one-cut}} \cup C^a_{\mathsf{crit}} \cup C^b_{\mathsf{crit}}$, see Figures 3 and 6(c), is positive when t belongs to the part of $O_{\mathsf{one-cut}}$ bounded by $C^a_{\mathsf{crit}} \cup C_{\mathsf{split}} \cup C^b_{\mathsf{crit}}$, and is negative otherwise. Thus, (4.1) describes the way T is separated from C_{birth} when it extends to the point at infinity.

For functions that depend both on the parameter t and variable z, we adopt the following conventions.

Definition 4.2. *We say that the equality $f_N(z;t) = \mathcal{O}(N^{-\alpha})$ holds (z,t)-locally uniformly for $z \in V$ as $N \to \infty$ if for each T such that $T \cap O_{\mathsf{one-cut}}$, $T \cap C_{\mathsf{split}}$, $T \cap C_{\mathsf{birth}}$ are compact, and any collection of compact sets $\{K_t\}_{t \in T}$ such that $K_t \subset V$, there exists $C(V;T) < \infty$ for which*

$$\big| f_N(z;t) \big| \leq C(V;T) N^{-\alpha}, \quad z \in K_t, \quad t \in T,$$

for all N large. Similarly, the notion of a property holding strongly (z,t)-locally uniformly for $z \in V$ should be straightforward in the view of Definition 4.1.

4.1. Asymptotic expansion of $F_N(t)$.

Theorem 4.3. *Let $F_N(t)$ be the free energy (1.29). Then it holds uniformly on subsets of $O_{\mathsf{one-cut}}$ satisfying (4.1) that*

$$F_N(t) \sim \sum_{k=0}^{\infty} F^{(2k)}(t) N^{-2k}, \tag{4.2}$$

where the functions $F^{(2k)}(t)$ are holomorphic in $O_{\mathsf{one-cut}}$ and extend holomorphically across each of the arcs C_{birth}, C^a_{split}, and C^b_{split}. In particular, it holds that

$$F^{(0)}(t) = 1 - \frac{2}{3}x^3(t) - \frac{1}{2}\log\left(-2x(t)\right) + \int_\infty^t \int_\infty^\tau \left(-\frac{1}{2x(\sigma)} + \frac{7x'(\sigma) + 2\sigma x''(\sigma)}{6}\right) \mathrm{d}\sigma \mathrm{d}\tau, \tag{4.3}$$

where $x'(t)$ is the derivative of $x(t)$ with respect to t and the integrals can be computed along any path in $O_{\mathsf{one-cut}}$.

Functions $F^{(2k)}(t)$ encode information on the number of certain graphs on a Riemann surface of genus k.

Remark 4.4. *If we relabel the functions $F^{(2g)}$ in (1.5) by $\widehat{F}^{(2g)}$, then it holds that*

$$\begin{cases} \widehat{F}^{(0)}(y) & := \frac{2}{3}t^{3/2} - \frac{1}{4}\log(4t) - F^{(0)}(t), \\ \widehat{F}^{(2g)}(y) & := F^{(2g)}(t), \quad g \geq 1, \end{cases} \qquad y^{-1} = 3(4t)^{3/4}.$$

We prove Theorem 4.3 in Section 8 using Toda equations (1.29) and the asymptotic expansion of the recurrence coefficients.

4.2. Asymptotic expansion of $\gamma_n^2(t;N)$ and $\beta_n(t;N)$.

Theorem 4.5. *Let $x(t)$ be as in Proposition 3.1. Assuming that $|n-N| \leq N_0$ for some absolute constant N_0, it holds that*

$$\begin{cases} \gamma_n^2(t,N) & \sim & -\dfrac{1}{2x(t)} + \displaystyle\sum_{k=1}^{\infty} G_{\alpha_t k}(t; n-N) N^{-\alpha_t k}, \\ \beta_n(t,N) & \sim & x(t) + \displaystyle\sum_{k=1}^{\infty} B_{\alpha_t k}(t; n-N) N^{-\alpha_t k}, \end{cases} \tag{4.4}$$

for some explicitly computable $G_{\alpha_t k}$ and $B_{\alpha_t k}$, where the expansions are t-locally uniform and

$$\alpha_t = 1, \quad \alpha_t = 1/2, \quad \text{and} \quad \alpha_t = 1/5 \tag{4.5}$$

for $t \in O_{\mathsf{one-cut}}$, $t \in C_{\mathsf{birth}} \cup C_{\mathsf{split}}$, and $t \in \left\{t_{\mathsf{cr}}, e^{2\pi i/3}t_{\mathsf{cr}}\right\}$, respectively. Moreover, the expansion of $\gamma_N^2(t,N)$ is strongly t-locally uniform. The functions $G_k(t;n-N)$ and $B_k(t;n-N)$ are holomorphic in $O_{\mathsf{one-cut}}$, can be holomorphically continued across each of the arcs C_{split}, C^a_{birth}, and C^b_{birth}, and

$$\begin{cases} G_{(2j-1)/5}(t;0) = 0, & t \in \left\{t_{\mathsf{cr}}, e^{2\pi \mathrm{i}/3}t_{\mathsf{cr}}\right\}, \\ G_{2j-1}(t;0) \equiv 0, & t \in O_{\mathsf{one-cut}}, \end{cases} \qquad j \in \mathbb{N}. \tag{4.6}$$

Using (4.4) we can deduce certain analyticity properties of $\gamma_N^2(t,N)$.

Remark 4.6. *The partition function $Z_N(t)$ is an entire function of the parameter t. Hence, it follows from Toda equation (1.29) that $\gamma_N^2(t,N)$ is a meromorphic function of t. Moreover,*

$$\gamma_N^2(t,N) = -\frac{1}{2x(t)} + \mathcal{O}(N^{-1})$$

strongly t-locally uniformly in $O_{\mathsf{one-cut}}$ by (4.4). Hence, for any closed set $T \subset O_{\mathsf{one-cut}}$ satisfying (4.1), there exists an integer $N(T)$ such that $\gamma_N^2(t,N)$ is holomorphic on T for all $N \geq N(T)$, i.e., on some neighborhood of T that belongs to $O_{\mathsf{one-cut}}$.

Observe that the recurrence coefficients for different parameters N are related. Indeed, given N_1 and N_2, it holds that

$$\left\{\begin{array}{lcl} P_n(z;t_1,N_1) & = & (N_2/N_1)^{n/3}P_n(w;t_2,N_2), \\ \gamma_n^2(t_1,N_1) & = & (N_2/N_1)^{2/3}\gamma_n^2(t_2,N_2), \\ \beta_n(t_1,N_1) & = & (N_2/N_1)^{1/3}\beta_n(t_2,N_2), \end{array}\right. \tag{4.7}$$

where $w := (N_1/N_2)^{1/3}z$ and $t_2 := (N_1/N_2)^{2/3}t_1$, as follows from (1.26) and (1.27). Hence, given N and $t \in O_{\mathsf{one-cut}}$, asymptotic expansions (4.4) can be expanded to a larger range of indices n than those covered by Theorem 4.5.

Remark 4.7. *Put $\mathcal{N} := \big\{(t,u):\ t\in O_{\mathsf{one-cut}},\ u\in\mathcal{N}_t\big\}$, where $\mathcal{N}_t$ is the largest open subset of $\mathbb{C}\setminus(-\infty,0]$ such that $ut\in O_{\mathsf{one-cut}}$ for all $u\in\mathcal{N}_t$ (clearly, $1\in\mathcal{N}_t$). Define*

$$\left\{\begin{array}{lcl} \widehat G_{2k}(t,u) & := & u^{3k-1}G_{2k}(ut;0), \\ \widehat B_k(t,u) & := & u^{(3k-1)/2}B_k(ut;0), \end{array}\right. \qquad t\in O_{\mathsf{one-cut}} \text{ and } u\in\mathcal{N}_t,$$

where we take the principal root of $u^{(3k-1)/2}$, which are holomorphic functions in $\mathcal{N}$. Then, it follows from (4.4), (4.6), and (4.7), applied with $N_1=N$ and $N_2=n$, that

$$\left\{\begin{array}{lcl} \gamma_n^2(t,N) & \sim & \dfrac{-1}{2ux(ut)} + \displaystyle\sum_{k=1}^\infty \widehat G_{2k}(t,u)N^{-2k}, \\ \beta_n(t,N) & \sim & \dfrac{x(ut)}{\sqrt{u}} + \displaystyle\sum_{k=1}^\infty \widehat B_k(t,u)N^{-k}, \end{array}\right. \qquad u=\left(\frac{n}{N}\right)^{-2/3}, \tag{4.8}$$

whenever $(n/N)^{-2/3}\in\mathcal{N}_t$, where the expansions are locally uniform in t and u.

In fact, following [8, Section 5], we can improve on the expansion of $\beta_n(t,N)$.

Theorem 4.8. *There exist holomorphic in $\mathcal{N}$ functions $\widetilde B_{2k}(t,v)$ such that*

$$\beta_n(t,N) \sim \sum_{k=0}^\infty \widetilde B_{2k}(t,v)N^{-2k}, \quad v=\left(\frac{n+1/2}{N}\right)^{-2/3}, \tag{4.9}$$

whenever $((n+1/2)/N)^{-2/3}\in\mathcal{N}_t$, where the expansion is locally uniform in t and v.

Theorems 4.5 and 4.8 are proven in Section 8.

4.3. **Strong asymptotics of $P_n(z;t,N)$.** To describe the asymptotics of the orthogonal polynomials themselves, we need to introduce complexified equilibrium potential

$$g(z;t) := \int \log(z-s)\mathrm{d}\mu_t(s), \quad z\in\mathbb{C}\setminus\Gamma_t\big(e^{\pi\mathrm{i}}\infty,b\big], \tag{4.10}$$

where we take the principal branch of $\log(\cdot-s)$ holomorphic outside of $\Gamma_t\big(e^{\pi\mathrm{i}}\infty,s\big]$ and μ_t is the equilibrium measure defined in (2.6). Since μ_t is a probability measure, it holds that

$$e^{g(z;t)} = z+\mathcal{O}(1) \quad \text{as} \quad z\to\infty \tag{4.11}$$

and this function is holomorphic in $\mathbb{C}\setminus J_t$. In fact, the function e^g can be written explicitly.

Proposition 4.9. *Let, as before, $a(t)$ and $b(t)$ be the endpoints of J_t, see (3.7). In what follows, we set*

$$\sqrt{(z-a(t))(z-b(t))}\sim z \quad \text{as} \quad z\to\infty$$

to be the square root with the branch cut along J_t. Then the function

$$D(z;t) := \exp\left\{\left(3V(z;t)-2x^3(t)+\big(z^2+zx(t)-2t\big)\sqrt{(z-a(t))(z-b(t))}\right)/6\right\} \tag{4.12}$$

is holomorphic in $\overline{\mathbb{C}}\setminus J_t$. Moreover, it is non-vanishing there, $D(\infty;t)=1$, and it has continuous traces on J_t that satisfy

$$D_+(s;t)D_-(s;t) = \exp\left\{V(s;t)-2x^3(t)/3\right\}, \quad s\in J_t. \tag{4.13}$$

That is, $D(z;t)$ is the Szegő function of $V(\cdot;t)_{|J_t}$ normalized to have value 1 at infinity. Furthermore, it holds that

$$e^{g(z;t)} = \frac{D(z;t)}{\sqrt{2x(t)}}\frac{A(z;t)}{B(z;t)}, \tag{4.14}$$

where the functions A and B are defined by

$$\left\{\begin{array}{lcl} A(z;t) & := & \dfrac{1}{2}\left(\left(\dfrac{z-b(t)}{z-a(t)}\right)^{1/4} + \left(\dfrac{z-a(t)}{z-b(t)}\right)^{1/4}\right), \\ B(z;t) & := & \dfrac{\mathrm{i}}{2}\left(\left(\dfrac{z-b(t)}{z-a(t)}\right)^{1/4} - \left(\dfrac{z-a(t)}{z-b(t)}\right)^{1/4}\right), \end{array}\right. \tag{4.15}$$

and the branches of the $1/4$-roots are principal and have the branch cuts along J_t; in particular, $A(\infty;t)=1$ and $B(\infty;t)=0$. The function $F(z;t) := -\mathrm{i}A(z;t)/B(z;t)$ can be equivalently written as

$$F(z;t) = \frac{2}{b(t)-a(t)}\left(z - \frac{b(t)+a(t)}{2} + \sqrt{(z-a(t))(z-b(t))}\right) \tag{4.16}$$

and is holomorphic and non-vanishing in $\mathbb{C}\setminus J_t$, has a simple pole at infinity, and its traces on J_t multiply to 1.

We prove Proposition 4.9 in Section 6.

Theorem 4.10. *Let α_t be as in Theorem 4.3 and $|N-n|\leq N_0$ for some fixed constant N_0. Then*

$$P_n(z;t,N) = \big(1+\mathcal{O}\big(N^{-\alpha_t}\big)\big)\,A(z;t)D^{N-n}(z;t)e^{ng(z;t)}, \tag{4.17}$$

(z,t)-locally uniformly for $z\in\mathbb{C}\setminus J_t$ (or $z\in\mathbb{C}\setminus\big(J_t\cup\{c\}\big)$ when $t\in C_{\mathsf{birth}}$), where $\mathcal{O}$-term vanishes at $z=\infty$. In particular, $\mathsf{deg}(P_n(\cdot;t,N))=n$ for all N large. Moreover,

$$\begin{aligned} P_n(s;t,N) = {} & \big(1+\mathcal{O}\big(N^{-\alpha_t}\big)\big)\,A_+(s;t)D_+^{N-n}(s;t)e^{ng_+(s;t)}+ \\ & + \big(1+\mathcal{O}\big(N^{-\alpha_t}\big)\big)\,A_-(s;t)D_-^{N-n}(s;t)e^{ng_-(s;t)}, \end{aligned} \tag{4.18}$$

(s,t)-locally uniformly for $s\in\Gamma_t(a,b)$ (or $s\in\Gamma_t(a,c)\cup\Gamma_t(c,b)$ when $t\in C_{\mathsf{split}}$). When $n=N$, $\mathcal{O}$-terms in (4.17) and (4.18) are strongly (z,t)-locally uniform.

Theorem 4.10 is proven in Section 7. Combining Theorem 4.10 with observation (4.7), we obtain the following corollary.

Corollary 4.11. *Given $t\in O_{\mathsf{one-cut}}$, assume that $u:=\lim_{N\to\infty}(n/N)^{-2/3}$ exists and $ut\in O_{\mathsf{one-cut}}$. Then it holds locally uniformly in $\mathbb{C}\setminus J_{ut}$ that*

$$u^{n/2}P_n\big(z/\sqrt{u};t,N\big) = \big(1+o(1)\big)A(z;ut)e^{ng(z;ut)}.$$

5. S-curves

For brevity, we set $\varpi_t := -Q(z;t)\mathrm{d}z^2$.

5.1. **Critical graphs: local structure.** The differential ϖ_t has two critical points of order 1, namely a,b, a critical point of order 2, namely c, (unless c coincides with either a or b in which case ϖ_t has critical points of orders 1 and 3), and a critical point of order -8 at infinity. All other points are regular with respect to ϖ_t (order 0).

Through each regular point of ϖ_t passes exactly one trajectory and one orthogonal trajectory, which are orthogonal to each other at the point. Two distinct (orthogonal) trajectories meet only at critical points [38, Theorem 5.5].

As $Q(z;t)$ is a polynomial, no finite union of (orthogonal) trajectories can form a closed Jordan curve while a trajectory and an orthogonal trajectory can intersect at most once [33, Lemmas 8.3]. Furthermore, (orthogonal) trajectories of ϖ_t cannot be recurrent (dense in two-dimensional regions) [28, Theorem 3.6].

If $z_0\in\{a,b,c\}$ has order m, there are $m+2$ critical trajectories emanating from z_0 at angles

$$\big((2k+1)\pi - \mathsf{arg}Q^{(m)}(z_0;t)\big)/(m+2), \quad k\in\{0,\dots,m+1\},$$

see [38, Theorem 7.1]. Thus, there are 3 critical trajectories of ϖ_t emanating from a, 3 critical trajectories emanating from b, and 4 emanating from c (under the condition $c \neq a, b$). Since the point at infinity is a pole of order 8, there are 6 distinguished directions, namely,

$$\pi/6 + k\pi/3, \quad k \in \{0, \dots, 5\},$$

in which the trajectories can approach it. Moreover, there is a neighborhood of infinity such that every trajectory entering this neighborhood necessarily tends to infinity, [38, Theorem 7.4]. The above discussion applies to the orthogonal trajectories as well. In particular, they can approach infinity only at the angles $k\pi/3$, $k \in \{0, \dots, 5\}$.

A geodesic polygon with respect to ϖ_t is a Jordan curve in $\overline{\mathbb{C}}$ that consists of a finite number of trajectories and orthogonal trajectories of ϖ_t. According to Teichmüller's lemma [38, Theorem 14.1], it holds that

$$\sum_{z \in P} \left(1 - \theta(z)\frac{\mathsf{ord}(z) + 2}{2\pi}\right) = 2 + \sum_{z \in \mathsf{int}(P)} \mathsf{ord}(z), \tag{5.1}$$

where P is a geodesic polygon, $\mathsf{ord}(z)$ is order of z with respect to ϖ_t, and $\theta(z) \in [0, 2\pi]$, $z \in P$, is the interior angle of P at z. Clearly, both sums in (5.1) are finite as only critical points of ϖ_t and vertices of the polygon have a non-zero contribution.

To simplify the forthcoming discussion, let us observe that the differential ϖ_t possesses several symmetries. Firstly, notice that when t belongs to the subregion of $O_{\mathsf{one-cut}}$ bounded by C^b_{birth} and C^b_{crit}, see Figures 3 and 6(c),we have

$$x(\bar t) = \overline{x(t)} \quad \Rightarrow \quad \varpi_t(z) = \overline{\varpi_{\bar t}(\bar z)}. \tag{5.2}$$

That is, for such t, the critical (orthogonal) graph coincides with the reflection across the real axis of the critical (orthogonal) graph for $\bar t$. Secondly, it holds that

$$x\big(\bar t e^{2\pi \mathrm{i}/3}\big) = \overline{x(t)}e^{4\pi \mathrm{i}/3} \quad \Rightarrow \quad \varpi_t(z) = \overline{\varpi_{\bar t e^{2\pi\mathrm{i}/3}}\big(\bar z e^{4\pi \mathrm{i}/3}\big)}. \tag{5.3}$$

That is, the critical (orthogonal) graph for t coincides with the reflection across the line $L_{2\pi/3}$, see (3.8), of the critical (orthogonal) graph for $\bar t e^{2\pi\mathrm{i}/3}$ (which is the reflection of t across the line $L_{\pi/3}$). Symmetries (5.2) and (5.3) yield that we need to concern ourselves only with the case

$$x \in \overline{\Omega}_{\mathsf{one-cut}} \quad \text{and} \quad 2\pi/3 \le \mathsf{arg}(x) \le \pi. \tag{5.4}$$

Notice also that (5.2) and (5.3) are precisely the symmetries described in Theorem 3.4.

5.2. **Critical graphs via level lines.** To continue, it will be convenient to observe the following. Let $R(z) := \sqrt{(z-a)(z-b)}$ be the branch holomorphic outside of some arc, say γ_{ab}, joining a and b and such that $R(z) = z + \mathcal{O}(1)$ as $z \to \infty$. Expressing a, b, and c through x via (3.7), we have

$$(z + x)R(z) = 2Q^{1/2}(z;t) = z^2 + \frac{1}{x} - x^2 + \frac{2}{z} + \mathcal{O}\big(z^{-2}\big) \tag{5.5}$$

as $z \to \infty$. Therefore, the function

$$\begin{aligned} I_x(z) \quad &:= \quad \int_b^z (s+x)R(s)\mathrm{d}s = 2\int_b^z Q^{1/2}(s;t)\mathrm{d}s \\ &= \quad \frac{1}{3}R^3(z) + x(z-x)R(z) + \log\left(\frac{z - x + R(z)}{z - x - R(z)}\right) \end{aligned} \tag{5.6}$$

is defined up to an addition of an integer multiple of $4\pi\mathrm{i}$ (depending on the path of integration) and is analytic (multi-valued) in $\mathbb{C} \setminus \gamma_{ab}$. From the previous subsection we know that there are 3 trajectories emanating from a and the three from b. As there are only three finite critical points, there always exists at least one trajectory out of a and at least one trajectory out of b that extends to infinity. Pick one such trajectory for a, say γ_a. Then $I_x(z)$ is a well-defined holomorphic function in $\mathbb{C} \setminus (\gamma_a \cup \gamma_{ab})$. Write,

$$U_x(z) := \mathsf{Re}\,(I_x(z)) \quad \text{and} \quad V_x(z) := \mathsf{Im}\,(I_x(z)).$$

Then we can see from (5.6) that $U_x(z)$ is a harmonic function in $\mathbb{C} \setminus \gamma_{ab}$ while $V_x(z)$ can be defined harmonically in $\mathbb{C} \setminus (\gamma_a \cup \gamma_{ab})$. Since $U_x(a) = U_x(b) = 0$, the zero level set of U_x contains the trajectories emanating from both a and b and is independent of the choice of

γ_{ab} (the analytic continuation of $I_x(z)$ across γ_{ab} is given by $-I_x(z)$ that preserves the zero level set of U_x). Similarly, the orthogonal trajectories out of b are part of the zero level set of the selected branch of V_x while the orthogonal trajectories out of a are part of 2π and -2π-level sets of V_x.

It is obvious from their definition that the harmonic functions U_x continuously depend on the parameter x. Hence, their corresponding level sets converge to each other in Hausdorff metric on any compact subset of $\mathbb{C}$ (to see this around γ_{ab}, recall that U_x can always be harmonically continued across γ_{ab}). Moreover, if we subtract from U_x the real part of the polynomial part of the first two terms in (5.6) and $\log|z|$, the obtained function will be harmonic at infinity and will continuously depend on x. Therefore, we can control the behavior of the level sets of U_x not only on compact subsets of $\mathbb{C}$ but around the point at infinity as well. Thus, if for some fixed x_0 all four critical trajectories out of $c=-x_0$ approach infinity, then the critical trajectories out of $c=-x$ will approach infinity in the same directions for all x in a small neighborhood of x_0. Hence, if x belongs to an open connected set on which $U_x(-x)\neq 0$ (this necessarily implies that trajectories out of c cannot end at a or b), then the trajectories out c approach infinity in the same directions for each x on this set. Similar considerations hold for V_x as well.

5.3. **Critical graphs: transitions.** It follows from the previous subsection the structure of the critical (orthogonal) graph can change only when $U_x(c)=0$ ($V_x(c)=0,\pm 2\pi$). Let us identify for which x these harmonic functions vanish at c. From the choice of the branch of the square root we have that $R(-x)=-2x\sqrt{1+1/2x^3}$, where the root is equal to 1 when $x=\infty$. Moreover, we see from (5.4) that we are interested only in the values $\mathsf{Im}\,(x^3)\geq 0$. Hence,

$$
\begin{aligned}
I_x(-x) &= -\frac{8x^3}{3}\left(1+\frac{1}{2x^3}\right)^{3/2}+4x^3\left(1+\frac{1}{2x^3}\right)^{1/2}+\log\left(\frac{1+\sqrt{1+1/2x^3}}{1-\sqrt{1+1/2x^3}}\right)\\
&= \frac{2}{3}\int_{-1}^{2x^3}\left(1+\frac{1}{s}\right)^{3/2}\mathrm{d}s, \qquad (5.7)
\end{aligned}
$$

where the path of integration lies in the upper half-plane. That is, we need to understand the integral (5.7) of the quadratic differential (3.5) in the upper half plane. From the general principles, we see that $U_x(-x)=0$ ($V_x(-x)=0$) if and only if $2x^3$ belongs to a (orthogonal) trajectory emanating from -1.

Differential (3.5) has a zero of order 3 at -1, a pole of order 3 at the origin, and a pole of order 4 at infinity. Thus, there are 5 trajectories emanating from -1, one of which is clearly $(-1,0)$. There is one distinguished approach of the origin, which is necessarily along the negative real axis since $(-1,0)$ is a trajectory. There are two distinguished directions at infinity, which are along the imaginary axis. Moreover, according to the three pole theorem [28, Theorem 3.6], this differential does not have any recurrent trajectories. The last fact implies that the four trajectories out of -1 (excluding $(-1,0)$) either approach infinity or form loops. Going through the possible cases and using Teichmüller's lemma, we see that the trajectories emanating from -1 at the angles $\pm 2\pi/5$ form a loop[2], and the other two approach infinity (it is a simple calculus exercise to see that they cannot touch the real line). Hence, Figure 2(a) is indeed correct.

On the other hand, the local structure of the critical orthogonal trajectories near critical points must be the same. It is obvious that $(-\infty,-1)$ and $(0,\infty)$ are critical orthogonal trajectories. Thus, by repeating the same analysis, we get that the critical orthogonal graph is as on Figure 6(a) with the orthogonal trajectory $(0,\infty)$ not displayed.

Combining the above analysis with explicit computations and (5.7), we see that the values of $U_x(c)$ and $V_x(c)$ are as displayed on Figure 7.

5.4. **Critical graph: global structure.** Let Δ_{split} and $\Delta^e_{\mathsf{birth}}$ be as defined before (3.6), while Δ_{crit} as defined before Theorem 3.4. Write $\Delta_{\mathsf{crit}}=\Delta^a_{\mathsf{crit}}\cup\Delta^b_{\mathsf{crit}}$, where Δ^e_{crit} and $\Delta^e_{\mathsf{birth}}$ are incident with the same point, see Figure 8. Then it follows from the preceding subsection

[2] The loop crosses the real line approximately at 0.6349131623.

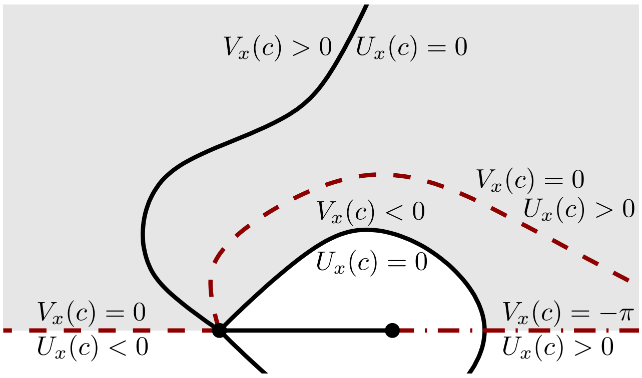

FIGURE 7. The values $U_x(c)$ and $V_x(c)$ $(c=-x)$ when $2x^3$ belongs to the displayed trajectories and orthogonal trajectories of (3.5).

that

$$U_x(c)=0 \quad \Leftrightarrow \quad x\in\Delta^a_{\mathsf{birth}}\cup\Delta^b_{\mathsf{birth}}\cup\Delta^a_{\mathsf{crit}}\cup\Delta^b_{\mathsf{crit}}\cup\big\{-2^{1/3}\big\}\cup\big\{2^{1/3}e^{\pi\mathrm{i}/3}\big\}.$$

Denote by Ω_e the subdomain of $\Omega_{\mathsf{one-cut}}$ bounded by $\Delta^e_{\mathsf{birth}}$ and Δ^e_{crit}, $e\in\{a,b\}$, and by Ω_{ab} the subdomain bounded by Δ^a_{crit}, Δ^b_{crit}, and Δ_{split}, see Figure 8. Recall that we only need to study the cases when x satisfies (5.4).

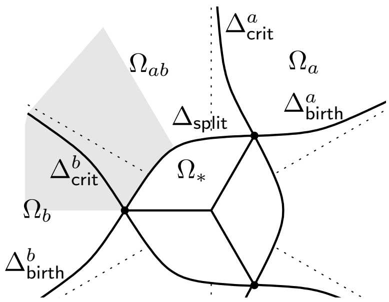

FIGURE 8. Domains Ω_*, Ω_a, Ω_b, Ω_{ab} and their boundaries; the shaded region is the one from (5.4).

Let $x\in\Omega_{ab}$. From what precedes we know that the trajectories out of c approach infinity in the same four directions. The case $x\in L_{2\pi/3}$, see (3.8), has been worked out in [27] (to obtain the setting of [27] one needs to perform the transformations $z\mapsto e^{\pi\mathrm{i}/6}z$ and $t\mapsto e^{\pi\mathrm{i}/3}K$ in (1.24)). It was shown that there exists r^* such that for $x=re^{2\pi/3}$, $r<r^*$, there are no critical trajectory of ϖ_t connecting a and b and for $r>r^*$ such a trajectory exists and the critical (orthogonal) graph is as on Figure 4(a). Thus, the trajectories out of c approach infinity at the angles

$$7\pi/6+k\pi/3, \quad k\in\{0,1,2,3\},$$

for each $x\in\Omega_{ab}$. Now, if there always exists a trajectory connecting a and b, the other two trajectories out of b must approach infinity at the angles $\pi/6$ and $\pi/2$ and the trajectories out of a must approach infinity at the angles $5\pi/6$ and $7\pi/6$ by Teichmüller's lemma (5.1), which would finish the description of the critical graph in this case.

Assume to the contrary that such a trajectory does not exist. It follows from Teichmüller's lemma that both a and b must belong to the sector of opening π at infinity delimited by the trajectories out of c. In this case two trajectories out one of the points a,b will approach infinity in the directions $5\pi/6$ and $7\pi/6$, forming a sector say X_a, and two trajectories of the other point will approach infinity in the directions $\pi/6$ and $\pi/2$, forming a sector say X_b, see Figure 9. We further can choose the arc γ_{ab} outside of $X_a\cup X_b$, that is, belonging to geodesic polygon with four corners a,b,∞,∞ and respective angles $2\pi/3,2\pi/3,0,0$, see the dashed arc on Figure 9. Denote by X the region that does not contain c and is bounded by γ_{ab} and a part of $\partial X_a\cup\partial X_b$. As trajectories cannot intersect, X contains one trajectory arc, say T.

Recall that around each simple zero of ϖ_t, the differential can be written as $(3/2)^2\zeta\mathrm{d}\zeta^2$ for some local parameter ζ, [38, Theorem 6.1]. This means that $U_x(z)$ has constant sign locally in X_a and X_b, but it also is continuous and cannot vanish there. Hence, it has a constant sign in each of these sectors. As we can always choose a branch of the logarithm in (5.6) so that $I_x(z)$ is holomorphic in the closure $X_a \cup X \cup X_b$, we have $I_x(z) = z^3(1/3 + o(1))$ as $z \to \infty$ uniformly in $X_a \cup X \cup X_b$. Thus, $U_x(z)$ has the same sign in X_a and X_b. The same local structure and continuity yield that $U_x(z)$ has the opposite sign in $X \setminus T$ and is zero on T by construction. As $U_x(z)$ is harmonic in X, the latter contradicts the maximum principle and therefore our assumption is false. Therefore, when $x \in \Omega_{ab}$ the critical graph of ϖ_t has indeed the structure as on Figure 4(a,b,c).

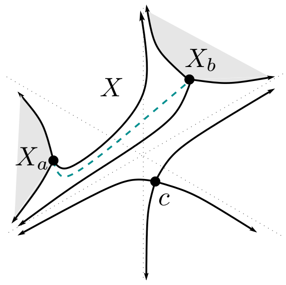

FIGURE 9. The dashed arc is γ_{ab}, the shaded regions are part of the open set $\{z : \; U_x(z) < 0\}$.

Let $x \in \Omega_b$, which is also connected. The case $x \in L_\pi \cap \overline{\Omega}_{\mathsf{one-cut}}$ has been investigated in [6]. It was shown that the critical and critical orthogonal graphs are as on Figure 4(f) when $x \in L_\pi \cap \Omega_b$ and Figure 5(a) when $x = -2^{1/3}$. This fixes the behavior of the trajectories out of c. Arguing as in the previous paragraph, we get that a and b must be connected by a trajectory and therefore the behavior of the whole critical graph is fixed, see Figure 4(e,f,g).

Let now $x \in \Delta^b_{\mathsf{crit}}$. Continuity with respect to parameter implies that the structure of the critical graph should be obtained through the limiting process from within both Ω_{ab} and Ω_b, which necessarily yields that it must be as on Figure 4(d).

Finally, let $x \in \Delta_{\mathsf{split}}$. Denote by Ω_* a domain whose boundary contains Δ_{split} that has empty intersection with $\Omega_{\mathsf{one-cut}}$ and Δ (defined after (3.4)), see Figure 8. It was shown in [27] that the critical graph and critical orthogonal graphs for $x \in L_{2\pi/3} \cap \Omega_*$ are as in Figure 10. Hence, the trajectories out of c approach infinity in the directions $-\pi/6$, $\pi/2$,

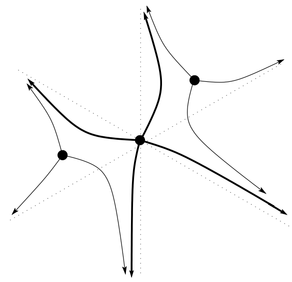

FIGURE 10. The critical graph of ϖ_t when t crosses Δ_{split}.

$5\pi/6$, and $3\pi/2$. It follows from Teichmüller's lemma (5.1) that the points a and b must be separated by the trajectories out of c. Hence, there are no trajectory joining a and b. Continuity with respect to the parameter immediately yields that the critical graph of ϖ_t is as on Figure 5(b) when $x \in \Delta_{\mathsf{split}}$.

5.5. **Critical orthogonal graph: global structure.** In what follows we shall refer to the *key observation*: given an unbounded domain whose boundary consists of critical trajectories that are consecutive at each point of intersection (such intersections have zero contribution to the left-hand side of (5.1) and the two trajectories extending to infinity necessarily form an angle of magnitude $\pi/3$ there), (5.1) implies that any orthogonal trajectory entering this domain will remain inside. Recall further, that orthogonal trajectories cannot intersect.

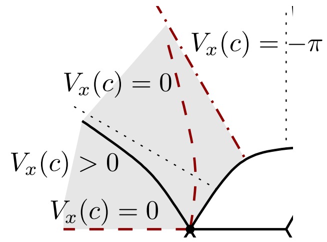

FIGURE 11. The dashed lines are those where $V_x(c) = 0$, solid lines are those where $U_x(c) = 0$, and $V_x(c) = -\pi$ on the dashed-dotted line. The shaded region is the one from (5.4).

Let $x \in L_{2\pi/3} \cap \overline{\Omega}_{\mathsf{one-cut}}$. In this case the graphs must be symmetric with respect to the line $L_{2\pi/3}$ by (5.3). This symmetry, the global structure of the critical graph, and the key observation yield that the orthogonal critical graph is as on Figure 4(a) or Figure 5(b).

When $x \in \Delta^b_{\mathsf{crit}}$, the global structure of the critical graph and key observation along fix the critical orthogonal graph to be as on Figure 4(d), see also Figure 4(c,e).

Consider x in the region bounded by $L_{2\pi/3}$ and Δ^b_{crit}. The critical graph is always the same, see Figure 4(a,b,c). The key observation fixes three orthogonal trajectories out of c except for the one, say T_x, that becomes $\Gamma(c, e^{2\pi\mathrm{i}/3}\infty)$ when $x \in L_{2\pi/3} \cap \overline{\Omega}_{\mathsf{one-cut}}$, see Figure 4(a), and becomes $\Gamma(c, e^{\pi\mathrm{i}/3}\infty)$ when $x \in \Delta^b_{\mathsf{crit}}$, see Figure 4(d). Notice also that fixing T_x fixes the entire critical orthogonal graph as orthogonal trajectories cannot intersect. Observe also that besides becoming a short orthogonal trajectory $\Gamma(c, b)$, those are the only options for T_x. Indeed, it could have happened that $T_x = \Gamma(c, e^{\pi\mathrm{i}}\infty)$, but then it would necessarily hold that $T_x = \Gamma(c, a)$ for some x by continuity. In this case we would have $V_x(c) = -2\pi$, which is impossible as $V_x(c) \geq -\pi$ in the considered region, see Figures 7 and 11. Thus, the critical graph is as on Figure 4(a,b,c) in the considered region. Since $V_x(c) < 0$ when $T_x = \Gamma(c, e^{2\pi\mathrm{i}/3}\infty)$, $V_x(c) = 0$ when $T_x = \Gamma(c, b)$, and $V_x(c) > 0$ when $T_x = \Gamma(c, e^{\pi\mathrm{i}/3}\infty)$, the corresponding claims of Theorem 3.4 follow.

When $x \in L_\pi \cap \overline{\Omega}_{\mathsf{one-cut}}$, we know that the critical orthogonal graph must be symmetric with respect to the real axis by (5.2). This symmetry, the global structure of the critical graph, and the key observation imply that the critical orthogonal graph must be as on Figure 4(f) or Figure 5(a).

Finally, let x belong to the region bounded by $L_\pi \cap \overline{\Omega}_{\mathsf{one-cut}}$ and Δ^b_{crit}. Since the critical graph is always the same, the critical orthogonal graph can be only as on Figure 4(e,f,g). Continuity considerations similar to the ones above imply that it is as on Figure 4(e) for the considered x. This finishes the proof of Theorem 3.4 and therefore of Theorem 3.2.

6. g-FUNCTION

In this section we discuss properties of $g(z;t)$ defined in (4.10). We consider the parameter $t \in \overline{O}_{\mathsf{one-cut}}$ to be fixed and stop indicating the dependence on t of the various quantities appearing below whenever this does not introduce ambiguity.

6.1. **Global properties.** It follows directly from definition (4.10) that

$$\partial_z g(z;t) = \int \frac{\mathrm{d}\mu_t(s)}{z-s},$$

where $\partial_z := (\partial_x - \mathrm{i}\partial_y)/2$. Therefore, we can deduce from (2.2) and (2.5) that

$$g(z;t) = \frac{V(z;t) - \ell^*_t}{2} + \int_b^z Q^{1/2}(s;t)\mathrm{d}s, \tag{6.1}$$

where, as usual, we take the branch $Q^{1/2}(z;t) = \frac{1}{2}z^2 + \mathcal{O}(z)$, and ℓ^*_t is a constant such that $\mathsf{Re}\,(\ell^*_t) = \ell_t$ (the explicit expression for ℓ^*_t can be obtained from (5.6) and the fact that

$g(z;t) = \log z + \mathcal{O}(z^{-1})$ as $z \to \infty$). In the view of (6.1), define

$$\phi_e(z) := 2\int_e^z Q^{1/2}(s;t)\mathrm{d}s, \quad e \in \{a,b\}, \tag{6.2}$$

holomorphically in $\mathbb{C}\setminus\Gamma_t\big[b, e^{\pi\mathrm{i}/3}\infty\big)$ when $e=b$, and in $\mathbb{C}\setminus\Gamma_t\big(e^{\pi\mathrm{i}}\infty, a\big]$ when $e=a$ (observe that (5.6) provides an explicit formula for these functions). It follows from (2.6) and (6.2) that

$$\begin{cases} \phi_b(z) = \phi_a(z) \pm 2\pi\mathrm{i}, & z \in \mathbb{C}\setminus\Gamma_t, \\ \phi_{b\pm}(s) = \pm 2\pi\mathrm{i}\mu_t\big(\Gamma_t[s,b]\big), & s \in \Gamma_t(a,b), \end{cases} \tag{6.3}$$

where, in the first relation, the plus sign is used if z lies to the left of Γ_t and the minus sign if z lies to the right of Γ_t. By combining (6.1) and (6.3) we get that

$$g_+(s;t) - g_-(s;t) = \begin{cases} 0, & s \in \Gamma_t\big(b, e^{\pi\mathrm{i}/3}\infty\big), \\ \pm\phi_{b\pm}(s), & s \in \Gamma_t(a,b), \\ 2\pi\mathrm{i}, & s \in \Gamma_t\big(e^{\pi\mathrm{i}}\infty, a\big), \end{cases} \tag{6.4}$$

and that

$$g_+(s;t) + g_-(s;t) - V(s;t) + \ell_t^* = \begin{cases} \phi_b(s), & s \in \Gamma_t\big(b, e^{\pi\mathrm{i}/3}\infty\big), \\ 0, & s \in \Gamma_t(a,b), \\ \phi_a(s), & s \in \Gamma_t\big(e^{\pi\mathrm{i}}\infty, a\big). \end{cases} \tag{6.5}$$

To control the error terms in Theorem 4.10, we need to have precise information on the behavior of ϕ_e around a, b, and c, when the latter belongs to Γ_t. This is exactly the goal of the following two subsections.

6.2. **Local analysis at** $e \in \{a,b\}$**,** $e \neq c$**.** Given $e \in \{a,b\}$, $e \neq c$, set

$$U_e := \big\{z : \ |z-e| < \delta_e\rho(t)|a-b|\big\}, \tag{6.6}$$

where $\delta_e \in (0,1]$ to be adjusted later and we shall specify the function $\rho(t)$ further below in Section 6.4. Define

$$J_e := U_e \cap J_t \quad \text{and} \quad I_e := U_e \cap (\Gamma_t \setminus J_t), \tag{6.7}$$

where the arcs J_e and I_e inherit their orientation from Γ_t. Since I_e is a subarc of the orthogonal trajectory of $\varpi_t = -Q(z;t)dz^2$, it holds that

$$\phi_e(s) < 0, \quad s \in I_e. \tag{6.8}$$

Moreover, we get from (6.3) that

$$\phi_{e\pm}(s) = \pm 2\pi\mathrm{i}\varepsilon_e\mu_t(J_{s,e}) = 2\pi e^{\pm\varepsilon_e\frac{3\pi}{2}\mathrm{i}}\mu_t(J_{s,e}), \tag{6.9}$$

where $J_{s,e}$ is the subarc of J_e with endpoints e and s,

$$\varepsilon_e := \begin{cases} 1, & e = b, \\ -1, & e = a, \end{cases} \tag{6.10}$$

and the second equality in (6.9) follows from (6.8) and the fact $|\phi_e(z)| \sim |z-e|^{3/2}$. Thus, we can define an analytic branch of $(-\phi_e)^{2/3}(z)$ that is positive on I_e. Then (6.9) yields that

$$(-\phi_e)_\pm^{2/3}(s) = -\big(2\pi\mu_t(J_{s,e})\big)^{2/3}, \quad s \in J_e,$$

that is, $(-\phi_e)^{2/3}(z)$ is holomorphic across J_e. Since $(-\phi_e)^{2/3}(z)$ has a simple zero at e, it is conformal in some region around e. It will be clear from the choice of $\rho(t)$ in Section 6.4 that this region contains U_e. Thus, we get that $(-\phi_e)^{2/3}(z)$ maps U_e conformally onto some neighborhood of the origin and satisfies

$$\begin{cases} (-\phi_e)^{2/3}(J_e) & \subset & (-\infty, 0), \\ (-\phi_e)^{2/3}(I_e) & \subset & (0, \infty). \end{cases} \tag{6.11}$$

Furthermore, if we define $(-\phi_e)^{1/6}(z)$ to be holomorphic in $U_e \setminus J_e$ and positive on I_e, then

$$(-\phi_e)_+^{1/6}(s) = \varepsilon_e\mathrm{i}(-\phi_e)_-^{1/6}(s), \quad s \in J_e. \tag{6.12}$$

6.3. **Local analysis at c.** Assume that $t \in \{t_{\mathsf{crit}}, e^{2\pi\mathrm{i}/3}t_{\mathsf{crit}}\}$. Then either $c = b$ or $c = a$. In any case c is a triple zero of Q. Define ε_c by (6.10). Define U_c by (6.6) with

$$\rho(t_{\mathsf{cr}}) := \rho(e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}) := 1/3, \tag{6.13}$$

where again $\delta_c \in (0,1]$. Let I_c and J_c be as in (6.7). It follows from (3.1) that $\phi_c(z) \sim |z-c|^{5/2}$ as $z \to c$. Since $\phi_c < 0$ on I_c and the angle between I_c and J_c is $3\pi/5$, we can define a branch of $\phi_c^{2/5}$ that is conformal around c, in fact, in U_c (see the analysis in the next section) and is negative on J_c. That is, $\phi_c^{2/5}$ maps U_c conformally onto some neighborhood of the origin and satisfies

$$\begin{cases} \phi_c^{2/5}(J_c) & \subset & \{z: \ \mathsf{arg}(z) = \varepsilon_c\pi\}, \\ \phi_c^{2/5}(I_c) & \subset & \{z: \ \mathsf{arg}(z) = \varepsilon_c 2\pi/5\}. \end{cases} \tag{6.14}$$

Moreover, (6.12) is replaced in this case by

$$\phi_{c+}^{1/10}(s) = \varepsilon_c\mathrm{i}\phi_{c-}^{1/10}(s), \quad s \in J_c. \tag{6.15}$$

Let now $t \in C_{\mathsf{split}}$. Determining the left and right sides of Γ_t by its orientation, set

$$\varphi(z) := \begin{cases} -\phi_b(z), & z \text{ is to the left of } \Gamma_t, \\ \phi_b(z), & z \text{ is to the right of } \Gamma_t. \end{cases} \tag{6.16}$$

It follows from (6.3) that $\varphi - \varphi(c)$ is holomorphic across J_t, vanishes at c, is negative purely imaginary on $\Gamma_t(a,c)$, and positive purely imaginary on $\Gamma_t(c,b)$. Moreover, (3.1) yields that $|\varphi(z) - \varphi(c)| \sim |z-c|^2$ as $z \to c$. Therefore, we can define a branch of $(\varphi - \varphi(c))^{1/2}$ that is conformal around c and satisfies

$$\begin{cases} (\varphi - \varphi(c))^{1/2}\,(\Gamma_t(c,b)) & \subset & \{z: \ \mathsf{arg}(z) = \pi/4\}, \\ (\varphi - \varphi(c))^{1/2}\,(\Gamma_t(a,c)) & \subset & \{z: \ \mathsf{arg}(z) = 3\pi/4\}. \end{cases} \tag{6.17}$$

As before, we attach a circular neighborhood to c of the form

$$U_c := \{z: \ |z-c| < \delta_c\rho(t)|a-b|\}, \tag{6.18}$$

where, as in (6.6), $\rho(t)$ is a function that will be specified in the next section (in particular, it will ensure conformality of $(\varphi - \varphi(c))^{1/2}$ in U_c), and $\delta_c \in (0,1]$.

Finally, let us consider the case $t \in C^e_{\mathsf{birth}}$, $e \in \{a,b\}$. Define $\phi_c := \phi_e - \phi_e(c)$. Notice that ϕ_c has a double zero at c and it is real negative on Γ_t around c. Hence, we can select a branch of $(-\phi_c)^{1/2}$ that is conformal in U_c of the form (6.18), satisfies

$$(-\phi_c)^{1/2}\big(\Gamma_t \cap U_c\big) \subset \mathbb{R}, \tag{6.19}$$

and preserves the orientation (positive direction on Γ_t is mapped into the positive direction on $\mathbb{R}$). As it will be important latter, let us also observe that $\phi_e(c)$ is purely imaginary.

6.4. **Neighborhoods U_e.** The goal of this section is to specify the function $\rho(t)$ appearing in the definition of the neighborhoods of U_e, $e \in \{a,b,c\}$, in (6.6) and (6.18). We would like to show that this function can be chosen in such a fashion that the corresponding map is conformal in U_e and the image of U_e under this map contains a disk

$$\{z: \ |z| < \delta_e\tilde\rho(t)/32\}, \tag{6.20}$$

where $\tilde\rho(t)$ is a continuous positive function in $\overline{O}_{\mathsf{one-cut}} \setminus \{t_{\mathsf{cr}}, e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}\}$ that is separated from zero when $t \to \infty$, and the constant $1/32$ is introduced for convenience only.

The main tool in showing that the above requirement can be met is the Basic Structure Theorem, see [28, Theorem 3.5]. It states in particular that the function $\phi_e(z)$, defined in (6.2), is conformal in each connected component of the complement of the joint critical graph (critical and critical orthogonal) of ϖ_t (see Figures 4 and 5 for the possible configurations of this graph). Recall that $\mathsf{Re}\,\phi_e(z)$ is constant on the critical trajectories and $\mathsf{Im}\,\phi_e(z)$ is constant on the critical orthogonal trajectories. Therefore, each connected component of the complement of the joint critical graph is mapped by $\phi_e(z)$ into a quadrant, semi-infinite strip, or a rectangle, see Figure 12. Moreover, if two such regions share a side that is the image of the same part of the joint critical graph, then $\phi_e(z)$ extends conformally through this side. As $|\phi_a(b)| = |\phi_b(a)| = 2\pi$ for any t, the inverse of the restriction of ϕ_e to any

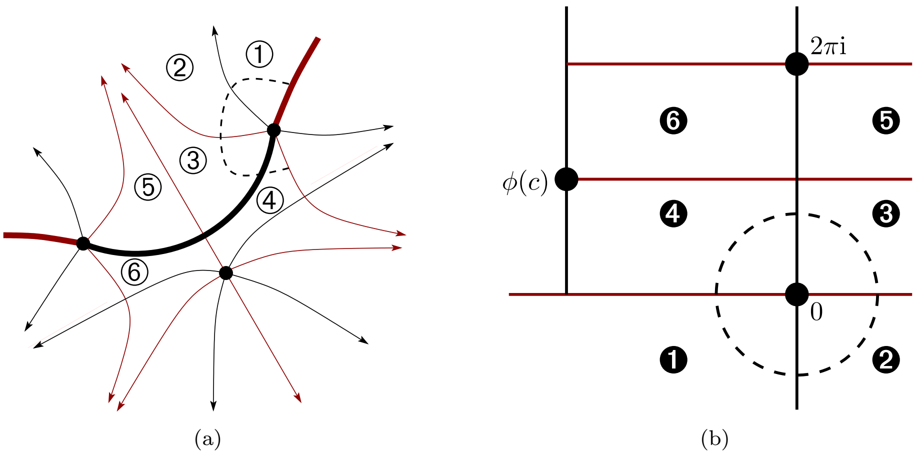

FIGURE 12. (a) The critical graph of ϖ_t with some of the connected components of its complement numbered; (b) The images of the numbered components from part (a) under a certain branch of the map $\phi_b(z)$. The interior of the dashed circle from panel (b) is mapped conformally by $\phi_b^{-1}(z)$ onto the corresponding set bounded by the dashed curve and joint critical graph on panel (a).

connected component is conformal in the intersection of the image of the component under $\phi_e(z)$ and any disk of radius at most $\min\{2\pi, |\phi_e(c)|\}$. In particular, the inverse of $(-\phi_e)^{2/3}$ is conformal in a disk

$$\{z:\ |z|<\tilde\rho_e(t)/2\},\quad \tilde\rho_e(t):=\big(\min\{2\pi,|\phi_e(c)|\}\big)^{2/3}.$$

It follows from a direct computation and (3.7) that

$$\left|\left((-\phi_e)^{2/3}\right)'(e)\right| = \frac{\sqrt[3]{96}}{|a-b|}\left|1+\frac{\mathrm{i}\varepsilon_e}{\sqrt{2}x^{3/2}}\right|^{2/3} \tag{6.21}$$

for any $t\neq t_{\mathsf{ct}}, e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}$. Koebe's Quarter Theorem then yields that $(-\phi_e)^{2/3}$ is conformal in any disk (6.6) with

$$\rho(t):=\min\left\{\frac13,\frac{\tilde\rho(t)}{8\sqrt[3]{96}}\left|1+\frac{\mathrm{i}\varepsilon_e}{\sqrt{2}x^{3/2}}\right|^{-2/3}\right\},\quad \tilde\rho(t):=\min\{\tilde\rho_a(t),\tilde\rho_b(t)\},$$

which, together with Koebe's Quarter Theorem used once more, implies that (6.20) indeed takes place (we bound $\rho(t)$ by $1/3$ for convenience only). Notice that the rate of decay to zero of $\tilde\rho(t)$ as $t\to\{t_{\mathsf{ct}},e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}\}$ can be deduced from (5.7) as $|\phi_b(c)|=|I_x(-x)|$ and the fact that an analogous formula holds for $|\phi_a(c)|$.

It is not hard to see that the above argument can be applied to the conformal map around c when $t\in C_{\mathsf{split}}\cup C_{\mathsf{birth}}$ to show that the conclusion (6.20) still holds for U_c as in (6.18) with $\tilde\rho(t)$ now defined as $\min\{\tilde\rho_a(t),\tilde\rho_b(t),\tilde\rho_c(t)\}$, where

$$\tilde\rho_c^2(t)=\begin{cases}\min\{|\phi_c(a)|,|\phi_c(b)|\}, & t\in C_{\mathsf{birth}},\\ \min\{|\varphi(a)-\varphi(c)|,|\varphi(b)-\varphi(c)|\}, & t\in C_{\mathsf{split}}.\end{cases}$$

6.5. **Functions $D(z;t)$, $A(z;t)$, and $B(z;t)$.** Let $D(z;t)$ be given by (4.12). As in (5.5) we can compute that

$$\sqrt{(z-a)(z-b)}=z-x+\frac{1}{xz}+\frac{1}{z^2}+\frac{1}{z^3}\left(x-\frac{1}{2x^2}\right)+\frac{1}{z^4}\left(x^2-\frac{3}{2x}\right)+\mathcal{O}\left(\frac{1}{z^5}\right)$$

and therefore

$$\left(z^2+zx-2t\right)\sqrt{(z-a)(z-b)}=z^3-3zt+2x^3+\frac{3}{2x^2z}+\mathcal{O}\left(\frac{1}{z^3}\right), \tag{6.22}$$

from which the analyticity and normalization at infinity follow. The equality in (4.13) is a trivial consequence of the behavior of the square root along the branch cut.

Define $F(z)$ as in (4.16). Since $\left(\frac{s-b}{s-a}\right)_+^{1/4} = \mathrm{i}\left(\frac{s-b}{s-a}\right)_-^{1/4}$ for $s \in J_t$, it holds that

$$A_\pm(s;t) = \pm B_\mp(s;t) \quad \Rightarrow \quad F_+(s)F_-(s) \equiv 1, \quad s \in J_t. \tag{6.23}$$

Notice also that neither A nor B is equal to zero in $\mathbb{C}$. Indeed, denote by $\mathcal{R}$ the Riemann surface realized as two copies of $\overline{\mathbb{C}} \setminus J_t$ glued to each other crosswise along J_t. Lift A^2 to one of the sheets of $\mathcal{R}$ (a copy of $\overline{\mathbb{C}} \setminus J_t$) and B^2 to another. It follows from (6.23) that thus defined function is rational on $\mathcal{R}$. As it only has two poles (at the places that project to a and b), it has exactly two zeros. Since B^2 has a double zero at infinity, the claim follows.

We further deduce from the previous paragraph that $F(z)$ is non-vanishing and finite in $\mathbb{C} \setminus J_t$ and has a simple pole at infinity. Hence, it follows from (6.23) that by lifting F to one of the sheets of $\mathcal{R}$ and F^{-1} to another, we construct a rational function with a single pole and a single zero, both projecting to the point at infinity. Clearly, a similar lift of the right-hand side of (4.16) and its reciprocal to $\mathcal{R}$ produces a rational function with the same properties. The normalization at infinity then gives (4.16). Observe also that the above argument applied to the ratio of the right- and left-hand sides of (4.14) together with (4.13), (6.5), (6.23), and the normalization at infinity implies the validity of (4.14) as well.

For further use, let us also record several estimates. By the very definition of U_a in (6.6) and (6.13), we have that

$$\frac{2}{3\delta_a\rho(t)} \le \frac{1}{\delta_a\rho(t)} - 1 \le \left|\frac{s-b}{s-a}\right| \le 1 + \frac{1}{\delta_a\rho(t)} \le \frac{4}{3\delta_a\rho(t)}, \quad s \in \partial U_a, \tag{6.24}$$

where we used the estimate $\delta_a\rho(t) \le 1/3$ and our convention (6.13). Since an analogous bound holds on ∂U_b, we get that

$$|A(z;t)|, |B(z;t)| \le \big(\min\{\delta_a, \delta_b\}\rho(t)\big)^{-1/4}, \quad z \in \overline{\mathbb{C}} \setminus \big(U_a \cup U_b\big), \tag{6.25}$$

where the bound extends outside of $U_a \cup U_b$ by the maximum modulus principle applied on the lift $\partial U_a \cup \partial U_b$ to $\mathcal{R}$ to the rational function on $\mathcal{R}$ comprised of the lifts of A^2 and B^2. Similarly, we deduce from (6.24) that

$$|A(s;t)|^{-1}, |B(s;t)|^{-1} \le 8\big(\max\{\delta_a, \delta_b\}\rho(t)\big)^{1/4}, \quad s \in \partial U_a \cup \partial U_b, \tag{6.26}$$

and therefore it follows from the very definition of $F(z)$ as the left-hand side of (4.16) that

$$|F^{\pm 1}(z)| \le 8, \quad z \in \overline{U}_a \cup \overline{U}_b, \tag{6.27}$$

independently of t, where we apply the argument with $\delta_a = \delta_b = 1$ on the corresponding $\partial U_a \cup \partial U_b$ and then extend the bound inside by the maximum modulus principle applied on $\mathcal{R}$. In fact, it also holds that

$$|F^{-1}(z)| = \mathcal{O}(1) \tag{6.28}$$

uniformly for $z \in \overline{\mathbb{C}} \setminus J_t$ and $t \in \overline{O}_{\mathsf{one-cut}}$. Indeed, by the maximum modulus principle and the analyticity of $F^{-1}(z)$ in $\overline{\mathbb{C}} \setminus J_t$, we only need to prove (6.28) for the traces $F_\pm^{-1}(s)$, $s \in J_t$. Moreover, the compactness argument shows that it is sufficient to consider only $|t|$ large. As explained in Section 6.4, in such situations the inverse of $(-\phi_e)^{2/3}$, $e \in \{a,b\}$, is conformal in the disk of radius $(2\pi)^{2/3}$. Moreover, $(-\phi_a)^{2/3}(b)$ or $(-\phi_b)^{2/3}(a)$, depending on whether $e = a$ or $e = b$, belongs to the boundary of this disk. Hence,

$$J_t \subset \left((-\phi_a)^{2/3}\right)^{-1}(U) \cup \left((-\phi_b)^{2/3}\right)^{-1}(U), \quad U := \left\{z : \ |z| < 2(2\pi)^{2/3}/3\right\}.$$

Then it follows from Koebe's distortion theorem and (6.21) that

$$|s - e| \le \mathsf{const}|a - b|, \quad s \in J_t \cap \left((-\phi_e)^{2/3}\right)^{-1}(U),$$

for some absolute constant. Therefore, $|s - e| \le \mathsf{const}|a-b|$ for all $s \in J_t$ and $e \in \{a,b\}$. This estimate and the explicit expression

$$F^{-1}(z) = \frac{2}{b-a}\left(z - \frac{b+a}{2} - \sqrt{(z-a)(z-b)}\right)$$

immediately imply the desired bound on J_t.

7. Asymptotic Analysis

7.1. **Initial Riemann-Hilbert problem.** In what follows, it will be convenient to set

$$\boldsymbol{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are seeking solutions of the following sequence of Riemann-Hilbert problems for 2×2 matrix-valued functions (RHP-$\boldsymbol{Y}$):

(a) $\boldsymbol{Y}$ is analytic in $\mathbb{C}\setminus\Gamma_t$ and $\lim_{\mathbb{C}\setminus\Gamma_t\ni z\to\infty}\boldsymbol{Y}(z)z^{-n\sigma_3} = \boldsymbol{I}$;
(b) $\boldsymbol{Y}$ has continuous traces on $\Gamma_t\setminus\{a,b,c\}$ that satisfy

$$\boldsymbol{Y}_+(s) = \boldsymbol{Y}_-(s)\begin{pmatrix} 1 & e^{-NV(s;t)} \\ 0 & 1 \end{pmatrix}.$$

The connection of RHP-$\boldsymbol{Y}$ to orthogonal polynomials was first demonstrated by Fokas, Its, and Kitaev in [24] and lies in the following. If the solution of RHP-$\boldsymbol{Y}$ exists, then it is necessarily of the form

$$\boldsymbol{Y}(z) = \begin{pmatrix} P_n(z) & \big(\mathcal{C}P_ne^{-NV}\big)(z) \\ -\frac{2\pi\mathrm{i}}{h_{n-1}}P_{n-1}(z) & -\frac{2\pi\mathrm{i}}{h_{n-1}}\big(\mathcal{C}P_{n-1}e^{-NV}\big)(z) \end{pmatrix}, \tag{7.1}$$

where P_n is the polynomial satisfying orthogonality relations (1.26), h_k are constants defined by (1.28), and $\mathcal{C}f$ is the Cauchy transform of a function f given on Γ_t, i.e.,

$$(\mathcal{C}f)(z) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{f(s)}{s-z}ds.$$

Below, we show the solvability of RHP-$\boldsymbol{Y}$ for all $|n-N|\leq N_0$ and N large enough following the framework of the steepest descent analysis introduced by Dieft and Zhou [19]. The latter lies in a series of transformations which reduce the initial problem to a problem with jumps asymptotically close to the identity.

7.2. **Renormalized Riemann-Hilbert problem.** Suppose that $\boldsymbol{Y}$ is a solution of RHP-$\boldsymbol{Y}$. Put

$$\boldsymbol{T} := e^{N\ell_t^*\sigma_3/2}\left(\frac{4}{b-a}\right)^{(n-N)\sigma_3}\boldsymbol{Y}(z)e^{-N\left(g+\ell_t^*/2\right)\sigma_3}F^{(N-n)\sigma_3}, \tag{7.2}$$

where the function g is defined by (4.10), ℓ_t^* is introduced in (6.1), and $F = -\mathrm{i}A/B$ is a function from (4.16). Then

$$\boldsymbol{T}_+ = \boldsymbol{T}_-\begin{pmatrix} (F_+/F_-)^{N-n}e^{-N(g_+-g_-)} & (F_+F_-)^{n-N}e^{N(g_++g_--V+\ell_t^*)} \\ 0 & (F_-/F_+)^{N-n}e^{-N(g_--g_+)} \end{pmatrix},$$

on Γ_t, and therefore we deduce from (4.11), (4.16), (6.4), (6.5), and (6.23) that $\boldsymbol{T}$ solves RHP-$\boldsymbol{T}$:

(a) $\boldsymbol{T}$ is analytic in $\mathbb{C}\setminus\Gamma_t$ and $\lim_{\mathbb{C}\setminus\Gamma_t\ni z\to\infty}\boldsymbol{T}(z) = \boldsymbol{I}$;
(b) $\boldsymbol{T}$ has continuous traces on $\Gamma_t\setminus\{a,b,c\}$ that satisfy

$$\boldsymbol{T}_+ = \boldsymbol{T}_-\begin{cases} \begin{pmatrix} 1 & F^{2(n-N)}e^{N\phi_b} \\ 0 & 1 \end{pmatrix}, & \text{on} \quad \Gamma_t\big(b, e^{\pi\mathrm{i}/3}\infty\big), \\ \begin{pmatrix} F_+^{2(N-n)}e^{-N\phi_{b+}} & 1 \\ 0 & F_-^{2(n-N)}e^{-N\phi_{b-}} \end{pmatrix}, & \text{on} \quad \Gamma_t(a,b), \\ \begin{pmatrix} 1 & F^{2(n-N)}e^{N\phi_a} \\ 0 & 1 \end{pmatrix}, & \text{on} \quad \Gamma_t\big(e^{\pi\mathrm{i}}\infty, a\big). \end{cases}$$

Clearly, if RHP-$\boldsymbol{T}$ is solvable and $\boldsymbol{T}$ is the solution, then by inverting (7.2) one obtains a matrix $\boldsymbol{Y}$ that solves RHP-$\boldsymbol{Y}$.

7.3. **Lens opening.** As usual in the steepest descent analysis of matrix Riemann-Hilbert problems for orthogonal polynomials, the next step is based on the identity

$$\begin{pmatrix} a_+ & 1 \\ 0 & a_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_- & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_+ & 1 \end{pmatrix}, \quad a_+a_- \equiv 1,$$

which is applicable by (6.3) and (6.23). To carry it out, we shall introduce two additional arcs. Denote by $J_\pm$ smooth homotopic deformations of J_t within the region $\mathsf{Re}\,(\phi_b(z)) > 0$ such that J_+ lies to the left and J_- to the right of J_t, see Figure 13. Moreover, we shall

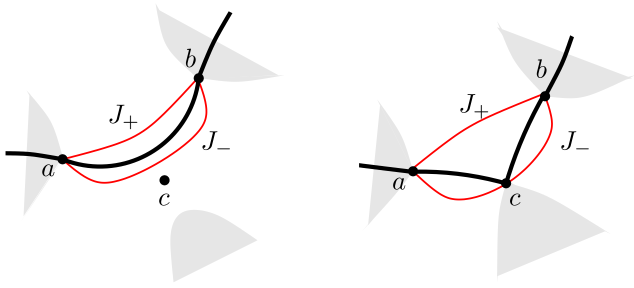

FIGURE 13. The thick arcs represent Γ_t and thiner arcs represent $J_\pm$. Shaded region is the set where $\mathsf{Re}\big(\phi_b(z)\big) < 0$.

fix the way these arcs emanate from $e \in \{a, b\}$. Namely, let U_e be a disk centered at e as described in Sections 6.2–6.4. Assume first that we are in a generic situation when $e \neq c$. Then we require that

$$\mathsf{arg}\big((-\phi_e)^{2/3}(z)\big) = \pm\varepsilon_e(2\pi/3), \quad z \in U_e \cap J_\pm, \tag{7.3}$$

where ε_e is defined by (6.10). The latter is always possible due to (6.11). Suppose now that $e = c \in \{a, b\}$. Then we require that

$$\mathsf{arg}\big(\phi_c^{2/5}(z)\big) = \pm\varepsilon_c(4\pi/5), \quad z \in U_c \cap J_\pm, \tag{7.4}$$

where ε_c is still defined by (6.10) and such a choice is possible according to (6.14). In addition, when $c \in J_t \setminus \{a, b\}$, it necessarily holds that J_- touches J_t at c. We shall choose J_- around c so that

$$(\varphi - \varphi(c))^{1/2}(U_c \cap J_-) \subset \mathbb{R}, \tag{7.5}$$

where φ is defined by (6.16) and such a choice is possible due to (6.17).

Denote by $O_\pm$ the open sets delimited by $J_\pm$ and J_t. Set

$$\boldsymbol{S}(z) := \boldsymbol{T}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp F^{2(N-n)}(z) e^{-N\phi_b(z)} & 1 \end{pmatrix}, & z \in O_\pm, \\ \boldsymbol{I}, & \text{otherwise.} \end{cases} \tag{7.6}$$

Then, if $\boldsymbol{T}$ solves RHP-$\boldsymbol{T}$, $\boldsymbol{S}$ solves RHP-$\boldsymbol{S}$:

(a) $\boldsymbol{S}$ is analytic in $\mathbb{C} \setminus (\Gamma_t \cup J_+ \cup J_-)$ and $\lim_{\mathbb{C}\setminus\Gamma_t \ni z \to \infty} \boldsymbol{S}(z) = \boldsymbol{I}$;

(b) $\boldsymbol{S}$ has continuous traces on $\Gamma_t \setminus \{a, b, c\}$ that satisfy RHP-$\boldsymbol{T}$(b) on $\Gamma_t\big(e^{\pi i}\infty, a\big)$ and $\Gamma_t\big(b, e^{\pi i/3}\infty\big)$, as well as

$$\boldsymbol{S}_+(s) = \boldsymbol{S}_-(s) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in J_t, \\ \begin{pmatrix} 1 & 0 \\ F^{2(N-n)}(s) e^{-N\phi_b(s)} & 1 \end{pmatrix}, & s \in J_\pm. \end{cases}$$

As before, since transformation (7.6) is invertible, a solution of RHP-$\boldsymbol{S}$ yields a solution of RHP-$\boldsymbol{T}$.

7.4. **Model solution.** The model Riemann-Hilbert problem RHP-$\boldsymbol{M}$ is obtained from RHP-$\boldsymbol{S}$ by removing from the jump matrices in RHP-$\boldsymbol{S}$(b) the quantities that are asymptotically zero. Thus, we are seeking the solution of RHP-$\boldsymbol{M}$:

(a) $\boldsymbol{M}$ is analytic in $\overline{\mathbb{C}}\setminus J_t$ and $\boldsymbol{M}(\infty)=\boldsymbol{I}$;
(b) $\boldsymbol{M}$ has continuous traces on $\Gamma_t(a,b)$ that satisfy

$$\boldsymbol{M}_+(s)=\boldsymbol{M}_-(s)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad s\in\Gamma_t(a,b).$$

Using (6.23), one can easily verify that RHP-$\boldsymbol{M}$ is solved by

$$\boldsymbol{M}(z):=\begin{pmatrix}A(z;t) & -B(z;t)\\ B(z;t) & A(z;t)\end{pmatrix}, \tag{7.7}$$

where $A(z;t)$ and $B(z;t)$ are given by (4.15). Observe also that $\det(\boldsymbol{M})\equiv 1$ in $\mathbb{C}$.

7.5. **Local parametrices.** The jumps discarded in RHP-$\boldsymbol{M}$ are not uniformly close to the identity around $e\in\{a,b,c\}$ (the point $e=c$ is included when $t\in\partial O_{\mathsf{one-cut}}$). The goal of this section is to solve RHP-$\boldsymbol{S}$ within the disks U_e introduced in Sections 6.2–6.4 with a certain matching condition on ∂U_e. More precisely, given $e\in\{a,b,c\}$, we are looking for a matrix-valued function $\boldsymbol{P}_e$ that solves RHP-$\boldsymbol{P}_e$:

(a) $\boldsymbol{P}_e$ has the same analyticity properties as $\boldsymbol{S}$ restricted to U_e, see RHP-$\boldsymbol{S}$(a);
(b) $\boldsymbol{P}_e$ satisfies the same jump relations as $\boldsymbol{S}$ restricted to U_e, see RHP-$\boldsymbol{S}$(b);
(c) $\boldsymbol{P}_e=\boldsymbol{M}\big(\boldsymbol{I}+\boldsymbol{\mathcal{O}}(N^{-\alpha_e})\big)$ holds uniformly on ∂U_e as $N\to\infty$ for some $\alpha_e>0$.

7.5.1. *Parametrix $\boldsymbol{P}_e$ around $e\in\{a,b\}$, $e\neq c$.* Let U_e, J_e, and I_e be as in (6.7). In this section we are looking for a matrix function $\boldsymbol{P}_e$ that is holomorphic in $U_e\setminus(\Gamma_t\cup J_+\cup J_-)$, fulfills RHP-$\boldsymbol{P}_e$(c), and whose traces satisfy

$$\boldsymbol{P}_{e+}(s)=\boldsymbol{P}_{e-}(s)\begin{cases}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & s\in J_e,\\ \begin{pmatrix}1 & 0\\ F^{2(N-n)}(s)e^{-N\phi_e(s)} & 1\end{pmatrix}, & s\in J_\pm\cap U_e,\\ \begin{pmatrix}1 & F^{2(n-N)}(s)e^{N\phi_e(s)}\\ 0 & 1\end{pmatrix}, & s\in I_e.\end{cases} \tag{7.8}$$

Notice that we replaced ϕ_b by ϕ_e as compared to RHP-$\boldsymbol{S}$(b). Such a substitution is possible due to the first relation in (6.3).

Let $\boldsymbol{A}$ be the Airy matrix [15, 16]. That is, it is analytic in $\mathbb{C}\setminus\big((-\infty,\infty)\cup L_-\cup L_+\big)$, $L_\pm:=\big\{z:\ \mathsf{arg}(z)=\pm 2\pi/3\big\}$, and satisfies

$$\boldsymbol{A}_+(s)=\boldsymbol{A}_-(s)\begin{cases}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & s\in(-\infty,0),\\ \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, & s\in L_\pm,\\ \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, & s\in(0,\infty),\end{cases}$$

where the real line is oriented from $-\infty$ to ∞ and the rays $L_\pm$ are oriented towards the origin. It is known that $\boldsymbol{A}$ has the following asymptotic expansion at infinity:

$$\boldsymbol{A}(\zeta)e^{\frac{2}{3}\zeta^{3/2}\sigma_3}\sim\frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\sum_{k=0}^{\infty}\begin{pmatrix}s_k & 0\\ 0 & t_k\end{pmatrix}\begin{pmatrix}(-1)^k & \mathrm{i}\\ (-1)^k\mathrm{i} & 1\end{pmatrix}\left(\frac{2}{3}\zeta^{3/2}\right)^{-k}, \tag{7.9}$$

where the expansion holds uniformly in $\mathbb{C}\setminus\big((-\infty,\infty)\cup L_-\cup L_+\big)$, and

$$s_0=t_0=1,\quad s_k=\frac{\Gamma(3k+1/2)}{54^k k!\Gamma(k+1/2)},\quad t_k=-\frac{6k+1}{6k-1}s_k,\quad k\geq 1.$$

Set $\boldsymbol{A}_b:=\boldsymbol{A}$ and $\boldsymbol{A}_a:=\sigma_3\boldsymbol{A}\sigma_3$. It can be easily checked that $\sigma_3\boldsymbol{A}\sigma_3$ has the same jumps as $\boldsymbol{A}$ only with the reversed orientation of the real line and the rays $L_\pm$. Moreover, one needs to replace each occurrence of i by $-\mathrm{i}$ in (7.9) when describing the asymptotic expansion $\sigma_3\boldsymbol{A}\sigma_3$. Then (6.3), (6.11), and (7.3) yield that the matrix function

$$\boldsymbol{P}_e(z)=\boldsymbol{E}_e(z)\boldsymbol{A}_e\left(\left[-(3/4)N\phi_e(z)\right]^{2/3}\right)e^{-N\phi_e(z)\sigma_3/2}F^{(N-n)\sigma_3}(z) \tag{7.10}$$

satisfies RHP-$\boldsymbol{P}_e$(a) and RHP-$\boldsymbol{P}_e$(b) for any matrix function $\boldsymbol{E}_e$ holomorphic in U_e. Thus, we only need to choose $\boldsymbol{E}_e$ so that RHP-$\boldsymbol{P}_e$(c) is fulfilled. Choose

$$\boldsymbol{E}_e(z):=\boldsymbol{M}(z)F^{(n-N)\sigma_3}(z)\begin{pmatrix}1 & -\varepsilon_e\mathrm{i}\\ -\varepsilon_e\mathrm{i} & 1\end{pmatrix}\frac{\left[-(3/4)N\phi_e(z)\right]^{\sigma_3/6}}{\sqrt{2}}$$

whose analyticity in U_e follows RHP-$\boldsymbol{M}$(b), (6.23), and (6.12) with ε_e given by (6.10). Then we deduce from (7.9) and (7.10) that

$$\boldsymbol{P}_e(z)\sim\boldsymbol{M}(z)F^{(n-N)\sigma_3}(z)\left(\boldsymbol{I}+N^{-\alpha_e}\sum_{k=0}^{\infty}\boldsymbol{P}_{e,k}(z)N^{-k}\right)F^{(N-n)\sigma_3}(z), \tag{7.11}$$

where the expansion inside the parenthesis holds uniformly on ∂U_e and locally uniformly for $t\in\overline{O}_{\mathsf{one-cut}}\setminus\left\{t_{\mathsf{cr}},e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}\right\}$ by (6.20), the parameter $\alpha_e=1$, and

$$\boldsymbol{P}_{e,k-1}(z)=\begin{pmatrix}1 & -\varepsilon_e\mathrm{i}\\ -\varepsilon_e\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}s_k & 0\\ 0 & t_k\end{pmatrix}\begin{pmatrix}(-1)^k & \varepsilon_e\mathrm{i}\\ \varepsilon_e(-1)^k\mathrm{i} & 1\end{pmatrix}\left(-\frac{\phi_e(z)}{2}\right)^{-k},\quad k\geq 1. \tag{7.12}$$

7.5.2. *Parametrix $\boldsymbol{P}_c$ around $c\in\{a,b\}$.* The local problem at $e=c$ is formulated exactly as before with the jumps as in (7.8). However, the above solution does not apply because $(-\phi_c)^{2/3}$ is no longer conformal (we shall replace it by $\phi_c^{2/5}$) and the arcs I_c and J_c no longer form an angle π at c (it is $3\pi/5$).

As in the previous subsection, we shall need an auxiliary matrix-valued function. This time it depends on two parameters: $\alpha,\lambda\in\mathbb{C}$, and solves the following Riemann-Hilbert problem (RHP-$\boldsymbol{\Psi}$):

(a) $\boldsymbol{\Psi}_\alpha$ is a holomorphic matrix function in $\mathbb{C}\setminus\big((-\infty,0]\cup L_{1-}\cup L_{1+}\cup L_{2-}\cup L_{2+}\big)$, where $L_{k\pm}:=\big\{z:\ \mathsf{arg}(z)=\pm 2k\pi/5\big\}$, the rays $(-\infty,0)$ and $L_{2\pm}$ are oriented towards the origin while $L_{1\pm}$ are oriented away from the origin;

(b) $\boldsymbol{\Psi}_\alpha$ has continuous traces on $(-\infty,0)\cup L_{1-}\cup L_{1+}\cup L_{2-}\cup L_{2+}$ that satisfy

$$\boldsymbol{\Psi}_{\alpha+}(s;\lambda)=\boldsymbol{\Psi}_{\alpha-}(s;\lambda)\begin{cases}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & s\in(-\infty,0),\\ \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, & s\in L_{2\pm},\\ \begin{pmatrix}1 & 1-\alpha\\ 0 & 1\end{pmatrix}, & s\in L_{1+},\\ \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}, & s\in L_{1-};\end{cases}$$

(c) $\boldsymbol{\Psi}_\alpha$ satisfies

$$\boldsymbol{\Psi}_\alpha(\zeta;\lambda)=\frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ 1 & \mathrm{i}\end{pmatrix}\left(\boldsymbol{I}+\boldsymbol{\mathcal{O}}\left(\zeta^{-1/2}\right)\right)e^{(\frac{4}{5}\zeta^{5/2}+\lambda\zeta^{1/2})\sigma_3}$$

uniformly in $\mathbb{C}\setminus\big((-\infty,0]\cup L_{1-}\cup L_{1+}\cup L_{2-}\cup L_{2+}\big)$.

RHP-$\boldsymbol{\Psi}$ characterizes *tronquée* solutions of Painlevé I equation [29]. That is, $\boldsymbol{\Psi}_\alpha(\zeta;\lambda)$ satisfies the following system of linear ODEs:

$$\begin{cases}\partial_\zeta(\boldsymbol{\Psi}_\alpha(\zeta;\lambda))\boldsymbol{\Psi}_\alpha(\zeta;\lambda)^{-1} &= \begin{pmatrix}-\partial_\lambda y_\alpha & 2\zeta^2+2\zeta y_\alpha+\lambda+2y_\alpha^2\\ 2\zeta-2y_\alpha & \partial_\lambda y_\alpha\end{pmatrix}\\ \partial_\lambda(\boldsymbol{\Psi}_\alpha(\zeta;\lambda))\boldsymbol{\Psi}_\alpha(\zeta;\lambda)^{-1} &= \begin{pmatrix}0 & \zeta+2y_\alpha\\ 1 & 0\end{pmatrix}\end{cases} \tag{7.13}$$

with the functions $y_\alpha(\lambda)$ forming a one parameter family of solutions to $y''(\lambda)=6y^2(\lambda)+\lambda$ and satisfying $y_\alpha(\lambda)=\sqrt{-\lambda/6}\big(1+\mathcal{O}\big((-\lambda)^{-5/2}\big)\big)$ as $|\lambda|\to\infty$, $|\mathsf{arg}(\lambda)-\pi|<2\pi/5$, where the parameter α appears when describing the more detailed asymptotics of y_α. In particular, each such solution is pole free in the sector $|\mathsf{arg}(\lambda)-\pi|<2\pi/5$ for $|\lambda|$ large, and therefore is a tronquée solution as designated by Boutroux [10]. Moreover, the functions $y_0(\lambda)$ and $y_1(\lambda)$ are known to be *tritronquée* solutions as they are asymptotically pole free in sectors $|\mathsf{arg}(\lambda)-7\pi/5|<4\pi/5$ and $|\mathsf{arg}(\lambda)-3\pi/5|<4\pi/5$, respectively.

It is known that RHP-$\boldsymbol{\Psi}$ is solvable if and only if λ is not a pole of the corresponding solution y_α [20, Section 4.6]. It is also known that tritronquée solutions are pole free in a disk around the origin [11, Theorem 1]. Hence, the matrices $\boldsymbol{\Psi}_0(\cdot;0)$ and $\boldsymbol{\Psi}_1(\cdot;0)$ exist and have the properties described by RHP-$\boldsymbol{\Psi}$.

Set $\boldsymbol{B}_b:=\boldsymbol{\Psi}_0(\cdot;0)$ and $\boldsymbol{B}_a:=\sigma_3\boldsymbol{\Psi}_1(\cdot;0)\sigma_3$. As before, one can check that $\sigma_3\boldsymbol{\Psi}_\alpha\sigma_3$ has the same jumps as $\boldsymbol{\Psi}_\alpha$ only with the reversed orientation of the rays. Moreover, one needs to replace the anti-diagonal elements in RHP-$\boldsymbol{\Psi}$(c) by their negatives when describing the behavior of $\sigma_3\boldsymbol{\Psi}_\alpha\sigma_3$ at infinity. Then (6.3), (6.14), and (7.4) yield that the matrix function

$$\boldsymbol{P}_c(z)=\boldsymbol{E}_c(z)\boldsymbol{B}_c\left(\big[(5/8)N\phi_c(z)\big]^{2/5}\right)e^{-N\phi_c(z)\sigma_3/2}F^{(N-n)\sigma_3}(z) \tag{7.14}$$

satisfies RHP-$\boldsymbol{P}_c$(a) and RHP-$\boldsymbol{P}_c$(b) for any matrix function $\boldsymbol{E}_c$ holomorphic in U_c. Thus, we only need to choose $\boldsymbol{E}_c$ so that RHP-$\boldsymbol{P}_c$(c) is fulfilled. Choose

$$\boldsymbol{E}_c(z):=\boldsymbol{M}(z)F^{(n-N)\sigma_3}(z)\begin{pmatrix}1 & \varepsilon_c\\ \varepsilon_c\mathrm{i} & -\mathrm{i}\end{pmatrix}\frac{\big[(5/8)N\phi_c(z)\big]^{\sigma_3/10}}{\sqrt{2}},$$

whose analyticity in U_c follows (6.15), (6.23), and RHP-$\boldsymbol{M}$(b). It can be readily verified that (7.14) satisfies (7.11) uniformly on ∂U_c with $\alpha_c=1/5$ and

$$\boldsymbol{P}_{c,k}(z)=\begin{pmatrix}1 & 0\\ 0 & \varepsilon_c\end{pmatrix}\boldsymbol{\Psi}_k\begin{pmatrix}1 & 0\\ 0 & \varepsilon_c\end{pmatrix}\left[\frac{5}{8}\phi_c(z)\right]^{\sigma_3/5} \tag{7.15}$$

where the $\boldsymbol{O}\big(\zeta^{-1/2}\big)\sim\sum_{k=1}^\infty\boldsymbol{\Psi}_k\zeta^{-k/2}$ is the error term from RHP-$\boldsymbol{\Psi}$(c) and the matrices $\boldsymbol{\Psi}_k$ can be found recursively using (7.13).

7.5.3. *Parametrix $\boldsymbol{P}_c$ around $c\in\Gamma_t(a,b)$.* Recall that U_c is given by (6.18). We always can adjust the constant δ_c so that $J_+\cap U_c=\varnothing$. In this case $\boldsymbol{P}_c$ is a holomorphic matrix in $U_c\setminus(J_\Gamma\cup J_-)$ that fulfills RHP-$\boldsymbol{P}_e$(c) and whose traces satisfy

$$\boldsymbol{P}_{c+}(s)=\boldsymbol{P}_{c-}(s)\begin{cases}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & s\in(U_c\cap J_t)\setminus\{c\},\\[2mm] \begin{pmatrix}1 & 0\\ F^{2(N-n)}(s)e^{-N\phi_b(s)} & 1\end{pmatrix}, & s\in(U_c\cap J_-)\setminus\{c\}.\end{cases} \tag{7.16}$$

Let $\boldsymbol{C}$ be the following matrix-valued function:

$$\boldsymbol{C}(\zeta):=\begin{pmatrix}e^{\zeta^2} & 0\\ b(\zeta) & e^{-\zeta^2}\end{pmatrix},\quad b(\zeta):=\frac{1}{2}e^{-\zeta^2}\begin{cases}\operatorname{erfc}\big(-\mathrm{i}\sqrt{2}\zeta\big), & \mathsf{Im}\,(\zeta)>0,\\ -\operatorname{erfc}\big(\mathrm{i}\sqrt{2}\zeta\big), & \mathsf{Im}\,(\zeta)<0.\end{cases} \tag{7.17}$$

Equivalently, we could have defined $b(\zeta)$ as $\pm(2\pi)^{-1/2}U(1/2;\mp2\mathrm{i}\zeta)$, $\pm\mathsf{Im}\,(\zeta)>0$, see [18, Eq. (12.7.5)], where $U(a;z)$ is a parabolic cylinder function solving [18, Eq. (12.2.2)]. Observe that

$$b_+(x)-b_-(x)=\frac{1}{2}e^{-x^2}\left(\operatorname{erfc}\big(-\mathrm{i}\sqrt{2}x\big)+\operatorname{erfc}\big(\mathrm{i}\sqrt{2}x\big)\right)=e^{-x^2},$$

for $x\in\mathbb{R}$ and therefore

$$\boldsymbol{C}_+=\boldsymbol{C}_-\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\quad\text{on}\quad\mathbb{R}. \tag{7.18}$$

Moreover, since $\mathsf{Re}\,(-\mathrm{i}\zeta)>0$ when $\mathsf{Im}\,(\zeta)>0$ and $\mathsf{Re}\,(\mathrm{i}\zeta)>0$ when $\mathsf{Im}\,(\zeta)<0$, it holds that

$$b(\zeta)\sim\frac{\mathrm{i}e^{\zeta^2}}{\sqrt{2\pi}}\sum_{k=0}^\infty\frac{\Gamma(k+1/2)}{2^{k+1}\Gamma(1/2)}\zeta^{-(2k+1)}=:e^{\zeta^2}\sum_{k=0}^\infty b_k\zeta^{-(2k+1)}$$

uniformly in the upper and lower half-planes by [18, Eq. (7.12.1) or Eq. (12.9.1)]. Thus, we deduce that

$$\boldsymbol{C}(\zeta)=\begin{pmatrix}1 & 0\\ e^{-\zeta^2}b(\zeta) & 1\end{pmatrix}e^{\zeta^2\sigma_3}\sim\left(\boldsymbol{I}+\sum_{k=0}^\infty\begin{pmatrix}0 & 0\\ b_k & 0\end{pmatrix}\zeta^{-(2k+1)}\right)e^{\zeta^2\sigma_3}\quad\text{as}\quad\zeta\to\infty, \tag{7.19}$$

where the expansion is uniform in the lower and upper half-planes. Set

$$\boldsymbol{J}(z) := \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in U_c^+, \\ \boldsymbol{I}, & z \in U_c^-, \end{cases}$$

where U_c^+ is the part of U_c that lies to the left of J_t and U_c^- that lies to the right of it. We claim that RHP-$\boldsymbol{P}_c$ is solved by

$$\boldsymbol{P}_c(z) = \boldsymbol{M}(z)F^{(n-N)\sigma_3}(z)\boldsymbol{J}(z)e^{N\varphi(c)\sigma_3/2}\boldsymbol{C}\left(\sqrt{N/2}\big[\varphi-\varphi(c)\big]^{1/2}(z)\right)\times \\ \times \boldsymbol{J}^{-1}(z)e^{-N\phi_b(z)\sigma_3/2}F^{(N-n)\sigma_3}(z). \tag{7.20}$$

Indeed, RHP-$\boldsymbol{P}_c$(a) is satisfied due to the choice of the branch of $(\varphi-\varphi(c))^{1/2}$, see (6.17), and the choice of J_-, see (7.5). Further, since $\boldsymbol{J}=\boldsymbol{I}$ in U_c^-, analyticity of $\boldsymbol{M}$, ϕ_b, and F across $J_-\setminus\{c\}$ as well as (7.5) and (7.18) imply that $\boldsymbol{P}_c$ has the jump there as in (7.16). Moreover, observe that

$$\boldsymbol{M}(z)F^{(n-N)\sigma_3}(z)\boldsymbol{J}(z) \quad \text{and} \quad e^{N\varphi(c)\sigma_3/2}\boldsymbol{C}\left(\sqrt{N/2}\big[\varphi-\varphi(c)\big]^{1/2}(z)\right)$$

are analytic across J_t by RHP-$\boldsymbol{M}$(b), (6.23), and the choice of $\boldsymbol{J}$. Hence, since $\phi_{b+}+\phi_{b-}\equiv 0$ on J_t by (6.3), it follows from the definition of $\boldsymbol{J}$ that $\boldsymbol{P}_c$ has the jumps across $J_t\setminus\{c\}$ as in (7.16). That is, RHP-$\boldsymbol{P}_c$(b) is fulfilled as well. Finally, we get from (6.16), (6.20), and (7.19) that (7.20) satisfies (7.11) uniformly on ∂U_c and locally uniformly on C_{split} with $\alpha_c = 1/2$ and $\boldsymbol{P}_{c,k}$ given by

$$\frac{2^{k+1/2}e^{-N\varphi(c)}}{(\varphi(z)-\varphi(c))^{k+1/2}}\begin{pmatrix} 0 & -b_k \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{2^{k+1/2}e^{-N\varphi(c)}}{(\varphi(z)-\varphi(c))^{k+1/2}}\begin{pmatrix} 0 & 0 \\ b_k & 0 \end{pmatrix} \tag{7.21}$$

in U_c^+ and U_c^-, respectively (recall also that $|e^{\varphi(c)}|=1$).

7.5.4. *Parametrix $\boldsymbol{P}_c$ around $c\in\Gamma_t\big(e^{\pi\mathrm{i}}\infty,a\big)\cup\Gamma_t\big(b,e^{\pi\mathrm{i}/3}\infty\big)$.* Put $e=a$ if $c\in\Gamma_t\big(e^{\pi\mathrm{i}}\infty,a\big)$ and $e=b$ if $c\in\Gamma_t\big(b,e^{\pi\mathrm{i}/3}\infty\big)$. We are seeking a matrix function holomorphic in $U_c\setminus\Gamma_t$ that fulfills RHP-$\boldsymbol{P}_c$(c) and whose traces satisfy

$$\boldsymbol{P}_{c+}(s)=\boldsymbol{P}_{c-}(s)\begin{pmatrix} 1 & F^{2(n-N)}(s)e^{N\phi_e(s)} \\ 0 & 1 \end{pmatrix}, \quad s\in\Gamma_t\cap U_c.$$

The Riemann-Hilbert problem RHP-$\boldsymbol{P}_c$ is solved by

$$\boldsymbol{P}_c(z)=\boldsymbol{M}(z)F^{(n-N)\sigma_3}(z)e^{N\phi_e(c)\sigma_3/2}\sigma_1\boldsymbol{C}\left(\big[-(N/2)\phi_c(z)\big]^{1/2}\right)\sigma_1\times \\ \times e^{-N\phi_e(z)\sigma_3/2}F^{(N-n)\sigma_3}(z), \tag{7.22}$$

where $\boldsymbol{C}$ is defined by (7.17) and $(-\phi_c)^{1/2}$ is the branch chosen in (6.19). Indeed, it can be readily verified using (7.18) that

$$(\sigma_1\boldsymbol{C}\sigma_1)_+ = (\sigma_1\boldsymbol{C}\sigma_1)_-\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad \mathbb{R}. \tag{7.23}$$

As $\boldsymbol{M}$ and F are holomorphic in U_c, $(-\phi_c)^{1/2}$ is conformal there, satisfies (6.19), and preserves the orientation, we see that RHP-$\boldsymbol{P}_c$(a) is fulfilled. The above properties and (7.23) yield that RHP-$\boldsymbol{P}_c$(b) is fulfilled as well. Finally, we get from (7.19) that

$$\sigma_1\boldsymbol{C}(\zeta)\sigma_1\sim\left(\boldsymbol{I}+\sum_{k=0}^{\infty}\begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix}\zeta^{-(2k+1)}\right)e^{-\zeta^2\sigma_3} \quad \text{as} \quad \zeta\to\infty$$

uniformly in the lower and upper half-planes. Therefore, by (6.20) and since $\phi_c(z)=\phi_e(z)-\phi_e(c)$, (7.22) satisfies (7.11) with the expansion in parenthesis being uniform on ∂U_c and closed subsets of $C^a_{\mathsf{birth}}\cup C^b_{\mathsf{birth}}$, $\alpha_c=1/2$, and

$$\boldsymbol{P}_{c,k}(z)=\frac{2^{k+1/2}e^{N\phi_e(c)}}{(-\phi_c(z))^{k+1/2}}\begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix} \tag{7.24}$$

(again, notice that $|e^{\phi_e(c)}|=1$).

7.6. **Riemann-Hilbert problem with small jumps.** Set $\Sigma_{\boldsymbol{R}} := \big[\big(\Gamma_t\big(e^{\pi\mathrm{i}}\infty,a\big)\cup J_+\cup J_-\cup\Gamma_t\big(b,e^{\pi\mathrm{i}/3}\infty\big)\big)\cap D\big]\cup\big[\cup_e\partial U_e\big]$, $D:=\mathbb{C}\setminus\cup_e\overline{U}_e$, where e runs over a and b as well as c when $t\in\partial O_{\mathsf{one-cut}}$ (in what follows, we shall always understand the symbol $\cup_e$ this way), see Figure 14. Consider RHP-$\boldsymbol{R}$:

(a) $\boldsymbol{R}$ is holomorphic in $\mathbb{C}\setminus\Sigma_{\boldsymbol{R}}$ and $\lim_{\mathbb{C}\setminus\Gamma_t\ni z\to\infty}\boldsymbol{R}(z)=\boldsymbol{I}$;

(b) $\boldsymbol{R}$ has continuous traces on $\Sigma_{\boldsymbol{R}}^\circ$ that satisfy

$$\boldsymbol{R}_+(s)=\boldsymbol{R}_-(s)\begin{cases}\boldsymbol{P}_e(s)\boldsymbol{M}^{-1}(s), & s\in\partial U_e,\\ \boldsymbol{M}(s)\begin{pmatrix}1 & 0\\ F^{2(N-n)}(s)e^{-N\phi_b(s)} & 1\end{pmatrix}\boldsymbol{M}^{-1}(s), & s\in J_\pm\cap D,\end{cases}$$

where ∂U_e is oriented clockwise, $\boldsymbol{M}$ is given by (7.7), and $\boldsymbol{P}_e$ is given by (7.10), (7.14), (7.20), or (7.22) depending on e; as well as

$$\boldsymbol{R}_+(s)=\boldsymbol{R}_-(s)\boldsymbol{M}(s)\begin{pmatrix}1 & F^{2(n-N)}(s)e^{N\phi_e(s)}\\ 0 & 1\end{pmatrix}\boldsymbol{M}^{-1}(s),$$

for $s\in\Gamma_t\big(b,e^{\pi\mathrm{i}/3}\infty\big)\cap D$ with $e=b$ and for $s\in\Gamma_t\big(e^{\pi\mathrm{i}}\infty,a\big)\cap D$ with $e=a$ (observe that $\boldsymbol{M}^{-1}$ is well defined since $\det(\boldsymbol{M})\equiv1$).

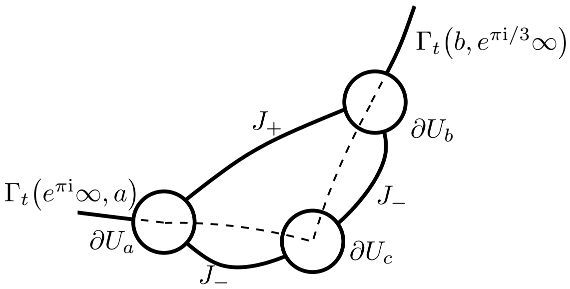

FIGURE 14. The contour $\Sigma_{\boldsymbol{R}}$ (continuous lines). The dashed lines represent the part of Γ_t that does not belong to $\Sigma_{\boldsymbol{R}}$.

Let us show that the jump matrices in RHP-$\boldsymbol{R}$(b) are uniformly close to $\boldsymbol{I}$. To this end, set

$$\boldsymbol{\Delta}:=\boldsymbol{R}_-^{-1}\boldsymbol{R}_+-\boldsymbol{I} \tag{7.25}$$

to be the deviation of the jumps of $\boldsymbol{R}$ from the identity matrix.

Firstly, it follows from RHP-$\boldsymbol{R}$(b) and (7.11) with $\boldsymbol{P}_{e,k}$ given by (7.12), (7.15), (7.21), or (7.24) that

$$\boldsymbol{\Delta}\sim N^{-\alpha_e}\sum_{k=0}^\infty\Big(\boldsymbol{M}F^{(n-N)\sigma_3}\boldsymbol{P}_{e,k}F^{(N-n)\sigma_3}\boldsymbol{M}^{-1}\Big)N^{-k}\quad\text{on}\quad\partial U_e. \tag{7.26}$$

The above expansions of $\boldsymbol{\Delta}(s)$ are uniform in s on each ∂U_e. Moreover, the expansions on ∂U_a and ∂U_b are also locally uniform in $\overline{O}_{\mathsf{one-cut}}\setminus\big\{t_{\mathsf{cr}},e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}\big\}$ by (6.25) and (6.27). Furthermore, the expansion on ∂U_c is uniform on compact subsets of C_{split}, C_{birth}^a, and C_{birth}^b by (6.25) and a compactness argument applied to $\max_{s\in\partial U_c}|F^{\pm1}(s)|$. In addition, the expansion on ∂U_c is uniform on closed subsets of C_{birth}^a and C_{birth}^b when $n=N$ because the term $F^{(n-N)\sigma_3}$ is no longer present. Altogether, we get by looking at the first term in expansion (7.26) that

$$\|\boldsymbol{\Delta}\|_{L^\infty(\cup_e\partial U_e)}\leq C_0(t,\delta)N^{-\alpha_t},\quad \alpha_t=\min_e\alpha_e,\quad \delta:=\min_e\delta_e, \tag{7.27}$$

where the constants δ_e were introduced in (6.6) and (6.18), $C_0(t,\delta)$ can be chosen to depend continuously on t and δ with additional property of being bounded as $t\to\infty$ for each fixed $\delta>0$ when $n=N$.

Secondly, since $\Gamma_t \setminus J_t$ consists of orthogonal trajectories of $-Q(z;t)\mathrm{d}z^2$, it holds by (6.2) that $\phi_e(s) < 0$ on the corresponding part of $(\Gamma_t \setminus J_t) \cap D$. More precisely, there exists a constant $0 < C_1(t,\delta) < 1$ such that

$$\left|F^{2(n-N)}(s)e^{N\phi_e(s)}\right| < C_1^N(t,\delta), \quad s \in \left[\Gamma_t \setminus J_t\right] \cap D, \tag{7.28}$$

for all N large. Since the quantities on the left-hand side of (7.28) depend on t continuously, one can clearly choose $C_1(t,N)$ to be a continuous function of t and δ. Hence, a simple compactness argument shows that the estimate (7.28) is (s,t)-locally uniform. In addition, notice that ϕ_e is monotone on each connected piece of $\Gamma_t \setminus (J_t \cup \{c\})$. Therefore, $|e^{N\phi_e(s)}|$ achieves its largest value on $(\Gamma_t \setminus J_t) \cap D$ either at c (when c belongs to Γ_t) or on ∂D. Then, assuming $n = N$, it follows from the properties of the function $\tilde{\rho}(t)$ in (6.20) and the way we construct the conformal maps in (6.11) and (6.19) that $C_1(t,N)$ is bounded away from 1 on closed subsets of C^a_{birth} and C^b_{birth}, and subsets $T \subset O_{\mathsf{one-cut}}$ satisfying (4.1). That is, (7.28) is strongly (s,t)-locally uniform in the notation of Definition 4.2.

Lastly, recall also that the arcs $J_\pm$ were chosen so that $\mathsf{Re}\,(\phi_b(s)) > 0$ on $J_\pm \cap D$. In fact, we can choose them to be level lines of $\mathsf{Re}\,(\phi_b)$. Clearly, the maximal level will depend on the maximal value of $\mathsf{Re}\,(\phi_b)$ on $\partial U_a \cup \partial U_b$ (also on ∂U_c when $t \in C_{\mathsf{split}}$) as well as $\mathsf{Re}\,(\phi_b(c))$. Therefore, there exists a constant $0 < C_2(t,\delta) < 1$ such that

$$\left|F^{2(N-n)}(s)e^{-N\phi_b(s)}\right| < C_2^N(t,\delta), \quad s \in J_\pm \cap D. \tag{7.29}$$

Again, it holds that $C_2(t,\delta)$ depends on its parameters continuously and is bounded away from 1 when t belongs to closed subsets of $\overline{O}_{\mathsf{one-cut}} \setminus \left\{t_{\mathsf{ct}}, e^{2\pi\mathrm{i}/3}t_{\mathsf{cr}}\right\}$ and $n = N$.

Summarizing, we get from (7.26), (6.25), (7.28), and (7.29) that

$$\boldsymbol{\Delta} = \boldsymbol{\mathcal{O}}\left(\max\left\{C_1^N(t,\delta), C_2^N(t,\delta)\right\}\right) \quad \text{on} \quad \Sigma_{\boldsymbol{R}} \setminus \bigcup \partial U_e. \tag{7.30}$$

Estimates (7.27) and (7.30) show that $\boldsymbol{\Delta}$ is uniformly close to zero on $\Sigma_{\boldsymbol{R}}$. Since the entries of $\boldsymbol{\Delta}$ are geometrically small as $\Gamma_t \ni s \to \infty$, $\boldsymbol{\Delta}$ is close to zero in L^2-norm as well. Then it follows from the same analysis as in [14, Corollary 7.108] that $\boldsymbol{R}$ exists for all N large and

$$\left|\boldsymbol{R} - \boldsymbol{I}\right| \leq c_0(t,\delta)N^{-\alpha_t} \tag{7.31}$$

in any matrix norm, where $c_0(t,\delta)$ continuously depends on t and δ, blows up as $\delta \to 0$ or $t \to \infty$, but is bounded as $t \to \infty$ along either C^a_{birth}, C^b_{birth}, or subsets $T \subset O_{\mathsf{one-cut}}$ satisfying (4.1) when $n = N$.

7.7. Solution of RHP-$\boldsymbol{Y}$.

Given $\boldsymbol{R}$, the solution of RHP-$\boldsymbol{R}$, it is straightforward to verify that RHP-$\boldsymbol{S}$ is solved by

$$\boldsymbol{S} = \begin{cases} \boldsymbol{RM} & \text{in} \quad D \setminus \left[(\Gamma_t \setminus J_t) \cup J_+ \cup J_-\right], \\ \boldsymbol{RP}_e & \text{in} \quad U_e. \end{cases} \tag{7.32}$$

Let K_t be a compact set in $\mathbb{C} \setminus \Gamma_t$. We can always adjust quantities δ_e in (6.6) and (6.18) as well as the arcs $J_\pm$ so that K_t lies entirely within one of the unbounded components of the complement of $\Sigma_{\boldsymbol{R}}$. Then it follows from (7.2), (7.6), and (7.32) that

$$\boldsymbol{Y}(z) = \left((b-a)/4\right)^{(n-N)\sigma_3} e^{-N\ell_t^*\sigma_3/2}\boldsymbol{R}(z)\boldsymbol{M}(z)e^{N(g(z;t)+\ell_t^*/2)\sigma_3}F^{(n-N)\sigma_3}(z) \tag{7.33}$$

on K_t. Subsequently, by using (4.14), (7.7), and the definition of F, we see that

$$\begin{aligned} [\boldsymbol{Y}]_{11}(z) &= \left([\boldsymbol{R}]_{11}(z)[\boldsymbol{M}]_{11}(z) + [\boldsymbol{R}]_{12}(z)[\boldsymbol{M}]_{21}(z)\right)D^{N-n}(z;t)e^{ng(z;t)} \\ &= \left([\boldsymbol{R}]_{11}(z) + [\boldsymbol{R}]_{12}(z)F^{-1}(z)\right)A(z;t)D^{N-n}(z;t)e^{ng(z;t)}. \end{aligned}$$

Equation (4.17) now follows from (6.28) and (7.31). To handle compact sets K_t in $\mathbb{C} \setminus (J_t \cup \{c\})$, it is enough to consider only the sets belonging to sufficiently small Hausdorff neighborhood of Γ_t. In this case the curve Γ_t can be deformed locally around K in such a fashion that (7.28) still holds on a deformed curve, perhaps with a different constant. As the rest of the analysis is the same, the full claim (4.17) follows. The (strongly) (z,t)-locally uniform character of (4.17) follows from the continuity properties of $c_0(t,\delta)$ in (7.31).

Take now $K_t \subset \Gamma_t(a,b) \setminus \{c\}$. Then it follows from (7.2), (7.6), (7.32), and (4.14) that

$$[\boldsymbol{Y}]_{11} = [\boldsymbol{R}]_{11}\left([\boldsymbol{M}]_{11\pm} \pm [\boldsymbol{M}]_{12\pm}F_\pm^{2(N-n)}e^{-N\phi_{b\pm}}\right)e^{ng_\pm}D_\pm^{N-n} + \\ + [\boldsymbol{R}]_{12}\left([\boldsymbol{M}]_{21\pm} \pm [\boldsymbol{M}]_{22\pm}F_\pm^{2(N-n)}e^{-N\phi_{b\pm}}\right)e^{ng_\pm}D_\pm^{N-n}.$$

Hence, we get from (7.7), (6.23), (6.4), (4.14), and some algebra that

$$[\boldsymbol{Y}]_{11} = [\boldsymbol{R}]_{11}\left(A_+D_+^{N-n}e^{ng_+} + A_-D_-^{N-n}e^{ng_-}\right) + \\ + [\boldsymbol{R}]_{12}\left(B_+D_+^{N-n}e^{ng_+} + B_-D_-^{N-n}e^{ng_-}\right).$$

Therefore, we deduce from the definition of F that

$$[\boldsymbol{Y}]_{11} = \left([\boldsymbol{R}]_{11} + [\boldsymbol{R}]_{12}F_+^{-1}\right)A_+D_+^{N-n}e^{ng_+} + \left([\boldsymbol{R}]_{11} + [\boldsymbol{R}]_{12}F_-^{-1}\right)A_-D_-^{N-n}e^{ng_-}.$$

Formula (4.18) now follows from (6.28) and (7.31).

8. Asymptotic Expansions

In this section, we first improve (7.31) to a full asymptotic expansion following the technique of [16, Theorem 7.8 and Theorem 7.10]. Then we show how recurrence coefficients appear within the matrix $\boldsymbol{Y}$ and use (7.33) and the expansion of $\boldsymbol{R}$ to prove Theorem 4.5. Next, we discuss the so-called *string equations* and prove Theorem 4.8 using them. Finally, we employ Theorem 4.5 and Toda equation (1.29) to prove Theorem 4.3.

8.1. **Error matrix $\boldsymbol{R}$.** Let us show that

$$\boldsymbol{R}(z) \sim \boldsymbol{I} + \sum_{i=1}^{\infty}\boldsymbol{r}_i(z;n-N)N^{-\alpha_t i}, \tag{8.1}$$

(strongly when $n=N$) (z,t)-locally uniformly in $\overline{\mathbb{C}}$, see Definitions 4.1 and 4.2. Indeed, as shown in [16, Theorem 7.8], it holds that

$$\boldsymbol{R} = \boldsymbol{I} + \sum_{k=0}^{\infty}\mathcal{C}\left(\mathcal{C}_{\boldsymbol{\Delta}}^k(\boldsymbol{I})\boldsymbol{\Delta}\right), \tag{8.2}$$

where, as before, $\mathcal{C}$ is the Cauchy transform, $\boldsymbol{\Delta}$ was defined in (7.25), and

$$\mathcal{C}_{\boldsymbol{\Delta}}(\boldsymbol{F}) := \mathcal{C}_-(\boldsymbol{\Delta F}), \quad \boldsymbol{F} \in L^2(\Sigma_{\boldsymbol{R}}). \tag{8.3}$$

It follows from (7.27) and (7.30) that

$$\|\mathcal{C}_{\boldsymbol{\Delta}}\| = \mathcal{O}\left(N^{-\alpha_t}\right) \tag{8.4}$$

(strongly when $n=N$) t-locally uniformly. Define

$$\boldsymbol{\Delta}_l(s) := N^{-\alpha_e}\sum_{k=0}^{l-1}\left(\boldsymbol{M}F^{(n-N)\sigma_3}\boldsymbol{P}_{e,k}F^{(N-n)\sigma_3}\boldsymbol{M}^{-1}\right)(s)N^{-k}, \quad s \in \partial U_e, \tag{8.5}$$

and set $\boldsymbol{\Delta}_l \equiv 0$ on $\Sigma_{\boldsymbol{R}} \setminus \bigcup \partial U_e$. Observe that the coefficient next to N^{-k} depends neither on n nor N individually, but does depend on the difference $n-N$. Clearly, it follows from (7.26) and (7.30) that $\boldsymbol{\Delta}_l$ approximate $\boldsymbol{\Delta}$, that is,

$$\|\boldsymbol{\Delta} - \boldsymbol{\Delta}_l\|_{L^1(\Sigma_{\boldsymbol{R}})} + \|\boldsymbol{\Delta} - \boldsymbol{\Delta}_l\|_{L^2(\Sigma_{\boldsymbol{R}})} + \|\boldsymbol{\Delta} - \boldsymbol{\Delta}_l\|_{L^\infty(\Sigma_{\boldsymbol{R}})} = \mathcal{O}\left(N^{-(\alpha_t+l)}\right) \tag{8.6}$$

(strongly when $n=N$) t-locally uniformly. Define $\mathcal{C}_{\boldsymbol{\Delta}_l}$ as in (8.3) with $\boldsymbol{\Delta}$ replaced by $\boldsymbol{\Delta}_l$. Clearly, $\mathcal{C}_{\boldsymbol{\Delta}_l}$ satisfies (8.4). Moreover, it holds that

$$\boldsymbol{R} - \boldsymbol{I} - \sum_{k=0}^{l-1}\mathcal{C}\left(\mathcal{C}_{\boldsymbol{\Delta}_l}^k(\boldsymbol{I})\boldsymbol{\Delta}_l\right) = \boldsymbol{\mathcal{O}}\left(N^{-\alpha_t(l+1)}\right) \tag{8.7}$$

(strongly when $n=N$) (z,t)-locally uniformly in $\overline{\mathbb{C}}$. Indeed, similarly to (7.31), equations (7.27), (8.4), and deformation of $\Sigma_{\boldsymbol{R}}$ technique yield that

$$\sum_{k=l}^{\infty}\mathcal{C}\left(\mathcal{C}_{\boldsymbol{\Delta}}^k(\boldsymbol{I})\boldsymbol{\Delta}\right) = \boldsymbol{\mathcal{O}}\left(N^{-\alpha_t(l+1)}\right) \tag{8.8}$$

(strongly when $n = N$) t-locally uniformly. Furthermore, it holds by (8.6) and an analogous argument that

$$\sum_{k=0}^{l-1} \mathcal{C}\left(\mathcal{C}^k_{\boldsymbol{\Delta}}(\boldsymbol{I})\left(\boldsymbol{\Delta}-\boldsymbol{\Delta}_l\right)\right) = \boldsymbol{\mathcal{O}}\left(N^{-(\alpha_t+l)}\right) \tag{8.9}$$

(strongly when $n = N$) t-locally uniformly. Finally, were deduce from (8.6) and deformation of $\Sigma_{\boldsymbol{R}}$ technique that

$$\sum_{k=0}^{l-1} \mathcal{C}\left(\left(\mathcal{C}^k_{\boldsymbol{\Delta}}(\boldsymbol{I})-\mathcal{C}^k_{\boldsymbol{\Delta}_l}(\boldsymbol{I})\right)\boldsymbol{\Delta}_l\right) = \boldsymbol{\mathcal{O}}\left(N^{-(2\alpha_t+l)}\right) \tag{8.10}$$

(strongly when $n = N$) t-locally uniformly. Estimates (8.8)–(8.10) imply (8.7). Now, to derive (8.1), it only remains to notice that $\mathcal{C}^k_{\boldsymbol{\Delta}_l}(\boldsymbol{I})\boldsymbol{\Delta}_l$ has an asymptotic expansion in powers of $N^{-\alpha_t}$ whose coefficients up to the order l do not depend on l or n, N individually.

8.2. **Recurrence coefficients.** It follows from (1.27) and (1.28) that

$$\left\{\begin{array}{rcl} \gamma_n^2(t,N) &=& h_n(t,N)/h_{n-1}(t,N), \\ \beta_n(t,N) &=& (P_n)_{n-1}-(P_{n+1})_n, \end{array}\right.$$

where we write $P_n(z;t,N) = z^n + \sum_{k=0}^{n-1}(P_n)_k z^k$. Hence, we get from (7.1) that

$$\boldsymbol{Y}(z)z^{-n\sigma_3} = \boldsymbol{I} + \frac{1}{z}\begin{pmatrix} (P_n)_{n-1} & -\frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_{n-1}} & * \end{pmatrix} + \boldsymbol{\mathcal{O}}\left(\frac{1}{z^2}\right) =: \boldsymbol{I} + \frac{\boldsymbol{Y}_1(n;t,N)}{z} + \boldsymbol{\mathcal{O}}\left(\frac{1}{z^2}\right).$$

Therefore,

$$\left\{\begin{array}{rcl} \gamma_n^2(t,N) &=& \big[\boldsymbol{Y}_1(n;t,N)\big]_{12}\big[\boldsymbol{Y}_1(n;t,N)\big]_{21}, \\ \beta_n(t,N) &=& \big[\boldsymbol{Y}_1(n;t,N)\big]_{11} - \big[\boldsymbol{Y}_1(n+1;t,N)\big]_{11}. \end{array}\right. \tag{8.11}$$

We deduce from (7.33), (4.14), (3.7), and the definition of F that

$$\begin{aligned}\boldsymbol{Y}(z)z^{-n\sigma_3} = \left(\frac{b-a}{4}\right)^{(n-N)\sigma_3} e^{-N\ell_t^*\sigma_3/2}\Bigg(\boldsymbol{I} + \frac{\boldsymbol{R}_1(n;t,N)+\boldsymbol{M}_1(t)}{z} + \\ + \frac{nG_1(t)\sigma_3+(N-n)D_1(t)\sigma_3}{z} + \boldsymbol{\mathcal{O}}\left(\frac{1}{z^2}\right)\Bigg) e^{N\ell_t^*\sigma_3/2}\left(\frac{b-a}{4}\right)^{(N-n)\sigma_3},\end{aligned}$$

where we write

$$\left\{\begin{array}{rcl} e^{g(z;t)} &=& z + G_1(t) + \mathcal{O}\big(z^{-1}\big), \\ D(z;t) &=& 1 + z^{-1}D_1(t) + \mathcal{O}\big(z^{-2}\big), \\ \boldsymbol{K}(z) &=& \boldsymbol{I} + z^{-1}\boldsymbol{K}_1(n;t,N) + \boldsymbol{\mathcal{O}}\big(z^{-2}\big), \quad \boldsymbol{K}\in\{\boldsymbol{M},\boldsymbol{R}\}. \end{array}\right.$$

In fact, it follows from (4.12) and (6.22) that $D_1(t) = 1/4x^2(t)$, and analogously we deduce from (4.14) and (4.16) that $G_1(t) = -x(t) + 1/4x^2(t)$. Further, using (7.7), (4.15), and (3.7) we see that

$$\boldsymbol{M}_1(t) = \begin{pmatrix} 0 & -1/\sqrt{2x(t)} \\ 1/\sqrt{2x(t)} & 0 \end{pmatrix}.$$

Therefore (8.11) can be rewritten as

$$\left\{\begin{array}{rcl} \gamma_n^2(t,N) &=& -1/(2x(t)) + \big([\boldsymbol{R}_1]_{12}-[\boldsymbol{R}_1]_{21}\big)/\sqrt{2x(t)} + [\boldsymbol{R}_1]_{12}[\boldsymbol{R}_1]_{21} \\ \beta_n(t,N) &=& x(t) + \big[\boldsymbol{R}_1(n;t,N)-\boldsymbol{R}_1(n+1;t,N)\big]_{11}. \end{array}\right. \tag{8.12}$$

Hence, (4.4) follows from (8.1) and (8.12). Moreover, as $\gamma_N^2(t,N)$ is expressed only through $\boldsymbol{R}_1(N;t,N)$, its expansion is strongly t-locally uniform.

To see the analyticity of $G_k(t;n-N)$ and $B_k(t;n-N)$ in $O_{\mathsf{one-cut}}$, we need to examine the dependence of $\boldsymbol{R}_1$ on t. To this end, write

$$\boldsymbol{R}_1 = -\frac{1}{2\pi i}\oint_{\cup\partial U_e}\left(\sum_{k=0}^{l-1}\mathcal{C}^k_{\boldsymbol{\Delta}_l}(\boldsymbol{I})\boldsymbol{\Delta}_l\right)(s)ds + \boldsymbol{\mathcal{O}}\left(N^{-(l+1)}\right), \tag{8.13}$$

which we can do according to (8.7), where $\boldsymbol{\Delta}_l$ is given by (8.5). Notice that on each ∂U_e the function $\boldsymbol{\Delta}_l$ is a trace on ∂U_e of a meromorphic matrix-valued function in U_e with a single pole at e. Indeed, it is clear that $\boldsymbol{\Delta}_l$ is holomorphic in $U_e \setminus J_t$ and on J_t it holds that

$$\begin{aligned}\boldsymbol{\Delta}_{l+} &= N^{-\alpha_e}\boldsymbol{M}_- F_-^{(n-N)\sigma_3}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\left(\sum_{k=0}^{l-1}\boldsymbol{P}_{e,k+}N^{-k}\right)F_+^{(N-n)\sigma_3}\boldsymbol{M}_+^{-1}\\ &= N^{-\alpha_e}\boldsymbol{M}_- F_-^{(n-N)\sigma_3}\left(\sum_{k=0}^{l-1}\boldsymbol{P}_{e,k-}N^{-k}\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}F_+^{(N-n)\sigma_3}\boldsymbol{M}_+^{-1} = \boldsymbol{\Delta}_{l-},\end{aligned}$$

where the first and the last equalities follow from RHP-$\boldsymbol{M}$(b) and (6.23) while the second can be verified by using the explicit expressions (7.12) and (6.9), (7.15) and (6.14), (7.21) and (6.17). Hence, $\boldsymbol{\Delta}_l$ is indeed meromorphic in each U_e with a pole at e. From this, it is easy to see that $\mathcal{C}^k_{\boldsymbol{\Delta}_l}(\boldsymbol{I})\boldsymbol{\Delta}_l$ is a trace on ∂U_e of a meromorphic matrix-valued function in U_e with a single pole at e. In particular, the integral in (8.13) does not depend on the radii of the disks U_e. Thus, its t-dependence comes only from the points a, b, c and the conformal maps constructed in (6.11), (6.14), (6.17), and (6.19). It follows from Proposition 3.1 that $a(t)$, $b(t)$, $c(t)$ are holomorphic functions of $t \in O_{\mathsf{one-cut}}$ with holomorphic continuations across each of the arcs C_{split}, C^a_{birth}, and C^b_{birth}. The conformal maps $(-\phi_a)^{2/3}$ and $(-\phi_b)^{2/3}$, see (6.11) and (6.2), have the same type of dependence on t. Thus, we indeed see that the functions $\boldsymbol{R}_1$ are analytic functions of $t \in O_{\mathsf{one-cut}}$.

Let now $n = N$. The first claim of (4.6) was derived in [7, Corollary 4.2]. Observe that to show the second claim, it is enough to prove that

$$\boldsymbol{R}_1 = \sum_{1\le 2j+1\le l-1}\begin{pmatrix}-p_{2j+1} & q_{2j+1}\\ q_{2j+1} & p_{2j+1}\end{pmatrix}N^{-2j-1} + \sum_{2\le 2j\le l-1}\begin{pmatrix}p_{2j} & q_{2j}\\ -q_{2j} & p_{2j}\end{pmatrix}N^{-2j} + \boldsymbol{\mathcal{O}}\left(N^{-(l+1)}\right) \tag{8.14}$$

for some constants p_k, q_k, as then it obviously holds that $[\boldsymbol{R}_1]_{12} - [\boldsymbol{R}_1]_{21}$ and $[\boldsymbol{R}_1]_{12}[\boldsymbol{R}_1]_{21}$ have asymptotic expansions only in even powers of N^{-1}. Using (7.12) and (7.7), it is tedious but straightforward to verify that the expansion for $\boldsymbol{\Delta}_l$ has exactly the same form as the right-hand side of (8.14) (without $\boldsymbol{\mathcal{O}}$-term), where

$$\begin{cases} p_{2j} & \mapsto \quad (s_{2j}+t_{2j})(-\phi_e/2)^{-2j},\\ q_{2j} & \mapsto \quad \mathrm{i}\varepsilon_e(s_{2j}-t_{2j})(-\phi_e/2)^{-2j},\\ p_{2j+1} & \mapsto \quad 2\mathrm{i}AB\big(s_{2j+1}+t_{2j+1}+\varepsilon_e(s_{2j+1}-t_{2j+1})\big)(-\phi_e/2)^{-2j-1},\\ q_{2j+1} & \mapsto \quad -2AB\big(\varepsilon_e(s_{2j+1}-t_{2j+1})+s_{2j+1}+t_{2j+1}\big)(-\phi_e/2)^{-2j-1},\end{cases}$$

on ∂U_e. Clearly, $\mathcal{C}_{\boldsymbol{\Delta}_l}(\boldsymbol{I}) = \mathcal{C}_-(\boldsymbol{\Delta}_l)$ also has the same form as the right-hand side of (8.14). Another boring computation shows that the product $\mathcal{C}_{\boldsymbol{\Delta}_l}(\boldsymbol{I})\boldsymbol{\Delta}_l$ has the same form as well. By induction, we get that all the summands under the integral sign in (8.13) have this form, from which (8.14) clearly follows.

8.3. **String Equations.** To prove Theorem 4.8, we need to introduce *discrete string equations* (see, e.g., [9]):

$$\begin{cases}\gamma_n[V'(\boldsymbol{Q})]_{n,n-1} = \dfrac{n}{N},\\ [V'(\boldsymbol{Q})]_{nn} = 0,\end{cases}$$

where $[\boldsymbol{A}]_{nm}$ is the (n,m)-th element of the matrix $\boldsymbol{A}$,

$$\boldsymbol{Q} = \begin{pmatrix}\beta_0 & \gamma_1 & 0 & 0 & \dots\\ \gamma_1 & \beta_1 & \gamma_2 & 0 & \dots\\ 0 & \gamma_2 & \beta_2 & \gamma_3 & \dots\\ 0 & 0 & \gamma_3 & \beta_3 & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

and γ_n and β_n are recurrence coefficients (1.27) for polynomials satisfying orthogonality relations (1.26) with respect to a potential V. For the potential V as in (1.24), the discrete string equations become

$$\begin{cases} \gamma_n\big[-\boldsymbol{Q}^2+t\boldsymbol{I}\big]_{nn-1}=\dfrac{n}{N}\,, \\ \big[-\boldsymbol{Q}^2+t\boldsymbol{I}\big]_{nn}=0, \end{cases}$$

where, this time, $\boldsymbol{I}$ is the semi-infinite identity matrix. This gives the equations

$$\begin{cases} \gamma_n^2(\beta_{n-1}+\beta_n)=-\dfrac{n}{N}\,, \\ \gamma_{n+1}^2+\gamma_n^2+\beta_n^2=t. \end{cases} \tag{8.15}$$

To prove (4.9) using (8.15), let us set

$$v:=\left(\frac{n+1/2}{N}\right)^{-2/3} \quad \text{and} \quad u_*:=\left(\frac{n}{N}\right)^{-2/3}=v\left(1-\frac{v^{3/2}}{2N}\right)^{-2/3}.$$

Then for all N large, the value $\widehat{B}_k(t,u_*)$ can be computed as a series

$$\widehat{B}_k(t,u_*)=\sum_{i=0}^\infty \frac{\partial^i B_k}{\partial u^i}(t,v)v^i\left(\left(1-\frac{v^{3/2}}{2N}\right)^{-2/3}-1\right)^i.$$

Using the Taylor expansion of $(1-x)^{-2/3}$ at the origin, we can rewrite the above expression as a series in powers of N with coefficients that are holomorphic in $\mathcal{N}$ functions. Thus, (4.8) can be equivalently written as

$$\beta_n(t,N)\sim\sum_{k=0}^\infty \widetilde{B}_k(t,v)N^{-k},$$

where the functions $\widetilde{B}_k(t,v)$ are holomorphic in $\mathcal{N}$. Hence, to prove (4.9) we need to show that $\widetilde{B}_{2j-1}(t,v)\equiv 0$ in $\mathcal{N}$. To this end, let us represent each $\widetilde{B}_k(t,v)$ as a series in powers of N^{-1}:

$$\widetilde{B}_k(t,v)=\sum_{j=0}^\infty \widetilde{B}_{k,j}(t,u_*)N^{-j}, \quad v=\left(\frac{n+1/2}{N}\right)^{-2/3}=u_*\left(1+\frac{u_*^{3/2}}{2N}\right)^{-2/3}.$$

Notice also that

$$\widetilde{B}_{k,0}(t,u)=\widetilde{B}_k(t,u) \quad \text{and} \quad \widetilde{B}_{0,0}(t,u)=x(tu)/\sqrt{u}. \tag{8.16}$$

Then it also holds that

$$\widetilde{B}_k(t,w)=\sum_{j=0}^\infty \widetilde{B}_{k,j}(t,u_*)(-N)^{-j}, \quad w=\left(\frac{n-1/2}{N}\right)^{-2/3}=u_*\left(1-\frac{u_*^{3/2}}{2N}\right)^{-2/3}.$$

Hence, we get that

$$\beta_n(t,N)+\beta_{n-1}(t,N)\sim 2\sum_{k=0}^\infty\left(\sum_{j=0}^\infty \widetilde{B}_{k,2j}(t,u_*)N^{-2j}\right)N^{-k}, \tag{8.17}$$

with the expansion valid locally uniformly in both variables. Then the constant term in the expansion of $\gamma_n^2(\beta_{n-1}+\beta_n)$ is equal to

$$-\frac{2\widetilde{B}_{0,0}(t,u_*)}{2u_*x(u_*t)}=-u_*^{-3/2}=-\frac{n}{N}$$

by (4.8) and (8.16). Thus, it follows from the first relation in (8.15) that the rest of the terms in the expansion of $\gamma_n^2(\beta_{n-1}+\beta_n)$ must be equal to zero. The N^{-1}-term is given by

$$-\frac{\widetilde{B}_{2l+1,0}(t,u_*)}{u_*x(u_*t)}=-\frac{\widetilde{B}_{2l+1}(t,u_*)}{u_*x(u_*t)}, \quad l=0, \tag{8.18}$$

by (4.8), (8.17), and (8.16). This implies that $\widetilde{B}_1(t,u_*)=0$. As we can vary n and N, the last equality holds on a set with a limit point in $\mathcal{N}_t$. Hence, $\widetilde{B}_1(t,\cdot)\equiv 0$ by holomorphy in the second variable. Assuming that $\widetilde{B}_{2l-1}(t,\cdot)\equiv 0$ for all $l\leq L$, we get from (4.8), (8.17), and (8.16) that the $N^{-(2L+1)}$-term in the expansion of $\gamma_n^2(\beta_{n-1}+\beta_n)$ is given by (8.18) with

$l = L$. Previous argument yields that $\widetilde{B}_{2L+1}(t,\cdot) \equiv 0$ and the desired claim now follows from the principle of mathematical induction.

8.4. **Free energy.** In [6, Proposition 5.1], it was shown that the free energy $F_N(t)$ and the recurrence coefficient $\gamma_N(t,N)$ satisfy Toda equation (1.29) for all real $t > t_{\mathsf{cr}}$. It was further shown in [6] that

$$F_N(t) = \frac{2}{3}t^{3/2} - \frac{1}{4}\log(4t) + \int_\infty^t \int_\infty^\tau \left(\gamma_N^2(\sigma,N) - \frac{1}{2\sqrt{\sigma}} - \frac{1}{4\sigma^2}\right) \mathrm{d}\sigma \mathrm{d}\tau, \tag{8.19}$$

where the integrals are taken along positive reals. It was also proved in [6] that an asymptotic expansion for $F_N(t)$ can be obtained by simply plugging the asymptotic expansion for $\gamma_N^2(t,N)$ into (8.19) and integrating term by term; that is, (4.2) is valid uniformly on closed subsets of (t_{cr},∞), where the functions $F^{(2k)}(t)$ can be computed via the following equations:

$$\begin{cases} F^{(0)}(t) & = & \dfrac{2}{3}t^{3/2} - \dfrac{1}{4}\log(4t) + \displaystyle\int_\infty^t \int_\infty^\tau \left(-\frac{1}{2x(\sigma)} - \frac{1}{2\sqrt{\sigma}} - \frac{1}{4\sigma^2}\right) \mathrm{d}\sigma \mathrm{d}\tau, \\ F^{(2k)}(t) & = & \displaystyle\int_\infty^t \int_\infty^\tau G_{2k}(\sigma;0)\mathrm{d}\sigma\mathrm{d}\tau, \quad k \geq 1, \end{cases} \tag{8.20}$$

(the integrals in (8.20) are well defined as it was shown that $G_{2k}(t;0) = \mathcal{O}\big(t^{-7/2}\big)$ and $x(t) = -\sqrt{t} + \frac{1}{2t} + \mathcal{O}\big(t^{-5/2}\big)$ uniformly as $O_{\mathsf{one-cut}} \ni t \to \infty$).

Since $Z_N(t)$ is an entire function of t, the free energy $F_N(t)$ is a meromorphic function of the parameter t. Hence, Toda equation (1.29) extends to the entire parameter plane. Recall that $\gamma_N^2(t,N)$ are holomorphic functions of the parameter t on each closed subsets of $O_{\mathsf{one-cut}}$ satisfying (4.1) for all N large enough (depending on the subset). Hence, using (3.4) and some algebra, we can rewrite (8.19) as

$$F_N(t) = 1 - \frac{2}{3}x^3(t) - \frac{1}{2}\log\big(-2x(t)\big) + \int_\infty^t \int_\infty^\tau \left(\gamma_N^2(\sigma,N) + \frac{7x'(\sigma) + 2\sigma x''(\sigma)}{6}\right) \mathrm{d}\sigma \mathrm{d}\tau,$$

where $x'(t)$ is the derivative of $x(t)$ with respect to t. Hence, for any closed subset $T \subset O_{\mathsf{one-cut}}$ satisfying (4.1) there exists a constant $N(T)$ such that the functions $F_N(t)$ are holomorphic on T. Similarly, we see that the functions $F^{(2k)}(t)$ are in fact holomorphic in $O_{\mathsf{one-cut}}$ and can be holomorphically extended across C_{split}, C^a_{birth}, and C^b_{birth}. This, in particular, gives formula (4.3). Thus,

$$F_N(t) - \sum_{k=0}^{K-1} F^{(2k)}(t)N^{-2k} = \int_\infty^t \int_\infty^\tau \left(\gamma_N^2(\sigma,N) - \sum_{k=0}^{K-1} G_{2k}(t;0)N^{-2k}\right) \mathrm{d}\sigma\mathrm{d}\tau = \mathcal{O}\big(N^{-2K}\big)$$

uniformly on closed subsets of $O_{\mathsf{one-cut}}$ satisfying (4.1), which implies that asymptotic expansion (4.2) does indeed hold in $O_{\mathsf{one-cut}}$ as claimed.

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