PADÉ APPROXIMANTS TO CERTAIN ELLIPTIC-TYPE FUNCTIONS

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ABSTRACT. Given non-collinear points a_1, a_2, a_3, there is a unique compact $\Delta \subset \mathbb{C}$ that has minimal logarithmic capacity among all continua joining a_1, a_2, and a_3. For a complex-valued non-vanishing Dini-continuous function h on Δ, we define
$$f_h(z) := \frac{1}{\pi i} \int_\Delta \frac{h(t)}{t-z} \frac{dt}{w^+(t)},$$
where $w(z) := \sqrt{\prod_{k=0}^3 (z-a_k)}$ and w^+ is the one-sided value according to some orientation of Δ. In this work we present strong asymptotics of diagonal Padé approximants to f_h and describe the behavior of the spurious pole and the regions of locally uniform convergence from a generic perspective.

1. INTRODUCTION

A truncation of continued fractions in the field of Laurent series in one complex variable, Padé approximants are among the oldest and simplest constructions in function theory [27]. These are rational functions of type[1] (m,n) that interpolate a function element at a given point with order $m+n+1$. They were introduced for the exponential function by Hermite [26], who used them to prove the transcendency of e, and later expounded more systematically by his student Padé [43]. Ever since their introduction, Padé approximants have been an effective device in analytic number theory [26, 49, 50, 29], and over the last decades they became an important tool in physical modeling and numerical analysis; the reader will find an introduction to such topics, as well as further references, in the monograph [7], see also [16, 17, 46].

Still, convergence properties of Padé approximants are not fully understood as yet. Henceforth, for simplicity, we only discuss approximants of type (n,n) (the so-called *diagonal* approximants) which are most natural since they treat poles and zeros on equal footing. For restricted classes of functions, such approximants were proven to converge, locally uniformly in the domain of analyticity, as n goes large. These classes include Markov functions and rational perturbation thereof [37, 23, 47, 12], Cauchy transforms of continuous non-vanishing functions on a segment [11, 42, 36] (in these examples interpolation takes place at infinity), and certain entire functions such as Polya frequencies or functions with smooth and fast decaying Taylor coefficients [5, 33, 34] (interpolation being now at the origin). However, such favorable cases do not reflect the general situation which is that Padé approximants often fail to converge locally uniformly, due to the occurrence of "spurious" poles that may wander about the domain of analyticity. The so-called Padé conjecture, actually raised by Baker, Gammel and Wills [6], laid hope for the next best thing namely convergence of a subsequence in the largest disk of holomorphy, but this was eventually settled in the negative by D. Lubinsky [35]. Shortly after, a weaker form of the conjecture due to H. Stahl [57], dealing with hyperelliptic functions, was disproved as well by V. Buslaev [15].

Nevertheless, spurious poles are no obstacle to some weaker type of convergence. Indeed, convergence in (logarithmic) capacity of Padé approximants to functions with singular set of capacity zero (the prototype of which is an entire function) was established by J. Nuttall and Ch. Pommerenke [41, 45]. Later, in

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[1]A rational function is said to be of type (m,n) if it can be written as the ratio of a polynomial of degree at most m and a polynomial of degree at most n.

his pathbreaking work [52, 53, 54, 55, 58] dwelling on earlier study by J. Nuttall and S. Singh [42, 39], H. Stahl proved an analogous result for (branches of) functions f having multi-valued meromorphic continuation over the plane deprived of a set of zero capacity (the prototype of which is an algebraic function). Here, there is an additional problem of identifying the convergence domain, since the approximants are single-valued in nature but the approximated function is not. It turns out to be characterized, among all domains on which f is meromorphic and single-valued, as one whose complement has minimal capacity.[2] As n tends to infinity, this complement attracts almost all the poles of the Padé approximant of order n (that is, all but at most $o(n)$ of them). However, the actual limit set of the poles can be significantly larger, possibly the whole complex plane, which is the reason why uniform convergence may fail.

When the singular set of f consists of finitely many branchpoints, the complement Δ of the convergence domain is a (generally branched) system of analytic cuts without loop, whose loose ends are branchpoints of f, which is called an S-contour. Here the prefix "S" stands for "symmetric", meaning that the equilibrium potential of Δ has equal normal derivative from each side at every smooth point. Actually, this symmetry property expresses that the first order variation of the capacity is zero under small distortions of the contour.

Stahl's work dwells on the classical and fruitful connection between Padé approximants and orthogonal polynomials: if f can be expressed as the Cauchy integral of a compactly supported (possibly complex) measure ν, then the denominator of the Padé approximant of type (n, n) to f at infinity is orthogonal to all polynomials of degree at most $n - 1$ for the non-Hermitian[3] scalar product defined by ν in $L^2(|\nu|)$. In the case of Markov functions, ν is a positive measure supported on a real segment (a segment is the simplest example of an S-contour), so the orthogonality is in fact Hermitian and the limiting behavior of the denominator can be addressed using classical asymptotics of orthogonal polynomials [59]. In this connection, the work [54] provides one with a non-Hermitian generalization to arbitrary S-contours of that part of the theory of orthogonal polynomials on a segment dealing with weak (i.e., n-th root) asymptotics.[4]

To determine subregions where uniform convergence of Padé approximants takes place, if any, one needs to analyze the behavior of *all* the poles when n goes large and not just $o(n)$ of them. For functions with finitely many branchpoints, in light of the previous discussion, it is akin to carrying over to a non-Hermitian context, over general S-contours, the Szegő theory of strong asymptotics for orthogonal polynomials.

On a segment K, strong asymptotics for non-Hermitian orthogonal polynomials q_n with respect to an absolutely continuous complex measure of the form $h d\omega_K$, with ω_K the equilibrium measure of K, was obtained in [11, 42, 39, 41] *via* the study of certain singular integral equations (see [1, 3, 9, 10] for generalizations to varying weights over analytic arcs). When the density h is smooth and does not vanish, the asymptotics is similar to classical one: up to normalization, q_n is equivalent for large n to Φ^n/S, locally uniformly outside of K, where Φ conformally maps the complement of K to the complement of the unit disk and S is an auxiliary function, the Szegő function of h, which solves a Riemann-Hilbert problem across K and has no zeros. In particular K attracts all zeros of q_n asymptotically, which is equivalent to saying that there are no spurious poles in Padé approximation to the Cauchy transform of $h d\omega_K$.

When h does have zeros (and even in the Hermitian case if it has non-convex support), spurious poles appear whose number can sometimes be estimated from the nature of the zeroing and the smoothness of $\arg h$ [42, 51, 56, 8]. But it is S.P. Suetin, for analytic non-vanishing densities on an S-contour comprised of finitely many disjoint arcs (in particular on a finite union of real intervals), who converted Nuttall's

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[2]This characterization is up to a set of zero capacity only, but the union of all such domains is again a convergence domain, maximal with respect to set-theoretic-inclusion, that we call *the* convergence domain in capacity of Padé approximants to f.

[3]This means there is no conjugation involved, *i.e.* the scalar product is $\langle g, h \rangle := \int g h \, d\nu$.

[4]Note that f can be written as the Cauchy integral of the difference between its values from each side of the S-contour, at least if the branchpoint have order > -1; if not, special treatment is needed at the endpoints, see [54].

singular equation approach to non-Hermitian orthogonality [41] into an affine Riemann-Hilbert problem on a hyperelliptic Riemann surface and who linked the occurrence of spurious poles to the outcome of a Jacobi inversion process [60]. In the case of two arcs the Riemann surface is elliptic (i.e., it has genus 1), and there is at most one spurious pole whose recurrent behavior can be explained by the rational independence of the equilibrium weights of the arcs, much for the same reason why a line with irrational slope embedded in the unit torus fills a dense subset of the latter [61] (see [2, 4] for a generalization). These results stress a parallel between non-Hermitian orthogonality and the theory of Hermitian orthogonal polynomials on a system of curves initiated by H. Widom [63, 22, 44].

Now, Suetin's work tells us about spurious poles of functions with four branchpoints of order 2 in special position, namely the associated S-contour should consist of two disjoint arcs. In the present paper, drawing inspiration from [61], we deal with the case of three branchpoints in arbitrary (non-collinear) position. The corresponding S-contour is a threefold, and thus we consider orthogonality on a non-smooth (in fact branched) contour. For a special class of Jacobi polynomials, such a setting was considered by Nuttall [40], using a different method, and just recently by Martínez Finkelshtein, Rakhmanov, and Suetin in [18]. Here, using classical properties of singular integrals, we handle at little extra cost Dini-continuous non-vanishing densities (that may not be analytic). Moreover, we put our results in generic perspective with respect to the location of the branchpoints employing differential geometric tools and properties of quadratic differentials.

We first identify a convenient Riemann surface $\mathfrak{R}$ and a suitable curve $L \subset \mathfrak{R}$ over which we can lift non-Hermitian orthogonality on the threefold into a Riemann-Hilbert problem. This step is somewhat more involved than in [60] but the surface we construct is still elliptic.

Next, to analyze the Riemann-Hilbert problem thus obtained in each degree n, we first solve it explicitly when the density is the reciprocal of a polynomial, in terms of (the two branches of) some auxiliary function S_n. Then, to handle an arbitrary Dini-continuous non-vanishing density, we approximate it by a sequence of reciprocal of polynomials and regard this case as a perturbation of the previous one, using some singular integral theory.

The function S_n is holomorphic in $\mathfrak{R} \backslash L$ and plays here the role of the product Φ^n / S which is the main term in the asymptotics of q_n in the segment case. It has at most one finite zero, given by the solution of a Jacobi inversion problem (which depends on n). When this zero belongs to the first sheet of the covering, it generates a spurious pole nearby, whereas there are no spurious poles when it belongs to the second sheet.

To describe the dynamics of this wandering zero, we proceed as in [61] by mapping the Jacobi inversion problem to an equation on the Jacobian variety of $\mathfrak{R}$ (which is a torus). There, the image of the zero evolves according to a discrete linear dynamical system whose coefficients depend on the equilibrium weights of the arcs of the threefold. The spurious pole recurs in a dense manner if these equilibrium weights are rationally independent, and eventually disappears or clusters to a union of disjoint arcs (resp. points) if they are rationally dependent but one of them is irrational (resp. if they are all rational). We establish that the recurrent case is generic in the measure-theoretic sense, and is one where the domain of convergence of the sequence of Padé approximants is empty although some subsequence converges locally uniformly in the complement of the threefold. Still, the clustering case densely occurs and is one where the domain of convergence is nonempty.

The paper is organized as follows. In Section 2 we construct the surface $\mathfrak{R}$, we introduce our main objects of study (sectionally holomorphic functions, Padé approximants), and we state our results. In Section 3, we discuss Cauchy integrals and use them to construct Szegő functions on the threefold. Section 4 contains basic facts on Abelian differentials, which are used in Section 5 to devise certain sectionally meromorphic functions on $\mathfrak{R}$. These are instrumental in Section 6 where we construct S_n, derive formulae for the product and ratio of its branches, and analyze the behavior of its wandering zero. At last, in Section 7,

we solve our initial Riemann-Hilbert problem in terms of S_n and prove the announced asymptotics for non-Hermitian orthogonal polynomials and Padé approximants to Cauchy integrals on the threefold.

   The author's motivation for writing the present paper has been twofold. On the one hand, we aimed at carrying over to more general geometries and putting in generic perspective the mechanism behind the dynamics of spurious poles, first unveiled by Suetin, which offers beautiful connections with classical function theory on Riemann surfaces. On the other hand, we wanted to illustrate that Szegő's theory of orthogonal polynomials can be generalized to both non-Hermitian and non-smooth context, and does not owe that much to positivity.

2. MAIN RESULTS

2.1. **Chebotarëv Continua.** Let a_1, a_2, and a_3 be three non-collinear points in the complex plane $\mathbb{C}$. There exists a unique connected compact $\Delta = \Delta(a_1, a_2, a_3)$, called Chebotarëv continuum [24, 31, 32, 30], containing these points and having minimal *logarithmic capacity* [48] among all continua connecting a_1, a_2, and a_3. It consists of three analytic arcs Δ_k, $k \in \{1, 2, 3\}$, emanating from a common endpoint a_0, called the Chebotarëv center, and ending at each of the given points a_k, respectively. It is also known that the tangents at a_0 of two adjacent arcs form an angle of magnitude $2\pi/3$. In what follows, we assume that the arcs Δ_k and the corresponding points a_k, are ordered clockwise with respect to a_0 (see Figure 1). The interiors $\Delta_k^\circ := \Delta_k \setminus \{a_0, a_k\}$ of the arcs Δ_k can be described (see *e.g.* [30, Thm. 1.1]) as the (negative) critical trajectories of the quadratic differential

$$(2.1) \qquad \frac{1}{\pi} \frac{z - a_0}{\prod_{k=1}^3 (z - a_k)} (dz)^2.$$

In other words, for any smooth parametrization $z_k(t)$, $t \in [0, 1]$, of Δ_k, it holds that

$$(2.2) \qquad \frac{1}{\pi} \frac{z_k(t) - a_0}{\prod_{k=1}^3 (z_k(t) - a_k)} (z_k'(t))^2 < 0 \quad \text{for all} \quad t \in (0, 1).$$

   In what follows, we denote by D the complement of Δ in the extended complex plane $\overline{\mathbb{C}}$ and orient each arc Δ_k from a_0 to a_k. According to this orientation we distinguish the left $(+)$ and right $(-)$ sides of each Δ_k° and therefore of $\Delta^\circ := \cup_{k=1}^3 \Delta_k^\circ = \Delta \setminus \{a_0, a_1, a_2, a_3\}$.
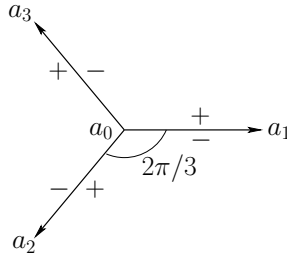

FIGURE 1. Contour Δ consists of the arcs Δ_k which are oriented from a_0 to a_k and are numbered clockwise. The left and right sides of the arcs Δ_k are labeled by signs $+$ and $-$, respectively.

   On some occasions, it will be more useful to consider the boundary of D as a limit of simple Jordan curves encompassing Δ and whose exterior domains exhaust D. Thus, we define ∂D to be the "curve" consisting of two copies of each arc Δ_k°, the left and right sides, three copies of a_0, and a single copy of each a_k. We assume ∂D to be oriented clockwise, that is, D lies to the left of ∂D when the latter is traversed in the positive direction.

The following function plays a prominent role throughout this work. We set

$$(2.3) \qquad w(z) := \sqrt{\prod_{k=0}^{3}(z - a_k)}, \quad \frac{w(z)}{z^2} \to 1 \quad \text{as} \quad z \to \infty,$$

which is a holomorphic function in $D \setminus \{\infty\}$. It is easy to see that w has continuous trace on ∂D and it holds that $w^+ = -w^-$, where w^+ and w^- are the traces of w from the left and right, respectively, on each Δ_k.

2.2. **An Elliptic Riemann Surface.** Let $\mathfrak{R}$ be the Riemann surface defined by w. The genus of the Riemann surface of an algebraic function is equal to the number of branch points divided by two, plus one, minus the order of branching. Thus, $\mathfrak{R}$ has genus 1, that is, $\mathfrak{R}$ is an elliptic Riemann surface. We represent $\mathfrak{R}$ as two-sheeted ramified cover of $\overline{\mathbb{C}}$ constructed in the following manner. Two copies of $\overline{\mathbb{C}}$ are cut along each arc Δ_k° comprising Δ. These copies are joint at each point a_k and along the cuts in such a manner that the right (left) side of each Δ_k° of the first copy, say $\mathfrak{R}^{(1)}$, is joined with the left (right) side of the respective Δ_k° of the second copy, $\mathfrak{R}^{(2)}$, Figure 3. Thus, to each arc Δ_k in $\mathbb{C}$ there corresponds a cycle L_k on $\mathfrak{R}$.

We denote by L the union $L_1 \cup L_2 \cup L_3$ and by π the canonical projection $\pi : \mathfrak{R} \to \overline{\mathbb{C}}$. In particular, it holds that $\pi(L_k) = \Delta_k$. Each point in $\overline{\mathbb{C}}$ has two preimages on $\mathfrak{R}$ under π except for the points a_k, $k \in \{0, 1, 2, 3\}$, which have only one preimage. For each $z \in D$ we set $z^{(k)} := \pi^{-1}(z) \cap \mathfrak{R}^{(k)}$, $k \in \{1, 2\}$, and write $\mathbf{z}$ for a generic point on $\mathfrak{R}$ such that $\pi(\mathbf{z}) = z$. We call two points *conjugate* if they have the same canonical projection and denote the conjugation operation by the superscript $*$. Furthermore, we put $D^{(k)} := \pi^{-1}(D) \cap \mathfrak{R}^{(k)}$. We orient each L_k in such a manner that $D^{(1)}$ remains on the left when L_k is traversed in the positive direction and these orientations induce an orientation on L, Figure 2.
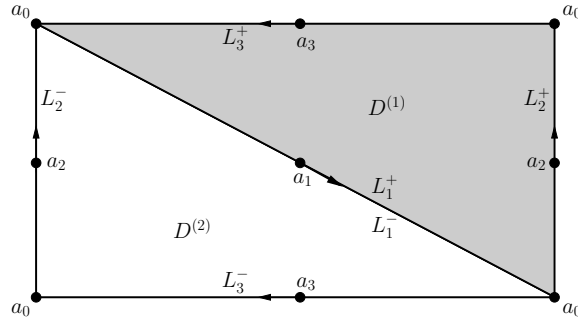

FIGURE 2. Elliptic Riemann surface $\mathfrak{R}$ has genus 1 and therefore is homeomorphic to a torus. We represent $\mathfrak{R}$ as a torus cut along curves L_2 and L_3. In this case domains $D^{(1)}$ and $D^{(2)}$ can be represented as the upper and lower triangles, respectively.

We identify $D^{(1)}$ with D and L^+ with ∂D. This means that we consider each function defined on D as a function defined on $D^{(1)}$. In particular, we set

$$(2.4) \qquad w(\mathbf{z}) = \begin{cases} w(z), & \mathbf{z} \in D^{(1)}, \\ -w(z), & \mathbf{z} \in D^{(2)}. \end{cases}$$

Clearly, w extends continuously to each side of L and the traces of w on L^+ and L^- coincide. Thus, w is holomorphic across L by the principle of analytic continuation. That is, w is in fact a rational function over $\mathfrak{R}$ (a holomorphic map from $\mathfrak{R}$ into $\overline{\mathbb{C}}$).

2.3. **Sectionally Meromorphic Functions.** A holomorphic (resp. meromorphic) function on $\mathfrak{R} \setminus L$ is called sectionally holomorphic (resp. meromorphic). In this section we discuss some properties of sectionally holomorphic functions on $\mathfrak{R}$ that we use further below.

Let r_h be a meromorphic function in $\mathfrak{R} \setminus L$ with continuous traces on L that satisfy

$$(2.5) \qquad\qquad r_h^- = r_h^+ \cdot (h \circ \pi),$$

where h is a continuous function on $\Delta \setminus \{a_0\}$ which extends continuously to each Δ_k. The function r_h gives rise to two meromorphic functions in D, namely,

$$(2.6) \qquad\qquad r_h(z) := r_h(z^{(1)}) \quad \text{and} \quad r_h^*(z) := r_h(z^{(2)}), \quad z \in D,$$

where we abuse notation in that we use r_h to stand for both a function on $\mathfrak{R}$ and its restriction to D (as mentioned before, we identify D with $D^{(1)}$). We call r_h and r_h^* the *conjugate functions derived from* r_h.
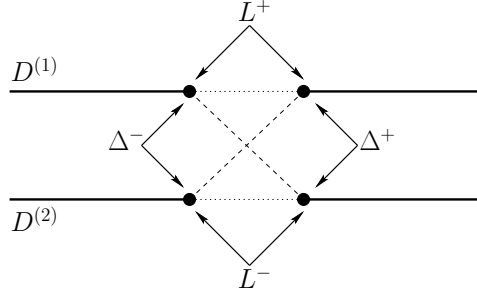

FIGURE 3. Domains $D^{(1)}$ and $D^{(2)}$ are represented as upper and lower layers, respectively (two thick horizontal lines each). Each pair of disks joint by a dotted line represents the same point on Δ as approached from the left (Δ^-) and from the right (Δ^+). Each pair of disk joint by a punctured line represents the same point on L as approached from the left (L^-) and from the right (L^+). The left and right sides are chosen according to the orientation of each contour in question.

Since the domains $D^{(k)}$ are "glued" to each other crosswise across Δ (see Figure 3), boundary value problem (2.5) gives rise to the following relation between the traces of r_h and r_h^* on Δ:

$$(2.7) \qquad\qquad (r_h^*)^\pm = r_h^\mp h.$$

Relations (2.7) have two useful consequences. Firstly, if h is holomorphic in some neighborhood of Δ then so is $r_h^* + h r_h$. Indeed, we only need to verify that this function has no jump on Δ. The latter follows from (2.7) and the computation:

$$(2.8) \qquad\qquad (r_h^* + h r_h)^\pm = (r_h^*)^\pm + h r_h^\pm = h r_h^\mp + (r_h^*)^\mp = (r_h^* + h r_h)^\mp.$$

Secondly, the product $r_h r_h^*$ is a rational function over $\overline{\mathbb{C}}$ as soon as h is continuous. Indeed, as $r_h r_h^*$ is clearly meromorphic in D, we need only check the behavior across Δ. If h vanishes on a subset of positive linear measure of Δ, then $r_h^* \equiv 0$ by (2.7) and Privalov's theorem. Otherwise (2.7) implies that $(r_h r_h^*)^+ = (r_h r_h^*)^-$ a.e. on Δ°, and since $r_h r_h^*$ is bounded we get by a standard continuation principle [21, Ch. II, Ex. 12]. that it extends holomorphically across each Δ_k°. Finally, because it has bounded behavior near each a_k, the latter are removable singularities, as desired.

When r_h as above is not constant, we define its *principal divisor* as

$$(2.9) \qquad\qquad (r_h) := \sum_l m_l \mathbf{z}_l - \sum_j k_j \mathbf{w}_j,$$

meaning that r_b has a pole (resp. zero) of multiplicity k_j (resp. m_l) at each $\mathbf{w}_j \in D^{(1)} \cup D^{(2)}$ (resp. $\mathbf{z}_l \in D^{(1)} \cup D^{(2)}$), and that $r_b r_b^*$ has a pole (resp. zero) of multiplicity k_j (resp. m_l) at each $\pi(\mathbf{w}_j)$ (resp. $\pi(\mathbf{z}_l)$) if $\mathbf{w}_j \in L$ (resp. $\mathbf{z}_l \in L$), while r_b has finite non-zero value at any other point of $\mathfrak{R}$ including its two-sided boundary values on L. Then it is easy to see that

$$(2.10) \qquad (r_b r_b^*)(z) = \text{const.} \prod_{|\pi(z_l)|<\infty} (z - \pi(\mathbf{z}_l))^{m_l} \prod_{|\pi(w_j)|<\infty} (z - \pi(\mathbf{w}_j))^{-k_j}.$$

In particular, $\sum_l m_l = \sum_j k_j$ as the number of poles and the number of zeros for a rational function over $\overline{\mathbb{C}}$ are the same. Thus definition (2.9) generalizes to sectionally meromorphic functions on $\mathfrak{R}$ meeting (2.5) the well-known fact that non-constant rational functions on a compact Riemann surface have as many zeros as poles, counting multiplicities. Note that w is such a rational function and that its principal divisor is $(w) = \sum_{k=0}^{3} a_k - 2\infty^{(1)} - 2\infty^{(2)}$.

2.4. **Szegő-type Functions.** The function S_n, introduced in this section, will provide the main term of the asymptotics of Padé approximants to functions of the form (2.20). Before stating our first proposition, recall that a function h is called Dini-continuous on Δ if

$$\int_{[0,\text{diam}(\Delta)]} \frac{\omega_h(\tau)}{\tau} d\tau < \infty, \quad \omega_h(\tau) := \max_{|t_1 - t_2| \leq \tau} |h(t_1) - h(t_2)|,$$

where $\text{diam}(\Delta) := \max_{t_1, t_2 \in \Delta} |t_1 - t_2|$.

Proposition 1. *Let h be a Dini-continuous non-vanishing function on Δ. Then there exists $\mathbf{z}_n \in \mathfrak{R}$ such that $\mathbf{z}_n + (n-1)\infty^{(2)} - n\infty^{(1)}$ is the principal divisor of a function S_n which is meromorphic in $\mathfrak{R} \setminus L$ and has continuous traces on L from both sides which satisfy*

$$(2.11) \qquad S_n^- = S_n^+ \cdot (h \circ \pi).$$

Moreover, under the normalization $S_n(\mathbf{z})z^{-k_n} \to 1$ as $\mathbf{z} \to \infty^{(1)}$, where $k_n = n - 1$ if $\mathbf{z}_n = \infty^{(1)}$ and $k_n = n$ otherwise, S_n is the unique function meromorphic in $\mathfrak{R} \setminus L$ with principal divisor of the form $\mathbf{w} + (n-1)\infty^{(2)} - n\infty^{(1)}$, $\mathbf{w} \in \mathfrak{R}$, and continuous traces on L that satisfy (2.11). Furthermore, if $\mathbf{z}_n = \infty^{(1)}$ then $\mathbf{z}_{n-1} = \infty^{(2)}$ and $S_n = S_{n-1}$.

Observe that when $h \equiv 1$ the principle of analytic continuation implies that S_n is simply a rational function over $\mathfrak{R}$ having n poles at $\infty^{(1)}$ and $n - 1$ zeros at $\infty^{(2)}$. The point $\mathbf{z}_n$ is then determined by the geometry of $\mathfrak{R}$ as one cannot prescribe all poles and zeros of rational functions over Riemann surfaces of non-trivial genus (see Section 4.5).

Note also that, when $h = 1/p$, where p is an algebraic polynomial non-vanishing on Δ, equation (2.8) yields that $S_n + pS_n^*$ is a monic polynomial of degree k_n if $2n > \deg(p) + 2$.

Denote by φ the conformal map of D onto $\{|z| > 1\}$ with $\varphi(\infty) = \infty$, $\varphi'(\infty) > 0$. Then

$$(2.12) \qquad \varphi(z) = \frac{z}{\text{cap}(\Delta)} + \dots,$$

where $\text{cap}(\Delta)$ is the *logarithmic capacity* of Δ, see [48] (often (2.12) serves as the definition of the logarithmic capacity of a continuum). Denote also by ω_Δ the *equilibrium (harmonic) measure* on Δ, see [48]. It is known[5] that ω_Δ has the form

$$(2.13) \qquad d\omega_\Delta(t) = \frac{i(t - a_0)dt}{\pi w^+(t)}, \quad t \in \Delta.$$

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[5] By (2.2) and the Cauchy formula, the right hand side of (2.13) is a probability measure on Δ, say μ. The differential along Δ_k° of its logarithmic potential $U^\mu(z) := -\int_\Delta \log|z - t| d\mu(t)$ is $\text{Re}\{(\int_\Delta d\mu(t)/(t - z))dz\} = \text{Re}\{(a_0 - z)dz/w^\pm(z)\} = 0$, using (2.2) and Cauchy's formula again. Hence U^μ is constant on Δ, which is characteristic of the equilibrium potential.

In fact, φ has an integral representation involving ω_Δ, see (4.21) in Section 4.4. Set

$$(2.14) \qquad G_h := \exp\left\{ \int \log h \, d\omega_\Delta \right\}.$$

It is clear, since $\omega_\Delta(\Delta) = 1$, that G_h is well-defined as long as a continuous branch of $\log h$ is used, which is possible since h is continuous and does not vanish on Δ and the latter is simply connected. Hereafter, we put for simplicity $z_n = \pi(\mathbf{z}_n)$.

Proposition 2. *In the setting of Proposition 1 we have that*

$$(2.15) \qquad \frac{(S_n S_n^*)(z)}{(\mathrm{cap}(\Delta))^{2n-1}} = \xi_n G_h \begin{cases} (z - z_n)/|\varphi(z_n)|, & \mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}, \\ \mathrm{cap}(\Delta), & \mathbf{z}_n = \infty^{(2)}, \\ (z - z_n)|\varphi(z_n)|, & \mathbf{z}_n \in L \cup D^{(1)} \setminus \{\infty^{(1)}\}, \end{cases}$$

where $|\xi_n| = 1$. Moreover, it holds that

$$(2.16) \qquad \frac{S_n^*(z)}{S_n(z)} = \frac{\xi_n G_h}{\varphi^{2n-1}(z)} \Upsilon(\mathbf{z}_n; z) \begin{cases} \dfrac{z - z_n}{\varphi(z)|\varphi(z_n)|}, & \mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}, \\ \mathrm{cap}(\Delta)/\varphi(z), & \mathbf{z}_n = \infty^{(2)}, \\ \dfrac{\varphi(z)|\varphi(z_n)|}{z - z_n}, & \mathbf{z}_n \in L \cup D^{(1)} \setminus \{\infty^{(1)}\}, \end{cases}$$

where S_n and S_n^* are the conjugate functions derived from S_n and $\{\Upsilon(\mathbf{a}; \cdot)\}_{\mathbf{a} \in \mathfrak{R}}$ is a normal family of non-vanishing functions in D.

For $\mathbb{N}_1$ an arbitrary subsequence of the natural numbers and $\mathbf{Z}_1$ the derived set of $\{\mathbf{z}_n\}_{n \in \mathbb{N}_1}$, the sequence $\{S_n^*/S_n\}_{n \in \mathbb{N}_1}$ converges to zero geometrically fast on closed subsets of $D \setminus \{\pi(\mathbf{Z}_1 \cap D^{(1)})\}$ by (2.16) (recall that $|\varphi| > 1$ in D). On the contrary, no convergence can take place on domains intersecting $\pi(\mathbf{Z}_1 \cap D^{(1)})$. Hence the convergence properties of $\{S_n^*/S_n\}$ depend on the geometry of $\mathbf{Z} = \mathbf{Z}(h)$, the set of the limit points of $\{\mathbf{z}_n\}$ in $\mathfrak{R}$.

Our next proposition qualitatively describes this geometry. The classification according to the rational independence of the numbers $\omega_\Delta(\Delta_k)$ is essentially due to Suetin [61] whose argument, originally developed to handle the case of two arcs rather than a threefold, applies here with little change. We complete the picture with generic properties of this classification which are intuitively as expected, although their proof is not so straightforward.

Proposition 3. *In the setting of Proposition 1 it holds that $\mathbf{Z} = \mathfrak{R}$ when the numbers $\omega_\Delta(\Delta_k)$, $k \in \{1, 2, 3\}$, are rationally independent; $\mathbf{Z}$ is the union of finitely many pairwise disjoint arcs when $\omega_\Delta(\Delta_k)$ are rationally dependent but at least one of them is irrational; $\mathbf{Z}$ is a finite set of points when $\omega_\Delta(\Delta_k)$ are all rational. All the points $\mathbf{z}_n$ are mutually distinct in the first two cases and $\{\mathbf{z}_n\} = \mathbf{Z}$ in the third one. The set of triples (a_1, a_2, a_3) for which the numbers $\omega_\Delta(\Delta_k)$ are rationally dependent form a dense subset of zero measure in $\mathbb{C}^3$. Triples (a_1, a_2, a_3) for which $\omega_\Delta(\Delta_k)$ are rational are also dense.*

We prove these propositions in Section 6 with all the preliminary work carried out in Sections 3, 4 and 5. Moreover, in these sections one can find integral representations for S_n and $\Upsilon(\mathbf{a}; \cdot)$.

2.5. Padé Approximation. Let f be a function holomorphic and vanishing at infinity. Then f can be represented as a power series

$$(2.17) \qquad f(z) = \sum_{k=1}^\infty \frac{f_k}{z^k},$$

which converges outside of some disk centered at the origin. A *diagonal Padé approximant* of order n to f is a rational function $\pi_n = p_n/q_n$ of type (n,n) such that

$$(2.18) \qquad q_n(z)f(z) - p_n(z) = \mathcal{O}\left(1/z^{n+1}\right) \quad \text{as} \quad z \to \infty$$

System (2.18) is always solvable since it consists of $2n+1$ homogeneous linear equations with $2n+2$ unknowns, whose coefficients are the moments f_k in (2.17), no solution of which can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic). A solution needs not be unique, but each pair $(\tilde{p}_n, \tilde{q}_n)$ meeting (2.18) yields the same rational function $\pi_n = \tilde{p}_n/\tilde{q}_n$. In particular, each solution of (2.18) is of the form $(l\,p_n, l\,q_n)$, where (p_n, q_n) is the unique solution of minimal degree. That is, a Padé approximant is the unique rational function π_n of type (n,n) satisfying

$$(2.19) \qquad f(z) - \pi_n(z) = \mathcal{O}\left(1/z^{n+1+\sigma(\pi_n)}\right) \quad \text{as} \quad z \to \infty.$$

where $\sigma(\pi_n)$ is the number of finite poles of π_n, counting multiplicity (see[43, 7]). Hereafter, when writing $\pi_n = p_n/q_n$, we always mean that (p_n, q_n) is the solution of minimal degree. In the generic case where $\deg q_n = n$, observe that the order of contact of π_n with f at infinity is $2n+1$, which is generically maximal possible as a rational function of type (n,n) has $2n+1$ free parameters. Notice here that $\deg(p_n) < \deg(q_n)$ as f vanishes at infinity. Equivalently, one could also regard π_n as a continued fraction of order n constructed from the series representation (2.17) [37], but we shall not dwell on this connection.

Let now h be a complex-valued integrable function given on Δ. We define the Cauchy integral of h as

$$(2.20) \qquad f_h(z) := \frac{1}{\pi i}\int_\Delta \frac{h(t)}{t-z}\frac{dt}{w^+(t)}, \quad z \in D,$$

where integration is taking place according to the orientation of each Δ_k, i.e., from a_0 to a_k. Clearly, f_h is a holomorphic function in D that vanishes at infinity and therefore can be represented as in (2.17). Thus, we can construct the sequence of Padé approximants to f_h whose asymptotic behavior is described by the following two theorems.

In the first theorem we assume that $h \equiv 1/p$, where p is a polynomial non-vanishing on Δ. This is not only a key step in our approach to the general case, but it is also of independent interest since this assumption assumption allows us to obtain non-asymptotic formula for the approximation error.

Theorem 4. *Let $\{\pi_n\}$, $\pi_n = p_n/q_n$, be the sequence of diagonal Padé approximants to $f_{1/p}$, where p is a polynomial non-vanishing on Δ. Then*

$$(2.21) \qquad \begin{cases} \left(f_{1/p} - \pi_n\right) &= \dfrac{2}{w}\dfrac{S_n^*}{S_n + pS_n^*}, \\[2ex] \qquad\qquad q_n &= S_n + pS_n^*, \end{cases}$$

for all $2n > \deg(p)+2$, where S_n and S_n^* are the conjugate functions derived from S_n granted by Proposition 1.

As we show in Section 7.1, the polynomial q_n is orthogonal (in the non-Hermitian sense) to all algebraic polynomials of degree at most $n-1$ with respect to the weight h/w^+ on Δ. Thus, polynomials q_n appearing in Theorem 4 stand analogous to the well-known Bernstein-Szegő polynomials on $[-1,1]$ [62, Sec. 2.6]. Moreover, since $1/w$ vanishes at infinity, Cauchy theorem yields that

$$\frac{1}{w(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{w(t)}\frac{dt}{z-t}$$

for z in the exterior of Γ, where Γ is any positively oriented Jordan curve encompassing Δ. By deforming Γ onto ∂D, one can easily verify that $1/w = f_1$ and therefore polynomials q_n for the case $h \equiv 1$ can be viewed as an analog of the classical Chebyshëv polynomials.

The second line in (2.21) yields that $q_n = S_n(1 + p S_n^*/S_n)$, and checking the behavior at infinity using Proposition 1 gives us $\deg(q_n) = n$ unless $\mathbf{z}_n = \infty^{(1)}$ in which case $\deg(q_n) = n - 1$, $q_n = q_{n-1}$, and $\mathbf{z}_{n-1} = \infty^{(2)}$. Hence, when analyzing the behavior of q_{n_k} along a subsequence $\{n_k\} \subset \mathbb{N}$, we may assume that $\mathbf{z}_{n_k} \neq \infty^{(1)}$ for all k upon replacing n_k by $n_k - 1$ if necessary.

Turning now to more general densities, let h be a Dini-continuous non-vanishing function. Denote by

$$(2.22) \qquad \omega_n = \omega_n(h) := \min_p \|1/h - p\|_\Delta,$$

where the minimum is taken over all polynomials p of degree at most n. Clearly $\omega_n \to 0$ as $n \to \infty$ by Mergelyan's theorem.

Theorem 5. *Let h be a complex-valued Dini-continuous non-vanishing function on Δ and $\{\pi_n\}$, $\pi_n = p_n/q_n$, be the sequence of Padé approximants to f_h. Then*

$$(2.23) \qquad (f_h - \pi_n) = \frac{2}{w} \frac{S_n^*}{S_n} \frac{1 + E_n^*}{1 + E_n + \mathcal{O}(|\varphi|^{-n})},$$

where $\mathcal{O}(|\varphi|^{-n})$ holds uniformly in D and E_n is a sectionally meromorphic function on $\mathfrak{R} \setminus L$ with at most one pole which is necessarily $\mathbf{z}_n$, and such that

$$(2.24) \qquad \left(\int_{\partial D} \left(|(E_n l_n)(t)|^2 + |(E_n^* l_n)(t)|^2 \right) \frac{|dt|}{|w(t)|} \right)^{1/2} \leq \text{const.}\omega_n$$

with $l_n(t) \equiv 1$ when $\mathbf{z}_n \notin O$ and $l_n(t) = t - z_n$ otherwise, where O is some fixed but arbitrary neighborhood of L in $\mathfrak{R}$ (the constant in (2.24) depends on O but is independent of n). Moreover, if $\mathbf{z}_n \neq \infty^{(1)}$ and n is large enough then $\deg(q_n) = n$.

Formulae (2.23) and (2.24) have the following ramifications for the uniform convergence of Padé approximants.

Corollary 6. *Assumptions being as in Theorem 5, let $\mathbb{N}_1 \subset \mathbb{N}$ be a subsequence such that $\{\mathbf{z}_n\}_{n \in \mathbb{N}_1}$ converges to $\mathbf{z} \in \mathfrak{R}$.*

- *If $\mathbf{z} \in D^{(2)} \cup L$, then the Padé approximants π_n converge to f geometrically fast on compact subsets of D as $\mathbb{N}_1 \ni n \to \infty$.*
- *If $\mathbf{z} \in D^{(1)}$, then the Padé approximants π_n converge to f geometrically fast on compact subsets of $D \setminus \{z\}$ as $\mathbb{N}_1 \ni n \to \infty$. Moreover, to each neighborhood O of Δ in $\mathbb{C}$, there is $n_O \in \mathbb{N}_1$ such that π_n has exactly one pole in $D \setminus O$ for $n \geq n_O$ and this pole converges to z as $\mathbb{N}_1 \ni n \to \infty$.*

Theorems 4, 5, and Corollary 6 are proven in Section 7. The following is an immediate consequence of Corollary 6.

Corollary 7. *Under conditions of Theorem 5, let $\{\pi_n\}_{n \in \mathbb{N}' \subset \mathbb{N}}$ be a sequence of diagonal Padé approximants to f and $\mathbf{Z}$ the set of accumulation points of $\{\mathbf{z}_n\}_{n \in \mathbb{N}'}$ on $\mathfrak{R}$. Then $\{\pi_n\}_{n \in \mathbb{N}'}$ converges locally uniformly to f on $D \setminus \pi(\mathbf{Z} \cap D^{(1)})$ and on no larger subdomain of D.*

Our last theorem puts the preceding results in a generic perspective.

Theorem 8. *There is dense subset of zero measure $E \subset \mathbb{C}^3$ such that, for f as in (2.20) with Δ the Chebotarëv continuum of $(a_1, a_2, a_3) \in \mathbb{C}^3$ and h a Dini-continuous non-vanishing function on Δ, the following holds:*

- *if $(a_1, a_2, a_3) \in \mathbb{C}^3 \setminus E$, then the sequence of Padé approximants to f converges on no subdomain of $\overline{\mathbb{C}} \setminus \Delta$ but some subsequence converges locally uniformly to f on $\overline{\mathbb{C}} \setminus \Delta$;*
- *if $(a_1, a_2, a_3) \in E$, then the sequence of Padé approximants to f converges locally uniformly on $\overline{\mathbb{C}} \setminus (\Delta \cup A)$, where A is either a finite (possibly empty) union of curves or finitely many points. In particular the domain of convergence is non-void.*

3. Cauchy Integrals

For an analytic Jordan arc F with endpoints e_1 and e_2, oriented from e_1 to e_2, define

$$(3.1) \qquad w_F(z) := \sqrt{(z - e_1)(z - e_2)}, \quad \frac{w_F(z)}{z} \to 1 \quad \text{as} \quad z \to \infty,$$

to be a holomorphic function outside of F with a simple pole at infinity. Then w_F has continuous traces w_F^+ and w_F^- on the left and right sides of F, respectively (sides are determined by the orientation in the usual manner). For an integrable function ϕ on F, set

$$(3.2) \qquad C_F(\phi; z) := \int_F \frac{\phi(t)}{t - z} \frac{dt}{2\pi i} \quad \text{and} \quad R_F(\phi; z) := w_F(z) C_F\left(\frac{\phi}{w_F^+}; z\right),$$

$z \in \overline{\mathbb{C}} \setminus F$. We also put

$$(3.3) \qquad C_\Delta(z) := \int_\Delta \frac{\theta(t)}{t - z} \frac{dt}{2\pi i} \quad \text{and} \quad R_\Delta(\theta; z) := w(z) C_\Delta\left(\frac{\theta}{w^+}; z\right),$$

$z \in D$, where θ is an integrable function on Δ. The following lemma will be needed later on.

Lemma 1. *Let θ be a Dini-continuous function on Δ and J be either $C_\Delta^\pm$ or $R_\Delta^\pm$. Then*

$$(3.4) \qquad \int_\Delta \frac{|J(t)|^2}{|w^+(t)|} |dt| \leq \text{const.} \int_\Delta \frac{|\theta(t)|^2}{|w^+(t)|} |dt|,$$

Proof. It was shown in [9, Sec. 3.2] that for a Dini-continuous function ϕ on Δ_k, the functions $R_{\Delta_k}(\phi; \cdot)$ and $C_{\Delta_k}(\phi; \cdot)$ have unrestricted boundary values on both sides of Δ_k and the traces $R_{\Delta_k}^\pm(\phi; \cdot)$ and $C_{\Delta_k}^\pm(\phi; \cdot)$ are continuous. It is known [14, Thm. 2.2] that $|w_{\Delta_k}|^{-1/2}$ is an A_2-weight on Δ_k. Hence, by [14, Thm. 4.15] it follows that

$$(3.5) \qquad \int_{\Delta_k} \frac{\left|C_{\Delta_k}^\pm(\phi; t)\right|^2}{|w_{\Delta_k}^+(t)|} |dt| \leq \text{const.} \int_{\Delta_k} \frac{|\phi(t)|^2}{|w_{\Delta_k}^+(t)|} |dt|,$$

where const. is a constant independent of ϕ. By the very definition, see [14, eq. (2.1)], $|w_{\Delta_k}|^{-1/2}$ is an A_2-weight if and only if $|w_{\Delta_k}|^{1/2}$ is also an A_2-weight. Thus, applying [14, Thm. 4.15] as in (3.5) only with $|w_{\Delta_k}|^{-1/2}$ replaced by $|w_{\Delta_k}|^{1/2}$ and ϕ replaced by $\phi/w_{\Delta_k}^+$, we get that

$$(3.6) \qquad \int_{\Delta_k} \frac{\left|R_{\Delta_k}^\pm(\phi; t)\right|^2}{|w_{\Delta_k}^+(t)|} |dt| \leq \text{const.} \int_{\Delta_k} \frac{|\phi(t)|^2}{|w_{\Delta_k}^+(t)|} |dt|.$$

Moreover, [14, Prop. 2.1] yields that not only $|w_{\Delta_k}|^{\pm 1/2}$ but also $|w|^{\pm 1/2}$ is an A_2-weight on each Δ_k. Thus, (3.4) is obtained by applying (3.5) and (3.6) on each Δ_k with $\phi = \theta_{|\Delta_k}$ and then taking the sum over k. $\qquad \square$

The main purpose of this section is to study the so-called *Szegő function* of a given function on Δ. Namely, let h be a Dini-continuous non-vanishing function on Δ. For a fixed $\varkappa \in \{1, 2, 3\}$ and an arbitrary continuous branch of $\log h$, we define the constant $G_{h,\varkappa}$ as

$$(3.7) \qquad G_{h,\varkappa} := \exp\left\{-m_1 + \frac{\beta_\varkappa^1}{\beta_\varkappa} m_0\right\}, \quad m_j := \frac{1}{\pi i} \int_\Delta \frac{t^j \log h(t) dt}{w^+(t)},$$

and the *Szegő function* of h as

$$(3.8) \qquad S_{h,\varkappa}(z) := \exp\left\{ w(z)\left(C_\Delta\left(\frac{\log h}{w^+};z\right) - \frac{m_0}{\beta_\varkappa}C_{\Delta_\varkappa}\left(\frac{1}{w^+};z\right)\right) - \frac{\log G_{h,\varkappa}}{2}\right\},$$

$z \in D$, where under $\log G_{h,\varkappa}$ we understand $-m_1 + m_0(\beta^1_\varkappa/\beta_\varkappa)$,

$$(3.9) \qquad \beta_k := \frac{1}{\pi i}\int_{\Delta_k}\frac{dt}{w^+(t)} \quad\text{and}\quad \beta^1_k := \frac{1}{\pi i}\int_{\Delta_k}\frac{t\,dt}{w^+(t)}, \quad k \in \{1,2,3\}.$$

As $1/w(z) = 1/z^2 + \ldots$, it holds by the Cauchy integral formula that

$$\beta_1 + \beta_2 + \beta_3 = \frac{1}{2\pi i}\int_{\partial D}\frac{dt}{w(t)} = 0 \quad\text{and}\quad \beta^1_1 + \beta^1_2 + \beta^1_3 = \frac{1}{2\pi i}\int_{\partial D}\frac{t\,dt}{w(t)} = -1.$$

Proposition 9. *For h and $\varkappa$ as above, the constant $G_{h,\varkappa}$ and the function $S_{h,\varkappa}$ do not depend on the continuous branch of $\log h$ used to define them through (3.7) and (3.8). Moreover $S_{h,\varkappa}$ is holomorphic in D, it has continuous boundary values on ∂D, and $S_{h,\varkappa}(\infty) = 1$.*

$$(3.10) \qquad h = \begin{cases} \widetilde{G}_{h,\varkappa}S^+_{h,\varkappa}S^-_{h,\varkappa} & on \quad \Delta^\circ_\varkappa, \\[2mm] G_{h,\varkappa}S^+_{h,\varkappa}S^-_{h,\varkappa}, & on \quad \Delta^\circ\setminus\Delta_\varkappa, \end{cases} \qquad \widetilde{G}_{h,\varkappa} := G_{h,\varkappa}\exp\left\{\frac{m_0}{\beta_\varkappa}\right\}.$$

Proof. As we mentioned in Lemma 1, it follows from [9, Sec. 3.2] that for any Dini-continuous function ϕ on an analytic arc F, $R_F(\phi;\cdot)$ has unrestricted boundary values on both sides of F, the traces $R^\pm_F(\phi;\cdot)$ are continuous, $R^+_F(\phi;e_k) = R^-_F(\phi;e_k)$, $k \in \{1,2\}$, and furthermore

$$(3.11) \qquad R^+_F(\phi;t) + R^-_F(\phi;t) = \phi(t), \quad t \in F,$$

where (3.11) is a consequence of Sokhotski-Plemelj formulae and the relation $w^+_F = -w^-_F$.

Let now θ be a Dini-continuous function on Δ. Observe that $w = w_{\Delta_k}w_{F_k}$ according to (3.1), where $F_k := (\Delta\setminus\Delta_k)\cup\{a_0\}$. Then

$$R_\Delta(\theta;z) = \sum_{k=1}^3 w_{F_k}(z)R_{\Delta_k}\left(\frac{\theta}{w_{F_k}};z\right)$$

according to (3.2). This immediately implies that $R_\Delta(\theta;\cdot)$ has continuous trace on ∂D with

$$(3.12) \qquad R^+_\Delta(\theta;t) + R^-_\Delta(\theta;t) = \theta(t), \quad t \in \Delta,$$

by (3.11). Moreover, applying the Cauchy integral formula to $1/w$ on ∂D, we get

$$(3.13) \qquad R_\Delta(\text{const.};z) = \frac{\text{const.}}{2}$$

for any constant. To describe the behaviour of $R_\Delta(\theta;\cdot)$ at infinity, define the moments

$$(3.14) \qquad m_k = m_k(\theta) := \frac{1}{\pi i}\int_\Delta\frac{t^k\theta(t)}{w^+(t)}dt, \quad k \in \{0,1\}.$$

Using the fact that $1/w = 1/z^2 + O(1/z^3)$ near infinity, one can readily verify that

$$(3.15) \qquad m_0(\theta + \text{const.}) = m_0(\theta) \quad\text{and}\quad m_1(\theta + \text{const.}) = m_1(\theta) + \text{const.}$$

for any constant. By developing $1/(t-z)$ at infinity in powers of z, we get that

$$(3.16) \qquad R_\Delta(\theta;z) = w(z)\left(-\frac{m_0}{2z} - \frac{m_1}{2z^2} + O\left(\frac{1}{z^3}\right)\right)$$

there. Analogously, one can check that

$$(3.17) \qquad w_{F_\varkappa}(z)R_{\Delta_\varkappa}\left(\frac{1}{w_{F_\varkappa}};z\right) = w(z)\left(-\frac{\beta_\varkappa}{2z} - \frac{\beta_\varkappa^1}{2z^2} + O\left(\frac{1}{z^3}\right)\right)$$

near infinity. Thus,

$$
\begin{aligned}
R_\varkappa(z) & := R_\Delta(\theta;z) - \frac{m_0}{\beta_\varkappa} w_{F_\varkappa}(z)R_{\Delta_\varkappa}\left(\frac{1}{w_{F_\varkappa}};z\right) \\
(3.18) \qquad & = w(z)\left(\frac{1}{2z^2}\left(m_0\frac{\beta_\varkappa^1}{\beta_\varkappa} - m_1\right) + O\left(\frac{1}{z^3}\right)\right) = \frac{1}{2}\left(m_0\frac{\beta_\varkappa^1}{\beta_\varkappa} - m_1\right) + O\left(\frac{1}{z}\right)
\end{aligned}
$$

near infinity by (3.16) and (3.17). Moreover, it follows from (3.11) and (3.12) that

$$(3.19) \qquad R_\varkappa^+ + R_\varkappa^- = \theta - \begin{cases} m_0/\beta_\varkappa, & \text{on} \quad \Delta_\varkappa^\circ, \\ 0, & \text{on} \quad \Delta^\circ \setminus \Delta_\varkappa. \end{cases}$$

Finally, let h be a Dini-continuous non-vanishing function on Δ. As explained in [9, Sec. 3.3], any continuous branch of $\log h$ is itself Dini-continuous. Fix such a branch and denote it by θ. Observe that the difference between any two continuous determinations of $\log h$ is an integer multiple of $2\pi i$ and therefore $G_{h,\varkappa}$ is well-defined by (3.15). Moreover, the Szegő function of h defined in (3.8) is nothing else but

$$(3.20) \qquad \exp\left\{R_\varkappa(z) - \frac{1}{2}\left(m_0\frac{\beta_\varkappa^1}{\beta_\varkappa} - m_1\right)\right\}, \quad z \in D.$$

As evident from (3.13) and (3.15), $S_{h,\varkappa}$ does not depend on the choice of the branch of $\log h$ as long as the branch is continuous and used in (3.3) and (3.14) simultaneously. Clearly, (3.10) follows from (3.19) and $S_{h,\varkappa}(\infty) = 1$ by (3.18). The continuity of $S_{h,\varkappa}$ on ∂D is a consequence of continuity of $R_\varkappa$ on ∂D. Obviously, $S_{h,\varkappa}$ is holomorphic and non-vanishing in D as it is an exponential of a holomorphic function. $\qquad \square$

4. ABELIAN DIFFERENTIALS AND THEIR INTEGRALS

The following material is expository on Abelian differentials on an elliptic Riemann surface. We use [13, 19] as primary sources, limiting ourselves to the case at hand (*i.e.* $\mathfrak{R}$).

For each $k \in \{1, 2, 3\}$ set $\widetilde{\mathfrak{R}}_k := \mathfrak{R} \setminus (L_k \cup L_{k+1})$ and $\widehat{\mathfrak{R}}_k := \mathfrak{R} \setminus L_k$, where indices are computed modulo 3. It is easy to see (*cf.* Figure 2), that each domain $\widetilde{\mathfrak{R}}_k$ is simply connected.

4.1. **Abelian Differentials of the First Kind.** A differential $d\Omega$ is called an *Abelian differential of the first kind* on $\mathfrak{R}$ if the integral $\int^z d\Omega$ defines a holomorphic multi-valued function on the whole surface. Since the genus of $\mathfrak{R}$ is 1, there exists exactly one Abelian differential of the first kind up to a multiplicative constant. This differential is given by

$$d\Omega(\mathbf{z}) := \frac{dz}{w(\mathbf{z})}$$

as the principal divisor of $d\Omega$ should be integral and since it is known that the principal divisor of the differential dz is given by $\sum_{k=0}^3 a_k - 2\infty^{(1)} - 2\infty^{(2)}$. By $d\Omega_1$ we denote the Abelian differential of the first kind normalized to have period 1 on L_2 (i.e., we choose L_2 to be the so-called **a**-cycle for $d\Omega$). That is,

$$(4.1) \qquad d\Omega_1(\mathbf{z}) := \frac{1}{2\pi i \beta_2}\frac{dz}{w(\mathbf{z})}, \quad \beta_k = \frac{1}{\pi i}\int_{\Delta_k}\frac{dt}{w^+(t)} = \frac{1}{2\pi i}\oint_{L_k} d\Omega,$$

$k \in \{1,2,3\}$. Moreover, it is known that

$$(4.2) \qquad \mathrm{Im}\left(\frac{\beta_3}{\beta_2}\right) > 0, \quad \frac{\beta_3}{\beta_2} = \oint_{L_3} d\Omega_1,$$

because (L_2, L_3) is positively oriented (*i.e.* we take L_3 to be the so-called **b**-cycle for $d\Omega$). The numbers 1 and β_3/β_2 are called the *periods* of $d\Omega_1$. It is known that for any Jordan curve Γ on $\mathfrak{R}$ the integral of $d\Omega_1$ along Γ is congruent to 0 ($\equiv 0$) modulo periods of $d\Omega_1$. That is,

$$\oint_{\Gamma} d\Omega_1 = l + j\frac{\beta_3}{\beta_2}, \quad l, j \in \mathbb{Z}.$$

It will be convenient for us to define[6]

$$(4.3) \qquad \Omega_1(\mathbf{z}) := \int_{a_1}^{\mathbf{z}} d\Omega_1, \quad \mathbf{z} \in \widetilde{\mathfrak{R}}_2,$$

where the path of integration except perhaps for the endpoint lies entirely in $\widetilde{\mathfrak{R}}_2$. Observe that Ω_1 is a well-defined holomorphic function in the simply connected domain $\widetilde{\mathfrak{R}}_2$ since $1/w$ has a double zero at infinity. Furthermore, Ω_1 has continuous traces on both sides of L_2 and L_3 and the jump of Ω_1 there can
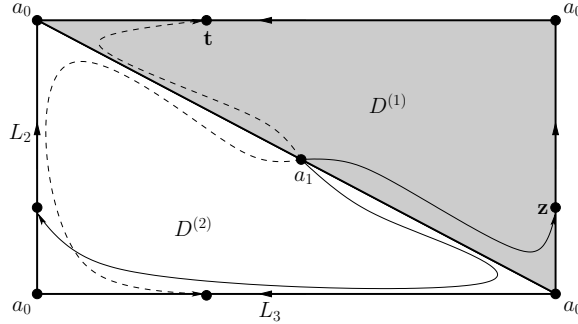

FIGURE 4. Paths of integration of $d\Omega_1$ that start at a_1 and end at $\mathbf{z} \in L_2$ (solid lines) and $\mathbf{t} \in L_3$ (dashed line).

be described by the relations

$$(4.4) \qquad \Omega_1^+ - \Omega_1^- = \begin{cases} -\beta_3/\beta_2, & \text{on} \quad L_2, \\ 1, & \text{on} \quad L_3, \end{cases}$$

as can be seen from Figure 4 (as shown on the figure, the dashed path in $D^{(2)}$ can be deformed into a concatenation of the dashed path in $D^{(1)}$ and the loop L_2 traversed in the negative direction; the solid path in $D^{(2)}$ can be deformed into a concatenation of the solid path in $D^{(1)}$ and the loop L_3 traversed in the positive direction).

---

[6]It is formally more appropriate but also more cumbersome to denote Ω_1 by $\Omega_{1,3}$, where $\Omega_{1,k}$ is defined as in (4.3) using the differential of the first kind that has L_{k-1} as the **a**-cycle, L_k as the **b**-cycle, and a_{k+1} as the initial bound for integration. This comment applies to all the differentials below where we do not explicitly specify the dependence on the choice of cycles.

4.2. Abelian Differentials of the Third Kind. An arbitrary Abelian differential is a differential of the form $rd\Omega$, where r is a rational function over $\mathfrak{R}$. The principal divisor of $rd\Omega$ coincides with the principal divisor of r. Thus, $rd\Omega$ has only poles as singularities and the residue of $rd\Omega$ at a pole $\mathbf{a}$ is equal to $\frac{1}{2\pi i}\oint_{\Gamma_{\mathbf{a}}} rd\Omega$, where $\Gamma_{\mathbf{a}}$ is a Jordan curve on $\mathfrak{R}$ that separates $\mathbf{a}$ from the rest of the poles of $rd\Omega$ and is oriented so that $\mathbf{a}$ lies to left of $\Gamma_{\mathbf{a}}$ when the latter is traversed in the positive direction.

In what follows, we are primarily interested in the following rational function:

$$(4.5) \qquad r(\mathbf{a};\mathbf{z}) := \frac{1}{2}\left(\frac{w(\mathbf{z})+w(\mathbf{a})}{z-a} + z - a + A\right), \quad A := \frac{1}{2}\sum_{j=0}^{3} a_j,$$

$\mathbf{a} \in \mathfrak{R}$, $|a| < \infty$. Since $w(z) = z^2 - Az + \cdots$ at infinity, it is an easy computation to verify by taking the appropriate limits that $r(\mathbf{a};\infty^{(2)}) = A - a$ and that $r(\infty^{(2)};\mathbf{z}) \equiv z$. That is, $r(\mathbf{a};\cdot)$ is, in fact, defined for all $\mathbf{a} \in \mathfrak{R}\setminus\{\infty^{(1)}\}$ and is bounded near $\infty^{(2)}$ for all $\mathbf{a}$ with finite canonical projection. Moreover,

$$(4.6) \qquad r(\mathbf{a};\mathbf{z}) + a - A \rightrightarrows 0 \quad \text{as} \quad \mathbf{a} \to \infty^{(1)}$$

in $\mathfrak{R}\setminus\{\infty^{(1)}\}$, where the sign $\rightrightarrows$ means "converges locally uniformly". Analogously, we get that

$$(4.7) \qquad r(\mathbf{b};\mathbf{z}) - r(\mathbf{a};\mathbf{z}) \rightrightarrows 0 \quad \text{as} \quad \mathbf{b} \to \mathbf{a}$$

in $\mathfrak{R}\setminus\{\mathbf{a}\}$. Summarizing, we have that

$$(4.8) \qquad r(\mathbf{a};\mathbf{z})d\Omega(\mathbf{z}), \quad \mathbf{a} \in \mathfrak{R}\setminus\{\infty^{(1)}\},$$

defines a differential with two poles, $\infty^{(1)}$ and $\mathbf{a}$. Let $\Gamma_{\infty^{(1)}}$ be a Jordan curve in $D^{(1)}$ that encompasses $\infty^{(1)}$ and separates it from $\mathbf{a}$. Assuming that $\Gamma_{\infty^{(1)}}$ is oriented clockwise, we can compute the residue of $r(\mathbf{a};\mathbf{z})d\Omega(\mathbf{z})$ at $\infty^{(1)}$ as

$$(4.9) \qquad \frac{1}{2\pi i}\int_{\Gamma_{\infty^{(1)}}} r(\mathbf{a};\mathbf{z})d\Omega(\mathbf{z}) = \frac{1}{4\pi i}\int_{\pi(\Gamma_{\infty^{(1)}})} \frac{dz}{z-a} + \frac{1}{4\pi i}\int_{\pi(\Gamma_{\infty^{(1)}})} \frac{zdz}{w(z)} = -1,$$

where all the integrals are evaluated by the Cauchy integral formula for unbounded domains. Respectively, the residue of $r(\mathbf{a};\mathbf{z})d\Omega(\mathbf{z})$ at $\mathbf{a}$ is equal to 1.

More generally, given two distinct points $\mathbf{b}_1$ and $\mathbf{b}_2$ on $\mathfrak{R}$, there is a differential $d\Omega(\mathbf{b}_1,\mathbf{b}_2;\mathbf{z})$ called an *Abelian differential of the third kind* having only two simple poles, $\mathbf{b}_1$ and $\mathbf{b}_2$, with residues 1 and -1, respectively. Such a differential is unique up to a differential of the first kind. Thus, for $\mathbf{b}_1,\mathbf{b}_2 \in \widetilde{\mathfrak{R}}_2$, there exists a unique differential of the third kind with period 0 on L_2, that we denote by $d\Omega_0(\mathbf{b}_1,\mathbf{b}_2;\mathbf{z})$ for brevity. It is known for such a differential that the L_3-period can be expressed through the Riemann relation:

$$(4.10) \qquad \oint_{L_3} d\Omega_0(\mathbf{b}_1,\mathbf{b}_2;\mathbf{z}) = -2\pi i \int_{\mathbf{b}_1}^{\mathbf{b}_2} d\Omega_1(\mathbf{z}),$$

where the path of integration for the integral on right-hand side of (4.10) lies entirely in $\widetilde{\mathfrak{R}}_2$. We shall also use another relation between the normalized Abelian integrals of the third kind, namely,

$$(4.11) \qquad \int_{\mathbf{b}_1}^{\mathbf{b}_2} d\Omega_0(\mathbf{b}_3,\mathbf{b}_4;\mathbf{z}) = \int_{\mathbf{b}_3}^{\mathbf{b}_4} d\Omega_0(\mathbf{b}_1,\mathbf{b}_2;\mathbf{z})$$

for $\mathbf{b}_k \in \widehat{\mathfrak{R}}_2$, $k \in \{1,2,3,4\}$, where the paths of integration again lie in $\widetilde{\mathfrak{R}}_2$.

Assume now that at least one of $\mathbf{b}_1,\mathbf{b}_2$ belongs to $L_2 \cup L_3$. Let L'_k, $k \in \{2,3\}$, be two Jordan curves on $\mathfrak{R}$ intersecting each other and L_1 once at the same point. Assume further that each L'_k is homologous to L_k and coincides with the latter except in a neighborhood of $\mathbf{b}_j$ if $\mathbf{b}_j \in L_k$ where they are disjoint. In particular, the periods of $d\Omega$ remain the same on these new curves. For definiteness, we suppose that

those of the points $\mathbf{b}_1, \mathbf{b}_2$ belonging to $L_2 \cup L_3$ lie to the left of L_2' and L_3'. Then (4.10) remains valid only with L_3 replaced by L_3', $d\Omega_0(\mathbf{b}_1, \mathbf{b}_2; \cdot)$ normalized to have zero period on L_2', and the path of integration for $d\Omega$ taken in $\mathfrak{R} \setminus \{L_2' \cup L_3'\}$. Clearly, (4.11) also holds only with the differentials of the third kind normalized to have zero period on L_2' and the paths of integration taken to lie in $\mathfrak{R} \setminus \{L_2' \cup L_3'\}$.

4.3. **Differentials** $d\Omega_0(\mathbf{a}, \infty^{(1)}; \cdot)$. In Section 5, we shall mainly work with differentials of the form $d\Omega_0(\mathbf{a}, \infty^{(1)}; \mathbf{z})$. It easily follows from (4.8) that

$$(4.12) \qquad d\Omega_0(\mathbf{a}, \infty^{(1)}; \mathbf{z}) = (r(\mathbf{a}; \mathbf{z}) + c(\mathbf{a})) \, d\Omega(\mathbf{z}),$$

where the constant $c(\mathbf{a})$ is chosen so that the period on L_2 (or L_2') of the differential is equal to 0 and is clearly a continuous function of $\mathbf{a}$. Moreover, it readily follows from a computation analogous to (4.9) that $c(\mathbf{a}) = a - A + \mathcal{O}(1/a)$ for $\mathbf{a}$ in the vicinity of $\infty^{(1)}$. That is, $d\Omega_0(\mathbf{a}, \infty^{(1)}; \mathbf{z})$ degenerates into a zero differential as $\mathbf{a} \to \infty^{(1)}$ by (4.6).

It is useful to observe that a general Abelian differential of the third kind is given by

$$(4.13) \qquad d\Omega_0(\mathbf{b}_1, \mathbf{b}_2; \mathbf{z}) = d\Omega_0(\mathbf{b}_1, \infty^{(1)}; \mathbf{z}) - d\Omega_0(\mathbf{b}_2, \infty^{(1)}; \mathbf{z}).$$

Clearly, (4.13) also provides a rational function representation for $d\Omega_0(\mathbf{b}_1, \mathbf{b}_2; \mathbf{z})$ via (4.12).

For $\mathbf{a} \in \mathfrak{R} \setminus (L_2 \cup \{a_1, \infty^{(1)}\})$, set

$$(4.14) \qquad \Omega_0(\mathbf{a}; \mathbf{z}) := \int_{a_1}^{\mathbf{z}} d\Omega_0(\mathbf{a}, \infty^{(1)}; \mathbf{t}), \quad \mathbf{z} \in \widetilde{\mathfrak{R}}_2,$$

where the path of integration, as usual, lies in $\widetilde{\mathfrak{R}}_2$ (or $\mathfrak{R} \setminus (L_2 \cup L_3')$ when $\mathbf{a} \in L_3$). Then $\Omega_0(\mathbf{a}; \cdot)$ is analytic and multi-valued (single-valued modulo $2\pi i$) on $\widetilde{\mathfrak{R}}_2 \setminus \{\mathbf{a}, \infty^{(1)}\}$ (or $\mathfrak{R} \setminus (L_2 \cup L_3' \cup \{\mathbf{a}, \infty^{(1)}\})$) with logarithmic singularities at $\mathbf{a}$ and $\infty^{(1)}$. Moreover, analyzing the boundary behavior of $\Omega_0(\mathbf{a}; \cdot)$ on L_2 and L_3 (or L_3') as in (4.4), we see that $\Omega_0(\mathbf{a}; \cdot)$ is analytic and multi-valued (single-valued modulo $2\pi i$) in $\widehat{\mathfrak{R}}_2 \setminus \{\mathbf{a}, \infty^{(1)}\}$, and that on L_2 it has the following jump:

$$(4.15) \qquad \Omega_0^+(\mathbf{a}; \cdot) - \Omega_0^-(\mathbf{a}; \cdot) \equiv - \oint_{L_3} d\Omega_0(\mathbf{a}, \infty^{(1)}; \mathbf{t}) = 2\pi i \left(\Omega_1(\infty^{(1)}) - \Omega_1(\mathbf{a}) \right) \pmod{2\pi i},$$

where the second equality follows from (4.10) and (4.3).

For $\mathbf{a} = \infty^{(1)}$, we formally set

$$(4.16) \qquad \Omega_0(\infty^{(1)}; \cdot) :\equiv 0 \Leftarrow \Omega_0(\mathbf{t}; \cdot) \pmod{2\pi i} \quad \text{as} \quad \mathbf{t} \to \infty^{(1)},$$

where convergence holds locally uniformly in $\mathfrak{R} \setminus \{\infty^{(1)}\}$ by (4.6). Observe that under this convention (4.15) still remains valid.

For $\mathbf{a} = a_1$, we simply change the initial bound of integration to some $b \in L_1 \setminus \{a_1, a_0\}$. Clearly, (4.15) remains valid in this case as well.

For $\mathbf{a} \in L_2$, the construction of $\Omega_0(\mathbf{a}; \cdot)$ is as follows. Define $\tilde{\Omega}_0(\mathbf{a}; \cdot)$ as in (4.14) with L_2 replaced by any admissible L_2'. This function is analytic and multi-valued in $\mathfrak{R} \setminus (L_2' \cup \{\mathbf{a}, \infty^{(1)}\})$ and has a jump across L_2' whose magnitude is described by (4.15). Observe that the magnitude of the jump does not depend on the choice of L_2' and that for any $\mathbf{z} \in D^{(1)}$ ($\mathbf{z} \in D^{(2)}$) the curve L_2' can be chosen not to separate $\mathbf{z}$ and $\infty^{(1)}$ ($\mathbf{z}$ and $\infty^{(2)}$). Hence, $\tilde{\Omega}_0(\mathbf{a}; \cdot)$ can be analytically continued to an analytic multi-valued function in $\widehat{\mathfrak{R}}_2 \setminus \{\mathbf{a}, \infty^{(1)}\}$, and we set $\Omega_0(\mathbf{a}; \cdot)$ to be this function. Notice that (4.15) is still at every point of $L \setminus \{\mathbf{a}\}$.

One more important property of functions $\Omega_0(\mathbf{a}; \cdot)$ is that

$$(4.17) \qquad \Omega_0(\mathbf{t}; \mathbf{z}) - \Omega_0(\mathbf{a}; \mathbf{z}) \rightrightarrows 0 \pmod{2\pi i} \quad \text{as} \quad \mathbf{t} \to \mathbf{a}$$

in $\mathfrak{R} \setminus \{\mathbf{a}\}$ for any $\mathbf{a} \in \mathfrak{R}$, which follows from (4.12) and (4.13) combined with (4.7).

4.4. **Green Differential.** Another important way to normalize a differential of the third kind is to make its periods to be purely imaginary. For instance, we shall be interested in the so-called *Green differential* given by

$$dG(\mathbf{z}) := (z - a_0) d\Omega(\mathbf{z}) = \frac{(z - a_0) dz}{w(\mathbf{z})}.$$

Computing as in (4.9), one can easily check that dG is a differential of the third kind having simple poles at $\infty^{(1)}$ and $\infty^{(2)}$ with residues -1 and 1 respectively. Moreover, by (2.13),

$$(4.18) \qquad \oint_{L_k} dG = -2\pi i \, \omega_\Delta(\Delta_k) =: -\omega_k,$$

where, as before, ω_Δ is the equilibrium measure on Δ. In particular, it follows from (3.9) that

$$(4.19) \qquad \omega_\Delta(\Delta_k) = a_0 \beta_k - \beta_k^1.$$

Observe also that

$$(4.20) \qquad d\Omega_0(\infty^{(2)}, \infty^{(1)}; \mathbf{z}) = dG(\mathbf{z}) + \omega_2 d\Omega_1(\mathbf{z})$$

by uniqueness of a normalized Abelian differential of the third kind with prescribed poles.

Set

$$(4.21) \qquad \varphi_{a_1}(\mathbf{z}) := \exp\left\{ \int_{a_1}^{\mathbf{z}} dG \right\}, \quad \mathbf{z} \in \widetilde{\mathfrak{R}}_2.$$

Then φ_{a_1} is a well-defined meromorphic function in $\widetilde{\mathfrak{R}}_2$ (the integral is defined modulo $2\pi i$) with a simple pole at $\infty^{(1)}$, a simple zero at $\infty^{(2)}$, otherwise non-vanishing, and with unimodular traces on $L_2 \cup L_3$ that satisfy

$$(4.22) \qquad \frac{\varphi_{a_1}^+}{\varphi_{a_1}^-} = \begin{cases} \exp\{\omega_3\}, & \text{on } L_2, \\ \exp\{-\omega_2\}, & \text{on } L_3, \end{cases}$$

where we obtain (4.22) exactly as we derived (4.4). In fact, φ_{a_1} is the conformal map of D onto $\{|z| > 1\}$, $\varphi_{a_1}(a_1) = 1$. It is known that

$$(4.23) \qquad z / \varphi_{a_1}(z^{(1)}) = z \varphi_{a_1}(z^{(2)}) \to \xi_{a_1} \mathrm{cap}(\Delta) \quad \text{as} \quad z \to \infty,$$

where $|\xi_{a_1}| = 1$ and $\mathrm{cap}(\Delta)$ is the logarithmic capacity of Δ. Here we indicate the dependence on the choice of the cycles and of the initial point of integration in (4.21), so that φ_{a_1} will not be confused with φ defined in (2.12). Clearly,

$$(4.24) \qquad \varphi_{a_1}(z^{(1)}) = \bar{\xi}_{a_1} \varphi(z) \quad \text{and} \quad \varphi_{a_1}(z^{(2)}) = \xi_{a_1} \varphi^{-1}(z)$$

for $z \in D$.

4.5. **Abel's Theorem and Jacobi Inversion Problem.** Given any arrangement of distinct points $\mathbf{z}_l, \mathbf{w}_j \in \mathfrak{R}$ and integers $m_l, k_j \in \mathbb{N}$, a *divisor* is a formal symbol

$$(4.25) \qquad d := \sum_l m_l \mathbf{z}_l - \sum_j k_j \mathbf{w}_j.$$

We define the degree of the divisor as $|d| := \sum_l m_l - \sum_j k_j$. By Abel's theorem, d is the principal divisor of a rational function on $\mathfrak{R}$ if, and only if $|d| = 0$ and

$$(4.26) \qquad \sum_l m_l \Omega_1(\mathbf{z}_l) - \sum_j k_j \Omega_1(\mathbf{w}_j) \equiv 0 \quad (\text{mod periods}).$$

When $\mathbf{z}_l$ (resp. $\mathbf{w}_j$) belongs to $L_2 \cup L_3$, we understand under $\Omega_1(\mathbf{z}_l)$ (resp. $\Omega_1(\mathbf{w}_j)$) its boundary values on either side of $L_2 \cup L_3$ as they are congruent to each other. It is also known that the range of Ω_1, as a multi-valued function on $\mathfrak{R}$, is the entire complex plane $\mathbb{C}$. Moreover, for any $a \in \mathbb{C}$ there uniquely exists $\mathbf{z}_a \in \mathfrak{R}$ such that

$$(4.27) \qquad \Omega_1(\mathbf{z}_a) \equiv a \quad (\text{mod periods}).$$

The problem of finding $\mathbf{z}_a$ from a is called the *Jacobi inversion problem*. In particular, the unique solvability of this problem implies that there are no rational functions with a single pole on $\mathfrak{R}$.

Using (4.27), we see that for each $n \in \mathbb{N} \setminus \{1\}$ and $\gamma \in \mathbb{C}$, there uniquely exists $\mathbf{z}_n = \mathbf{z}_n(\gamma) \in \mathfrak{R}$ such that

$$(4.28) \qquad \Omega_1(\mathbf{z}_n) + (n-1)\Omega_1(\infty^{(2)}) - n\Omega_1(\infty^{(1)}) + \gamma \frac{\beta_3}{\beta_2} =: l_n + j_n \frac{\beta_3}{\beta_2} \equiv 0 \quad (\text{mod periods}),$$

$l_n, j_n \in \mathbb{Z}$. Observe that when γ is an integer or an integer multiple of β_2/β_3, the constant $\gamma\beta_3/\beta_2$ is congruent to 0 modulo periods and therefore $\mathbf{z}_n + (n-1)\infty^{(2)} - n\infty^{(1)}$ is the principal divisor of a rational function on $\mathfrak{R}$ by (4.26). In this case, notice also that if $\mathbf{z}_n = \infty^{(2)}$ then necessarily $\mathbf{z}_{n+1} = \infty^{(1)}$ and $l_{n+1} = l_n$, $j_{n+1} = j_n$.

Due to the integral expressions for $\Omega_1(\mathbf{z}_n)$ and $\Omega_1(\infty^{(2)})$, (4.28) can be easily rewritten as

$$(4.29) \qquad n\left(\Omega_1(\infty^{(2)}) - \Omega_1(\infty^{(1)})\right) + (\gamma - j_n)\frac{\beta_3}{\beta_2} = l_n - \int_{\infty^{(2)}}^{\mathbf{z}_n} d\Omega_1.$$

Again, by the very definition of Ω_1, we have that

$$\Omega_1(\infty^{(2)}) - \Omega_1(\infty^{(1)}) = \int_{\infty^{(1)}}^{\infty^{(2)}} d\Omega_1 = \frac{1}{2\pi i} \oint_{L_3} d\Omega_0(\infty^{(2)}, \infty^{(1)}; \mathbf{t}),$$

where the second equality follows from the Riemann relation (4.10). Now, using (4.20), (4.18), and (4.2), we get that

$$(4.30) \qquad \Omega_1(\infty^{(2)}) - \Omega_1(\infty^{(1)}) = \frac{1}{2\pi i} \oint_{L_3} (dG + \omega_2 d\Omega_1) = \omega_\Delta(\Delta_2)\frac{\beta_3}{\beta_2} - \omega_\Delta(\Delta_3).$$

Hence, by plugging (4.30) into (4.29) and rearranging the summands, we arrive at the equality

$$(4.31) \qquad \left(n\omega_\Delta(\Delta_2) - j_n + \gamma\right)\frac{\beta_3}{\beta_2} = n\omega_\Delta(\Delta_3) + l_n - \int_{\infty^{(2)}}^{\mathbf{z}_n} d\Omega_1.$$

In particular, comparing the imaginary parts on both sides of (4.31), we get that

$$(4.32) \qquad \left(n\omega_\Delta(\Delta_2) - j_n + \mathrm{Re}(\gamma)\right)\mathrm{Im}\left(\frac{\beta_3}{\beta_2}\right) = -\mathrm{Im}(\gamma)\mathrm{Re}\left(\frac{\beta_3}{\beta_2}\right) - \mathrm{Im}\left(\int_{\infty^{(2)}}^{\mathbf{z}_n} d\Omega_1\right).$$

Thus, we obtain from (4.32) that

$$(4.33) \qquad \lambda_n := 2\pi i\left(n\omega_\Delta(\Delta_2) - j_n + \gamma\right) = \lambda(\mathbf{z}_n)$$

where

$$(4.34) \qquad \lambda(\mathbf{z}) := -2\pi i\left(\mathrm{Im}(\gamma)\overline{\left(\frac{\beta_3}{\beta_2}\right)} + \mathrm{Im}\left(\int_{\infty^{(2)}}^{\mathbf{z}} d\Omega_1\right)\right)\Big/\mathrm{Im}\left(\frac{\beta_3}{\beta_2}\right).$$

It follows from the definition of Ω_1 and (4.4) that λ is a continuous function in $\widehat{\mathfrak{R}}_2$ with continuous traces on both sides of L_2 that satisfy

$$(4.35) \qquad\qquad \lambda^+ - \lambda^- = 2\pi i.$$

Moreover, it holds that

$$(4.36) \qquad\qquad |\lambda_n| \leq \text{const.}$$

independently of n since $\text{Im}(\beta_3/\beta_2) > 0$ by (4.2) and $|\Omega_1|$ is uniformly bounded above in $\widetilde{\mathfrak{R}}_2$.

4.6. Linear Functions.
Here, we obtain several auxiliary representations for the linear function $z - a_1$ and its multiples. It holds that

$$(4.37) \qquad (z - a_1) \;=\; C_* \exp\left\{ 2 \int_{\infty^{(2)}}^{\mathbf{z}} d\Omega_0(a_1, \infty^{(1)}; \mathbf{t}) - \omega_2 \Omega_1(\mathbf{z}) \right\} \varphi_{a_1}^{-1}(\mathbf{z})$$

$$(4.38) \qquad\qquad\quad =\; C_* \exp\left\{ 2 \int_{\infty^{(1)}}^{\mathbf{z}} d\Omega_0(a_1, \infty^{(2)}; \mathbf{t}) + \omega_2 \Omega_1(\mathbf{z}) \right\} \varphi_{a_1}(\mathbf{z})$$

where

$$(4.39) \qquad\qquad C_* := \xi_{a_1} \text{cap}(\Delta) \exp\left\{ -\omega_2 \Omega_1(\infty^{(1)}) \right\}.$$

To verify (4.37), denote the right-hand side of this expression by E. Then E is a meromorphic function in $\widetilde{\mathfrak{R}}_2$ whose primary divisor is equal to $2a_1 - \infty^{(1)} - \infty^{(2)}$. Moreover, E has continuous traces on both sides of L_2 and L_3. Examining these traces as in (4.4), this time with the help of Figure 5, we get on L_2
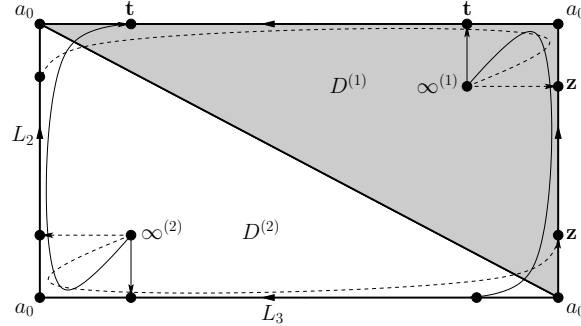

FIGURE 5. Paths of integration for $d\Omega_0(a_1, \infty^{(1)}; \cdot)$ $(d\Omega_0(a_1, \infty^{(2)}; \cdot))$ that start at $\infty^{(2)}$ ($\infty^{(1)}$) and end at $\mathbf{t} \in L_3$ (solid lines) and $\mathbf{z} \in L_2$ (dashed line).

that

$$\frac{E^+}{E^-} \;=\; \exp\left\{ -2 \oint_{L_3} d\Omega_0(a_1, \infty^{(1)}; \mathbf{t}) + \omega_2 \frac{\beta_3}{\beta_2} \right\} \frac{\varphi_{a_1}^-}{\varphi_{a_1}^+}$$

$$(4.40) \qquad\qquad =\; \exp\left\{ 2\pi i \left(2\Omega_1(\infty^{(1)}) + \omega_\Delta(\Delta_2) \frac{\beta_3}{\beta_2} - \omega_\Delta(\Delta_3) \right) \right\} \equiv 1$$

where the first equality follows from (4.4), the second from (4.10) and (4.22), while the last is a consequence of (4.30) and the fact that $\Omega_1(\infty^{(2)}) = -\Omega_1(\infty^{(1)})$. Moreover on L_3

$$(4.41) \qquad \frac{E^+}{E^-} = \exp\left\{ 2 \oint_{L_2} d\Omega_0(a_1, \infty^{(1)}; \mathbf{t}) - \omega_2 \right\} \frac{\varphi_{a_1}^-}{\varphi_{a_1}^+} \equiv 1$$

by (4.4), (4.22) and since $d\Omega_0(a_1, \infty^{(1)}; \cdot)$ has zero period on L_2. Hence, E is a rational function over $\mathfrak{R}$ such that $(E) = 2a_1 - \infty^{(1)} - \infty^{(2)}$. That is, $E(\mathbf{z}) = \mathrm{const}.(z - a_1)$. Now, it is easy to verify by considering the behavior of E near $\infty^{(2)}$ and using (4.23) that C_* is chosen exactly so (4.37) holds.

The validity (4.38) can be shown following exactly the same steps.

In another connection, using properties of $\Omega_0(\mathbf{a}; \cdot)$ it is easy to show by analyzing the boundary behavior on L_2 that

$$(4.42) \qquad \frac{z - a}{a_1 - a} = \exp\left\{\Omega_0(\mathbf{a}; \mathbf{z}) + \Omega_0(\mathbf{a}; \mathbf{z}^*)\right\},$$

where, as usual, $a = \pi(\mathbf{a})$. Let now b be a point in a punctured neighborhood of a_1 with respect to which we defined $\Omega_0(a_1; \cdot)$. Then

$$(4.43) \qquad C(b)\frac{z - a_1}{b - a_1} = \exp\left\{\Omega_0(a_1; \mathbf{z}) + \Omega_0(a_1; \mathbf{z}^*)\right\},$$

where $C(b)$ is the normalizing constant. Clearly,

$$
\begin{aligned}
C(b) \quad &:= \quad \exp\left\{\Omega_0(a_1; b) + \Omega_0(a_1; b^*)\right\} = \exp\left\{\Omega_0(a_1; b^*)\right\} \\
&= \quad \exp\left\{\int_b^{b^*} d\Omega_0(a_1, \infty^{(1)}; \mathbf{t})\right\} = \exp\left\{\left(\int_{\infty^{(2)}}^{b^*} - \int_{\infty^{(2)}}^{b}\right) d\Omega_0(a_1, \infty^{(1)}; \mathbf{t})\right\} \\
(4.44) \qquad &= \quad -\exp\left\{-\omega_2\Omega_1(b)\right\}/\varphi_{a_1}(b),
\end{aligned}
$$

where we used (4.37) as well as the continuity of $C(b)$ as a function of b in a neighborhood of a_1 and the fact that $C(a_1) = -1$ to derive (4.44). To see that $C(a_1) = -1$, pick b on L_1. The integration path $\Gamma(b)$ from b^* till b can be chosen so that $\pi(\Gamma(b))$ is a Jordan curve through $\pi(b)$ since b and b^* have the same canonical projection. Using (4.12) and (4.5) we can write mod $2\pi i$ that

$$\int_b^{b^*} d\Omega_0(a_1, \infty^{(1)}; \mathbf{t}) = \int_{\pi(\Gamma(b))}\left(\frac{w(t)}{2(t - a_1)} + t + c\right)\frac{dt}{w(t)} = \pm\pi i + \int_{\pi(\Gamma(b))}\frac{t + c}{w(t)}dt$$

where c is some constant and the choice of $+$ or $-$ in front of πi depends of the orientation of $\pi(\Gamma(b))$. Finally, it is easy to see that the last integral approaches 0 as b tends to a_1.

4.7. Cauchy-type Integrals on L.

Let χ be a continuous function on L. Define

$$(4.45) \qquad F_\chi(\mathbf{z}) := \frac{1}{2\pi i}\oint_L \chi(\mathbf{t})\frac{w(\mathbf{t}) + w(\mathbf{z})}{t - z}\frac{dt}{2w(\mathbf{t})},$$

$\mathbf{z} \in \mathfrak{R} \setminus (L \cup \{\infty^{(1)}, \infty^{(2)}\})$. It is known [64], and can be verified easily by projecting onto the complex plane using (4.49)–(4.50) below, together with the classical Sokhotski-Plemelj formulae, [20, Sec. I.4.2], see also (3.11), that F_χ is a sectionally holomorphic function in $\mathfrak{R} \setminus (L \cup \{\infty^{(1)}, \infty^{(2)}\})$ with simple poles at $\infty^{(1)}$ and $\infty^{(2)}$ that satisfies

$$(4.46) \qquad F_\chi^+ - F_\chi^- = \chi \quad \text{a.e. on} \quad L.$$

By developing $1/(t - z)$ and $w(z)$ in powers of z at infinity as was done after (4.5), we get that $F_\chi(z^{(1)}) - \ell_\chi(z) \to 0$ and $F_\chi(z^{(2)}) + \ell_\chi(z) \to 0$ as $z \to \infty$, where

$$(4.47) \qquad \ell_\chi(z) = u_\chi z + v_\chi := -z\oint_L\frac{\chi}{2w}\frac{dt}{2\pi i} + \oint_L(A - t)\frac{\chi}{2w}\frac{dt}{2\pi i}$$

and A was defined in (4.5). As for the traces of F_χ on L, it holds that

$$(4.48) \qquad \|F_\chi^\pm\|_{2,L} \le \text{const.}\|\chi\|_{2,L}, \quad \|\cdot\|_{2,L} := \left(\oint_L |\cdot|^2 |d\Omega|\right)^{1/2},$$

where const. is independent of χ. Indeed, we have that

$$
\begin{aligned}
F_\chi(z^{(1)}) &= \frac{w(\mathbf{z})}{4\pi i} \oint_L \frac{\chi(\mathbf{t})}{t-z} \frac{dt}{w(\mathbf{t})} + \frac{1}{4\pi i} \oint_L \frac{\chi(\mathbf{t})}{t-z} dt \\
&= \frac{w(z)}{4\pi i} \int_\Delta \frac{\chi^+(t)+\chi^-(t)}{t-z} \frac{dt}{w^+(t)} + \frac{1}{4\pi i} \int_\Delta \frac{\chi^+(t)-\chi^-(t)}{t-z} dt \\
(4.49) \qquad &= \frac{1}{2}\left(R_\Delta(\chi^+;z) + R_\Delta(\chi^-;z) + C_\Delta(\chi^+;z) - C_\Delta(\chi^-;z)\right), \\
(4.50) \qquad F_\chi(z^{(2)}) &= \frac{1}{2}\left(-R_\Delta(\chi^+;z) - R_\Delta(\chi^-;z) + C_\Delta(\chi^+;z) - C_\Delta(\chi^-;z)\right),
\end{aligned}
$$

where $\chi^\pm(t) := \chi(\mathbf{t})$, $t \in \Delta^\pm$, and the functions R_Δ and C_Δ are defined in (3.3). Hence,

$$
\begin{aligned}
\|F_\chi^+\|_{2,L}^2 &\le \int_\Delta \left(|R_\Delta^+(\chi^+;t)|^2 + |R_\Delta^-(\chi^+;t)|^2 + |R_\Delta^+(\chi^-;t)|^2 + |R_\Delta^-(\chi^-;t)|^2\right) \frac{|dt|}{|w^+(t)|} \\
&\quad + \int_\Delta \left(|C_\Delta^+(\chi^+;t)|^2 + |C_\Delta^-(\chi^+;t)|^2 + |C_\Delta^+(\chi^-;t)|^2 + |C_\Delta^-(\chi^-;t)|^2\right) \frac{|dt|}{|w^+(t)|} \\
&\le \text{const.}\left(\int_\Delta \frac{|\chi^+(t)|^2 + |\chi^-(t)|^2}{|w^+(t)|}|dt|\right) = \text{const.}\|\chi\|_{2,L}^2
\end{aligned}
$$

by Lemma 1.

5. Boundary Value Problems on $\mathfrak{R}$

On an elliptic Riemann surface it is possible to prescribe all but one elements of the zero/pole set of a sectionally meromorphic function with given jump. The following proposition deals with the case when we prescribe n poles at $\infty^{(1)}$ and $n-1$ zeros at $\infty^{(2)}$.

In what follows, we construct a function φ_n which should rather be denoted by $\varphi_{n,\varkappa}$. However, we alleviate the notation and drop the subscript $\varkappa$.

Proposition 10. *Let $\varkappa \in \{1,2,3\}$ be fixed.*
(i) For each $n \in \mathbb{N}\setminus\{1\}$ and $\gamma \in \mathbb{C}$ there exists $\mathbf{z}_n \in \mathfrak{R}$ such that $\mathbf{z}_n + (n-1)\infty^{(2)} - n\infty^{(1)}$ is the principal divisor of a function φ_n which is meromorphic in $\widehat{\mathfrak{R}}_\varkappa$ and has continuous traces on $L_\varkappa$ that satisfy

$$(5.1) \qquad \varphi_n^+ = \varphi_n^- e^{2\pi i \gamma}.$$

Under the normalization $\varphi_n(z)z^{-k_n} \to 1$ as $z \to \infty$, φ_n is the unique function meromorphic in $\widehat{\mathfrak{R}}_\varkappa$ with principal divisor of the form $\mathbf{w} + (n-1)\infty^{(2)} - n\infty^{(1)}$, $\mathbf{w} \in \mathfrak{R}$, and continuous traces on $L_\varkappa$ that satisfy (5.1). Moreover, if $\mathbf{z}_n = \infty^{(1)}$ then $\mathbf{z}_{n-1} = \infty^{(2)}$ and $\varphi_n = \varphi_{n-1}$.
(ii) It holds that

$$(5.2) \qquad \frac{(\varphi_n \varphi_n^*)(z)}{G_\varkappa(\text{cap}(\Delta))^{2n-1}} = \xi_{n,\varkappa} \begin{cases} (z-z_n)/|\varphi(z_n)|, & \mathbf{z}_n \in D^{(2)}\setminus\{\infty^{(2)}\}, \\ \text{cap}(\Delta), & \mathbf{z}_n = \infty^{(2)}, \\ (z-z_n)|\varphi(z_n)|, & \mathbf{z}_n \in L\cup D^{(1)}\setminus\{\infty^{(1)}\}, \end{cases}$$

where $|\xi_{n,\varkappa}| = 1$ and $G_\varkappa := \exp\{2\pi\omega_\Delta(\Delta_\varkappa)\mathrm{Im}(\gamma)\}$.
(iii) *It holds that*

$$(5.3) \qquad \frac{\varphi_n^*(z)}{\varphi_n(z)} = \frac{\xi_{n,\varkappa}G_\varkappa}{\varphi^{2n-1}(z)}\Upsilon_\varkappa(\mathbf{z}_n;z)\begin{cases} \dfrac{z-z_n}{\varphi(z)|\varphi(z_n)|}, & \mathbf{z}_n \in D^{(2)}\setminus\{\infty^{(2)}\}, \\[2mm] \mathrm{cap}(\Delta)/\varphi(z), & \mathbf{z}_n = \infty^{(2)}, \\[2mm] \dfrac{\varphi(z)|\varphi(z_n)|}{z-z_n}, & \mathbf{z}_n \in L \cup D^{(1)}\setminus\{\infty^{(1)}\}, \end{cases}$$

where $\{\Upsilon_\varkappa(\mathbf{a};\cdot)\}_{\mathbf{a}\in\mathfrak{R}}$ is a normal family of non-vanishing functions in D that are uniformly bounded in D for $\mathbf{a}$ outside of any fixed neighborhood of L.

Proof. For definiteness, we put $\varkappa = 3$. Cases where $\varkappa = 1, 2$ are handled similarly upon choosing $L_{\varkappa-1}$ to be the **a**-cycle, $L_\varkappa$ to be the **b**-cycle, and $a_{\varkappa+1}$ to be the initial bound for integration.
(i) Let $\mathbf{z}_n$ be the unique point satisfying (4.28). Set

$$(5.4) \qquad \varphi_n(\mathbf{z}) := \gamma_n \exp\left\{\Omega_0(\mathbf{z}_n;\mathbf{z}) + (n-1)\Omega_0(\infty^{(2)};\mathbf{z}) + 2\pi i(\gamma - j_n)\Omega_1(\mathbf{z})\right\}, \quad \mathbf{z}\in\widetilde{\mathfrak{R}}_2,$$

where γ_n is some constant to be chosen later. Then φ_n is a holomorphic and non-vanishing function in $\widetilde{\mathfrak{R}}_2$ except for a pole of order n at $\infty^{(1)}$, a zero of order $n-1$ at $\infty^{(2)}$, and a simple zero at $\mathbf{z}_n$. Denote by χ the multiplicative jump of φ_n on $L_2 \cup L_3$. That is, $\varphi_n^+ = \varphi_n^-\chi$. Then it follows from (4.4) that

$$(5.5) \qquad \chi = \exp\{2\pi i(\gamma - j_n)\} = \exp\{2\pi i\gamma\} \quad \text{on} \quad L_3.$$

Moreover, we deduce from (4.15), (4.4), and (4.28) that

$$\begin{aligned} \chi &= \exp\left\{-2\pi i\left(\Omega_1(\mathbf{z}_n) + (n-1)\Omega_1(\infty^{(2)}) - n\Omega_1(\infty^{(1)}) + (\gamma - j_n)\frac{\beta_3}{\beta_2}\right)\right\} \\ (5.6) \qquad &= \exp\{-2\pi i l_n\} = 1 \quad \text{on} \quad L_2\setminus\{\mathbf{z}_n\}. \end{aligned}$$

Clearly, (5.6) extends to $\mathbf{z}_n$ as well by continuity when the latter belongs to L_2. Thus, φ_n is, in fact, meromorphic in $\widehat{\mathfrak{R}}_3$ and satisfies (5.1).

Choose γ_n so that $\varphi_n(z)z^{-n} \to 1$ as $z \to \infty$. Let $\widetilde{\varphi}_n$ be a function meromorphic in $\widehat{\mathfrak{R}}_3$ with continuous traces on L_3 satisfying (5.1), such that $(\widetilde{\varphi}_n) = \mathbf{w} + (n-1)\infty^{(1)} - n\infty^{(2)}$ for some $\mathbf{w} \in \mathfrak{R}$, and normalized so that $\widetilde{\varphi}_n(\mathbf{z})z^{-k_n} \to 1$ as $z \to \infty$, where $k_n = n$ if $\mathbf{w} \neq \infty^{(1)}$ and $k_n = n-1$ otherwise. Then the ratio $\varphi_n/\widetilde{\varphi}_n$ is continuous across L_3 and therefore is a rational function over $\mathfrak{R}$. Since $(\varphi_n/\widetilde{\varphi}_n) = \mathbf{z}_n - \mathbf{w}$ and there are no rational functions over $\mathfrak{R}$ with only one pole, $\mathbf{w} = \mathbf{z}_n$ and $\widetilde{\varphi}_n$ is a constant multiple of φ_n. Due to the normalization at $\infty^{(1)}$, it holds that $\widetilde{\varphi}_n = \varphi_n$. That is, φ_n is the unique function with the prescribed properties. The claim for $\mathbf{z}_n = \infty^{(1)}$ follows from (5.4) and the remark made after (4.28).
(ii) Suppose that $\mathbf{z}_n \in \mathfrak{R}\setminus\{a_1, \infty^{(1)}, \infty^{(2)}\}$. Then (5.4) combined with (4.20) and (4.33) yields that

$$(5.7) \qquad \varphi_n(\mathbf{z}) = \gamma_n\varphi_{a_1}^{n-1}(\mathbf{z})\exp\{\Omega_0(\mathbf{z}_n;\mathbf{z}) + (\lambda_n - \omega_2)\Omega_1(\mathbf{z})\}.$$

For brevity, let us put $\gamma_\Delta := \mathrm{cap}(\Delta)$. Using (4.23) and (4.42) we can equivalently write

$$(5.8) \qquad \varphi_n(\mathbf{z}) = \frac{(z - z_n)\varphi_{a_1}^{n-1}(\mathbf{z})\exp\{(\lambda_n - \omega_2)\Omega_1(\mathbf{z}) - \Omega_0(\mathbf{z}_n;\mathbf{z}^*)\}}{(\xi_{a_1}\gamma_\Delta)^{1-n}\exp\{(\lambda_n - \omega_2)\Omega_1(\infty^{(1)}) - \Omega_0(\mathbf{z}_n;\infty^{(2)})\}}.$$

Then it follows from the symmetries $\varphi_{a_1}(\mathbf{z})\varphi_{a_1}(\mathbf{z}^*) \equiv 1$ and $\Omega_1(\mathbf{z}) + \Omega_1(\mathbf{z}^*) \equiv 0$ that

$$(5.9) \qquad \varphi_n(\mathbf{z})\varphi_n(\mathbf{z}^*) = \frac{(z - z_n)(a_1 - z_n)(\xi_{a_1}\gamma_\Delta)^{2n-1}}{\xi_{a_1}\gamma_\Delta\exp\{2(\lambda_n - \omega_2)\Omega_1(\infty^{(1)}) - 2\Omega_0(\mathbf{z}_n;\infty^{(2)})\}}$$

where we used (4.42) once more. Set

$$\alpha(\mathbf{z}) := C_*^{-1}(a_1 - z)\exp\left\{(\omega_2 - 2\lambda(\mathbf{z}))\Omega_1(\infty^{(1)}) + 2\Omega_0(\mathbf{z};\infty^{(2)})\right\},$$

where C_* was defined in (4.39). Clearly, we can rewrite (5.9) as

$$\varphi_n(\mathbf{z})\varphi_n(\mathbf{z}^*) = (\xi_{a_1}\gamma_\Delta)^{2n-1}(z - z_n)\alpha(\mathbf{z}_n). \tag{5.10}$$

Now, we deduce from (4.11) and (4.38) that

$$\begin{aligned}
\alpha(\mathbf{z}) &= C_*^{-1}(a_1 - z)\exp\left\{(\omega_2 - 2\lambda(\mathbf{z}))\Omega_1(\infty^{(1)}) - 2\int_{\infty^{(1)}}^{\mathbf{z}} d\Omega_0(a_1,\infty^{(2)};\mathbf{t})\right\} \\
&= -\exp\left\{-2\lambda(\mathbf{z})\Omega_1(\infty^{(1)}) + \omega_2\left(\Omega_1(\mathbf{z}) - \Omega_1(\infty^{(2)})\right)\right\}\varphi_{a_1}(\mathbf{z}). \tag{5.11}
\end{aligned}$$

Representation (5.11) combined with (4.35), (4.4), and (4.22) yields that α is a continuous function in $\mathfrak{R} \setminus \{\infty^{(1)}\}$ that vanishes at $\infty^{(2)}$, blows up at $\infty^{(1)}$, and is otherwise non-vanishing and finite. It further follows from (5.11) and (4.30) that

$$\left|\frac{\alpha(\mathbf{z})}{\varphi_{a_1}(\mathbf{z})}\right| = \exp\left\{-2\pi\omega_\Delta(\Delta_2)\mathrm{Im}\left(\int_{\infty^{(2)}}^{\mathbf{z}} d\Omega_1\right) - \mathrm{Re}\left(\lambda(\mathbf{z})\left[\omega_\Delta(\Delta_3) - \omega_\Delta(\Delta_2)\frac{\beta_3}{\beta_2}\right]\right)\right\}.$$

The latter expression can be simplified using (4.34), (4.30), and elementary algebra to

$$\left|\alpha(\mathbf{z})/\varphi_{a_1}(\mathbf{z})\right| = \exp\left\{2\pi\omega_\Delta(\Delta_3)\mathrm{Im}(\gamma)\right\} = G_3. \tag{5.12}$$

Hence, (5.2) holds by (5.10) and (5.12) with

$$\xi_{n,3} := \xi_{a_1}^{2n-1}\alpha(\mathbf{z}_n)/|\alpha(\mathbf{z}_n)|. \tag{5.13}$$

When $\mathbf{z}_n = \infty^{(2)}$, we get as in (5.7) and (5.8) that

$$\varphi_n(\mathbf{z}) = (\xi_{a_1}\gamma_\Delta)^n \varphi_{a_1}^n(\mathbf{z})\exp\left\{\lambda_n\Omega_1(\mathbf{z}) - \lambda_n\Omega_1(\infty^{(1)})\right\} \tag{5.14}$$

and therefore

$$\varphi_n(\mathbf{z})\varphi_n(\mathbf{z}^*) = (\xi_{a_1}\gamma_\Delta)^{2n}\exp\left\{-2\lambda(\infty^{(2)})\Omega_1(\infty^{(1)})\right\} =: \gamma_\Delta^{2n}\xi_{n,3}G_3, \tag{5.15}$$

where it can be shown as in (5.12) that $\left|\exp\left\{-2\lambda(\infty^{(2)})\Omega_1(\infty^{(1)})\right\}\right| = G_3$.

Finally, suppose that $\mathbf{z}_n = a_1$. Then we deduce as in (5.8) only using (4.43) instead of (4.42) that

$$\varphi_n(\mathbf{z}) = \frac{(z - a_1)\varphi^{n-1}(\mathbf{z})\exp\left\{(\lambda_n - \omega_2)\Omega_1(\mathbf{z}) - \Omega_0(a_1;\mathbf{z}^*)\right\}}{(\xi_{a_1}\gamma_\Delta)^{1-n}\exp\left\{(\lambda_n - \omega_2)\Omega_1(\infty^{(1)}) - \Omega_0(a_1;\infty^{(2)})\right\}}. \tag{5.16}$$

Further, we get as in (5.9) only by using (4.43) again, that

$$\varphi_n(\mathbf{z})\varphi_n(\mathbf{z}^*) = \frac{(z - a_1)(b - a_1)(\xi_{a_1}\gamma_\Delta)^{2(n-1)}}{C(b)\exp\left\{2(\lambda_n - \omega_2)\Omega_1(\infty^{(1)}) - 2\Omega_0(a_1;\infty^{(2)})\right\}}.$$

Since $\Omega_0(a_1;\infty^{(2)}) = -\int_{\infty^{(2)}}^b d\Omega_0(a_1,\infty^{(1)};\mathbf{t})$ by definition, we get from (4.37), (4.44), and (5.11) that

$$\varphi_n(\mathbf{z})\varphi_n(\mathbf{z}^*) = \frac{-(z - a_1)(\xi_{a_1}\gamma_\Delta)^{2(n-1)}C_*}{\exp\left\{2(\lambda_n - \omega_2)\Omega_1(\infty^{(1)})\right\}} = (\xi_{a_1}\gamma_\Delta)^{2n-1}(z - a_1)\alpha(a_1), \tag{5.17}$$

which finishes the proof of (5.2) upon setting

$$\xi_{n,3} := \xi_{a_1}^{2n-1}\alpha(a_1)/|\alpha(a_1)|. \tag{5.18}$$

(iii) For $\mathbf{z}_n$ with finite canonical projection, we deduce from (5.8) and (5.16) that

(5.19)
$$\frac{\varphi_n(\mathbf{z}^*)}{\varphi_n(\mathbf{z})} = \frac{\exp\{2(\omega_2 - \lambda_n)\Omega_1(\mathbf{z}) - \Omega_0(\mathbf{z}_n;\mathbf{z}) + \Omega_0(\mathbf{z}_n;\mathbf{z}^*)\}}{\varphi_{a_1}^{2(n-1)}(\mathbf{z})}.$$

Suppose first that $\mathbf{z}_n = a_1$. Then we get from (5.19) by using (4.24) and (4.43) that

$$\frac{\varphi_n^*(z)}{\varphi_n(z)} = \left(\frac{\xi_{a_1}}{\varphi(z)}\right)^{2n-2} \frac{1}{C(b)} \frac{b-a_1}{z-a_1} \exp\left\{2(\omega_2 - \lambda_n)\Omega_1(z^{(1)}) + 2\Omega_0(a_1;z^{(2)})\right\}.$$

Hence, (5.3) takes place with

$$\Upsilon_3(a_1;z) := \frac{1}{C(b)} \frac{b-a_1}{\xi_{a_1}\alpha(a_1)} \exp\left\{2(\omega_2 - \lambda(a_1))\Omega_1(z^{(1)}) + 2\Omega_0(a_1;z^{(2)})\right\}$$

by (5.18). Since

$$\frac{1}{C(b)} \frac{b-a_1}{\xi_{a_1}\alpha(a_1)} = \gamma_\Delta \exp\left\{-2(\omega_2 - \lambda(a_1))\Omega_1(\infty^{(1)}) + 2\int_{\infty^{(2)}}^{b} d\Omega_0(a_1,\infty^{(1)};\mathbf{t})\right\}$$

by (4.44), (4.37), (4.39), and (5.11) and $\Omega_0(a_1;z^{(2)}) = \int_b^{z^{(2)}} d\Omega_0(a_1,\infty^{(1)};\mathbf{t})$ by the very definition, we get that

$$\Upsilon_3(a_1;z) = \gamma_\Delta \exp\left\{2(\omega_2 - \lambda(a_1))\left(\Omega_1(z^{(1)}) - \Omega_1(\infty^{(1)})\right) + 2\int_{\infty^{(2)}}^{z^{(2)}} d\Omega_0(a_1,\infty^{(1)};\mathbf{t})\right\}.$$

Suppose now that $\mathbf{z}_n \in (L \setminus \{a_1\}) \cup (D^{(1)} \setminus \{\infty^{(1)}\})$. Then we obtain from (5.19) that

$$\frac{\varphi_n^*(z)}{\varphi_n(z)} = \left(\frac{\xi_{a_1}}{\varphi(z)}\right)^{2n-2} \frac{a_1 - z_n}{z - z_n} \exp\left\{2(\omega_2 - \lambda_n)\Omega_1(z^{(1)}) + 2\Omega_0(\mathbf{z}_n;z^{(2)})\right\},$$

where we used (4.24) and (4.42). Therefore, (5.3) holds with

$$\begin{aligned}
\Upsilon_3(\mathbf{a};z) &:= \frac{a_1 - a}{\xi_{a_1}\alpha(\mathbf{a})} \exp\left\{2(\omega_2 - \lambda(\mathbf{a}))\Omega_1(z^{(1)}) + 2\Omega_0(\mathbf{a};z^{(2)})\right\} \\
&= \gamma_\Delta \exp\left\{2(\omega_2 - \lambda(\mathbf{a}))\left(\Omega_1(z^{(1)}) - \Omega_1(\infty^{(1)})\right) + 2\int_{\infty^{(2)}}^{z^{(2)}} d\Omega_0(\mathbf{a},\infty^{(1)};\mathbf{t})\right\}
\end{aligned}$$

by (5.13) and where we used the definition of α (see the line above (5.11)), (4.39), and (4.13) to derive the second equality. Treating $d\Omega_0(\mathbf{a};\infty^{(1)};\mathbf{t})$ as being identically zero when $\mathbf{a} = \infty^{(1)}$, we can define

$$\Upsilon_3(\infty^{(1)};z) := \gamma_\Delta \exp\left\{2\left(\omega_2 - \lambda(\infty^{(1)})\right)\left(\Omega_1(z^{(1)}) - \Omega_1(\infty^{(1)})\right)\right\}.$$

Then for each $\mathbf{a} \in L \cup D^{(1)}$ the function $\Upsilon_3(\mathbf{a};\cdot)$ is holomorphic and non-vanishing in D such that $\Upsilon(\mathbf{a};\infty) = \gamma_\Delta$. Moreover, the continuity of λ as a function of $\mathbf{a}$ in $L \cup D^{(1)}$, (4.16), and (4.17) imply that

(5.20)
$$\Upsilon_3(\mathbf{t};\cdot) \rightrightarrows \Upsilon_3(\mathbf{a};\cdot)$$

in D as $\mathbf{t} \to \mathbf{a}$, $\mathbf{a}, \mathbf{t} \in L \cup D^{(1)}$.

Assume next that $\mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}$. Then we deduce from (5.19) that

$$\frac{\varphi_n^*(z)}{\varphi_n(z)} = \left(\frac{\xi_{a_1}}{\varphi(z)}\right)^{2n} \frac{z - z_n}{a_1 - z_n} \varphi_{a_1}^2(z^{(1)}) \exp\left\{2(\omega_2 - \lambda_n)\Omega_1(z^{(1)}) - 2\Omega_0(\mathbf{z}_n;z^{(1)})\right\},$$

where as before we used (4.24) and (4.42). Thus, (5.3) holds with

$$\Upsilon_3(\mathbf{a};z) := \frac{\xi_{a_1}}{\alpha(\mathbf{a})(a_1 - a)} \exp\left\{-2\lambda(\mathbf{a})\Omega_1(z^{(1)}) - 2\Omega_0(\mathbf{a};z^{(1)}) + 2\Omega_0(\infty^{(2)};z^{(1)})\right\}$$

due to (5.13) and (4.20). Because

$$\frac{\xi_{a_1}}{\alpha(\mathbf{a})(a_1 - a)} = \frac{1}{\gamma_\Delta} \exp\left\{2\lambda(\mathbf{a})\Omega_1(\infty^{(1)}) + 2\int_{a_1}^{\infty^{(1)}} d\Omega_0(\mathbf{a}, \infty^{(2)};\mathbf{t})\right\}$$

by (5.11), (4.37), (4.39), (4.11), and since $\Omega_0(\mathbf{a};z^{(1)}) - \Omega_0(\infty^{(2)};z^{(1)}) = \int_{a_1}^{z^{(1)}} d\Omega_0(\mathbf{a}, \infty^{(2)};\mathbf{t})$ by (4.11) again, we get that

$$(5.21) \qquad \Upsilon_3(\mathbf{a};z) = \frac{1}{\gamma_\Delta} \exp\left\{-2\lambda(\mathbf{a})\left(\Omega_1(z^{(1)}) - \Omega_1(\infty^{(1)})\right) - 2\int_{\infty^{(1)}}^{z^{(1)}} d\Omega_0(\mathbf{a}, \infty^{(2)};\mathbf{t})\right\}.$$

Finally, assume that $\mathbf{z}_n = \infty^{(2)}$. Then we get from (5.14) that

$$\frac{\varphi_n(\mathbf{z}^*)}{\varphi_n(\mathbf{z})} = \frac{1}{\varphi_{a_1}^{2n}(\mathbf{z})} \exp\left\{-2\lambda(\infty^{(2)})\Omega_1(\mathbf{z})\right\}$$

and therefore (5.3) holds with

$$\Upsilon_3(\infty^{(2)};z) = \frac{1}{\gamma_\Delta} \exp\left\{-2\lambda(\infty^{(2)})\left(\Omega_1(z^{(1)}) - \Omega_1(\infty^{(1)})\right)\right\}$$

by (5.15). Clearly, for each $\mathbf{a} \in D^{(2)}$ the function $\Upsilon_3(\cdot;\infty^{(j)})$ is holomorphic and non-vanishing in D such that $\Upsilon_3(\mathbf{a};\infty) = 1/\gamma_\Delta$. Moreover, the continuity of λ as a function of $\mathbf{a}$ in $D^{(2)}$, (4.16), and (4.17) imply that (5.20) holds for $\mathbf{a},\mathbf{t} \in D^{(2)}$ as well. It only remains to observe that if $\mathbf{t} \to \mathbf{a} \in L$, $\mathbf{t} \in D^{(2)}$, then the limiting function is given by (5.21) used with this given $\mathbf{a}$. $\qquad\square$

In Section 4.5 we explained that $\mathbf{z}_n + (n-1)\infty^{(2)} - n\infty^{(1)}$ is the principal divisor of a rational function over $\mathfrak{R}$ when γ is an integer or an integer multiple of $\beta_\varkappa/\beta_{\varkappa+1}$. In the former case φ_n is exactly this rational function ($e^{2\pi i\gamma} = 1$ and therefore φ_n has no jump across $L_\varkappa$), but in the latter case it is not. In fact, φ_n is then the product of the rational function with principal divisor $\mathbf{z}_n + (n-1)\infty^{(2)} - n\infty^{(1)}$ by a function holomorphic in $\widehat{\mathfrak{R}}_\varkappa$ and having multiplicative jump across L_k as in (5.1).

6. Szegő-type Functions on $\mathfrak{R}$

6.1. **Proof of Proposition 1.** The ground work for the proof of Proposition 1 was done in Sections 3 and 5. Here, we only need to combine the results of these sections.

Fix $\varkappa \in \{1,2,3\}$ and let $G_{h,\varkappa}$, $S_{h,\varkappa}$, and $\varphi_n(= \varphi_{n,\varkappa})$ be as in Propositions 10 and 9, where γ in Proposition 10 equals to $-m_0/(2\pi i\beta_\varkappa)$ with m_0 defined in (3.7). Set

$$(6.1) \qquad S_n(\mathbf{z}) = \begin{cases} \varphi_n(\mathbf{z})/S_{h,\varkappa}(\mathbf{z}), & \mathbf{z} \in D^{(1)} \\ G_{h,\varkappa}\varphi_n(\mathbf{z})S_{h,\varkappa}(\mathbf{z}), & \mathbf{z} \in D^{(2)}. \end{cases}$$

Fix $\mathbf{t} \in L$ and let $D^{(2)} \ni \mathbf{z} \to \mathbf{t}$ so that $z \to t \in \Delta^{\mp}$ (recall that $z = \pi(\mathbf{z})$ and $t = \pi(\mathbf{t})$). Then

$$(6.2) \qquad S_n(\mathbf{z}) \to S_n^-(\mathbf{t}) \quad\text{and}\quad S_n(\mathbf{z}) = \varphi(\mathbf{z})G_{h,\varkappa}S_{h,\varkappa}(\mathbf{z}) \to \varphi^-(\mathbf{t})G_{h,\varkappa}S_{h,\varkappa}^{\mp}(t)$$

by the very definition of S_n. Let now $D^{(1)} \ni \mathbf{a} \to \mathbf{t}$. Then $a \to t \in \Delta^{\pm}$, $S_n(\mathbf{a}) \to S_n^+(\mathbf{t})$, and

$$
\begin{aligned}
S_n(\mathbf{a}) &= \frac{\varphi(\mathbf{a})}{S_{h,\varkappa}(a)} \to \frac{\varphi_n^+(\mathbf{t})}{S_{h,\varkappa}^{\pm}(t)} = \left\{ \begin{array}{ll} \varphi_n^-(\mathbf{t}) G_{h,\varkappa} S_h^{\mp}(t)/h(t), & t \in \Delta \setminus \Delta_{\varkappa}, \\[2mm] \varphi_n^-(\mathbf{t}) \exp\left\{ -\frac{m_0}{\beta_{\varkappa}} \right\} \widetilde{G}_{h,\varkappa} S_{h,\varkappa}^{\mp}(t)/h(t), & t \in \Delta_{\varkappa} \end{array} \right\} \\[3mm]
&= \frac{S_n^-(\mathbf{t})}{h(t)}
\end{aligned}
$$

by (5.1), (3.10), and (6.2). Moreover, it follows easily from Propositions 9 and 10(i) that S_n satisfies all the functional properties required by Proposition 1. Hence, we are left to show uniqueness of S_n. Suppose that $\tilde{S}_n$ is another such function with principal divisor of the form $\mathbf{w} + (n-1)\infty^{(2)} - n\infty^{(1)}$. Then $S_n/\tilde{S}_n$ is a rational function on $\mathfrak{R}$ by the principle of analytic continuation, and it has at most one pole namely $\mathbf{w}$. Therefore it is a constant as there are no rational functions over $\mathfrak{R}$ with one pole. The fact that $S_n = S_{n-1}$ and $\mathbf{z}_{n-1} = \infty^{(2)}$ whenever $\mathbf{z}_n = \infty^{(1)}$ follows from the analogous claim in Proposition 10(i). □

6.2. **Proof of Proposition 2.** By the very definition of S_n, we have that

$$
S_n S_n^* = \varphi_n S_{h,\varkappa}^{-1} G_{h,\varkappa} \varphi_n^* S_{h,\varkappa} = G_{h,\varkappa} \varphi_n \varphi_n^*.
$$

Observe that

$$
G_{h,\varkappa} = \exp\left\{ -m_1 + m_0 \frac{\beta_{\varkappa}^1}{\beta_{\varkappa}} \right\} = G_h \exp\left\{ -\omega_{\Delta}(\Delta_{\varkappa}) \frac{m_0}{\beta_{\varkappa}} \right\}
$$

by (4.19) and since

$$
-m_1 + m_0 a_0 = \frac{1}{\pi i} \int_{\Delta} (a_0 - t) \frac{\log h(t)}{w^+(t)} dt = \int \log h \, d\omega_{\Delta}
$$

by (2.13). As we use Proposition 10 with $\gamma = -m_0/(2\pi i \beta_{\varkappa})$, it holds that

$$
|G_{h,\varkappa}/G_h| = \exp\left\{ -\mathrm{Re}\left(\omega_{\Delta}(\Delta_{\varkappa}) \frac{m_0}{\beta_{\varkappa}} \right) \right\} = \exp\{ -2\pi \omega_{\Delta}(\Delta_{\varkappa}) \mathrm{Im}(\gamma) \} = G_{\varkappa}^{-1}.
$$

Hence, (2.15) follows from (5.2) with $\xi_n := \xi_{n,\varkappa} G_{h,\varkappa} G_{\varkappa}/G_h$. The fact that ξ_n does not depend on $\varkappa$ follows from uniqueness of S_n.

By the same token, we get that $S_n^*/S_n = (G_{h,\varkappa} S_{h,\varkappa}^2)(\varphi_n^*/\varphi_n)$. As $S_{h,\varkappa}$ is a holomorphic and non-vanishing function in D with continuous and non-vanishing trace on ∂D, (2.16) follows from (5.3) with $\Upsilon(\mathbf{a}; \cdot) := S_{h,\varkappa}^2(\cdot) \Upsilon_{\varkappa}(\mathbf{a}; \cdot)$. Again, $\Upsilon(\mathbf{a}; \cdot)$ does not depend on $\varkappa$ by uniqueness of S_n. □

6.3. **An Auxiliary Estimate.** For the proof of Theorem 5 in the case of $\mathbf{z}_n$ approaching L, we need an estimate of the ratio S_n^*/S_n on L^+ (L approached from $D^{(1)}$), see (6.8) below.

We start by constructing a special rational functions of degree 2 on $\mathfrak{R}$. The unique solvability of (4.27) means that Ω_1 and its boundary values from each side on $L_2 \cup L_3$ define an isomorphism from $\mathfrak{R}$ onto the quotient surface $\mathbb{C}/(\mathbb{Z} + (\beta_3/\beta_2)\mathbb{Z})$. In particular,

$$
\delta := \min\left\{ \left| \Omega_1(\mathbf{t}) - \Omega_1(\infty^{(1)}) \right| : \mathbf{t} \in L^+ \right\} > 0.
$$

Let O be any neighborhood of L which is disjoint from $O_{\infty^{(1)}}$, the neighborhood of $\infty^{(1)}$ given by

$$
\tag{6.3} O_{\infty^{(1)}} := \left\{ \mathbf{z} \in \mathfrak{R} : \left| \Omega_1(\mathbf{z}) - \Omega_1(\infty^{(1)}) \right| < \delta/6 \right\},
$$

and such that for any $\mathbf{z} \in O$ there exists $\mathbf{t} \in \mathfrak{R} \setminus O$ satisfying

$$
\tag{6.4} \left| \Omega_1(\mathbf{z}) - \Omega_1(\mathbf{t}) \right| = \delta/3.
$$

Assume that $n \in \mathbb{N}$ is such that $\mathbf{z}_n \in O$. Denote by $\mathbf{t}_{n1}$ a point in $\mathfrak{R} \setminus O$ satisfying (6.4) with $\mathbf{z}$ replaced by $\mathbf{z}_n$. Then, since $\Omega_1(O_{\infty^{(1)}})$ is a disk of diameter $\delta/3$, there exist $\mathbf{w}_n, \mathbf{t}_{n2} \in \partial O_{\infty^{(1)}}$ such that

$$(6.5) \qquad \Omega_1(\mathbf{z}_n) - \Omega_1(\mathbf{t}_{n1}) = \Omega_1(\mathbf{t}_{n2}) - \Omega_1(\mathbf{w}_n).$$

In other words, $\mathbf{z}_n + \mathbf{w}_n - \mathbf{t}_{n1} - \mathbf{t}_{n2}$ is a principal divisor by (4.26), see Figure 6 for the geometric interpretation of the above construction. Let U_n be the rational function over $\mathfrak{R}$ with this principal divisor
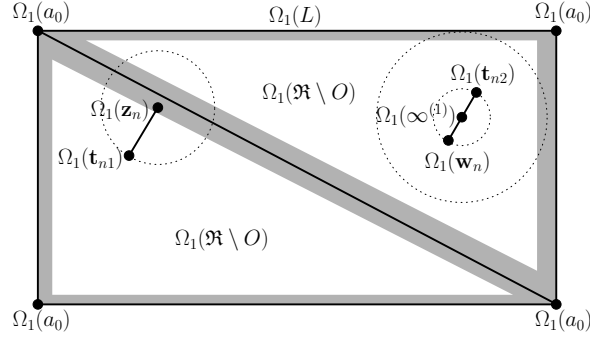

FIGURE 6. Schematic geometric interpretation of the construction of the principle divisor $\mathbf{z}_n + \mathbf{w}_n - \mathbf{t}_{n1} - \mathbf{t}_{n2}$ representing the image of $\mathfrak{R}$ under Ω_1 as a rectangle (in general, it is a centrally symmetric curvilinear rectangle). The shaded region is $\Omega_1(O)$ and the boundary and the diagonal represent the image of L under Ω_1.

normalized so $U_n(\infty^{(1)}) = 1$. Let us stress that the zero of U_n which is not $\mathbf{z}_n$ and the two poles do not belong to O. Observe also that

$$U_n(\mathbf{z}) / U_n(\mathbf{z}^*) = \prod_{\mathbf{a} \in \{\mathbf{z}_n, \mathbf{w}_n\}} \exp\{\Omega_0(\mathbf{a}; \mathbf{z}) - \Omega_0(\mathbf{a}; \mathbf{z}^*)\} \prod_{j=1}^{2} \exp\{\Omega_0(\mathbf{t}_{nj}; \mathbf{z}^*) - \Omega_0(\mathbf{t}_{nj}; \mathbf{z})\}$$

by the properties of $\Omega_0(\mathbf{a}; \cdot)$ (see the paragraph between (4.14) and (4.15)). In particular, it follows from (5.19) that

$$(6.6) \qquad \left| \frac{\varphi_n(\mathbf{t}^*) U_n(\mathbf{t})}{\varphi_n(\mathbf{t}) U_n(\mathbf{t}^*)} \right| \leq \text{const.} \exp\{\Omega_0(\mathbf{w}_n; \mathbf{t}) - \Omega_0(\mathbf{w}_n; \mathbf{t}^*)\} \prod_{j=1}^{2} \exp\{\Omega_0(\mathbf{t}_{nj}; \mathbf{t}^*) - \Omega_0(\mathbf{t}_{nj}; \mathbf{t})\}$$

for $\mathbf{t} \in L^+$, where we used (4.36) and the fact that $|\varphi_{a_1}^{\pm}| = 1$ on L. Since $\Omega_0(\mathbf{a}; \mathbf{t})$ is a continuous function of $\mathbf{t} \in L^+$ for $\mathbf{a} \notin L$ and differentials continuously depend on the parameter $\mathbf{a}$, see (4.7), a compactness argument shows that

$$(6.7) \qquad 1/C_O \leq |\Omega_0(\mathbf{a}; \mathbf{t}) - \Omega_0(\mathbf{a}; \mathbf{t}^*)| \leq C_O$$

on L^+ for all $\mathbf{a} \notin O$ and some finite non-zero constant $C_O = C_0(O)$. Combining (6.6) and (6.7) with the definition of S_n and the boundedness of $S_{b,\varkappa}$, see (6.1), we get that

$$(6.8) \qquad \left| \frac{S_n(\mathbf{t}^*) U_n(\mathbf{t})}{S_n(\mathbf{t}) U_n(\mathbf{t}^*)} \right| \leq \text{const.}$$

for all $t \in L^+$ and n such that $\mathbf{z}_n \in O$, where the constant depends only on O.

6.4. **Proof of Proposition 3.** By definition, see (4.31), the point $\mathbf{z}_n$ is the solution of the Jacobi inversion problem

$$(6.9) \qquad \int_{\infty^{(2)}}^{\mathbf{z}_n} d\Omega_1 \equiv n\omega_\Delta(\Delta_3) - (n\omega_\Delta(\Delta_2) + \gamma)\frac{\beta_3}{\beta_2} = v(n), \quad (\text{mod periods}),$$

where $v(t) := t\omega_\Delta(\Delta_3) - (t\omega_\Delta(\Delta_2) + \gamma)(\beta_3/\beta_2)$. Denote by $\mathbf{V}$ the set of the limit points of $\{v(n)\}$ in the Jacobi variety $\mathrm{Jac}(\mathfrak{R}) := \mathbb{C}/\{n + m(\beta_3/\beta_2)\}$. Since the function $\int_{\infty^{(2)}}^{\mathbf{z}} d\Omega_1$ is a holomorphic bijection of $\mathfrak{R}$ onto $\mathrm{Jac}(\mathfrak{R})$, the set $\mathbf{Z}$ is equal to $\mathfrak{R}$, is a finite set of points in $\mathfrak{R}$, or is a finite union of pairwise disjoint arcs on $\mathfrak{R}$ if and only if $\mathbf{V} = \mathrm{Jac}(\mathfrak{R})$, is a finite set of points in $\mathrm{Jac}(\mathfrak{R})$, or is a finite union of pairwise disjoint arcs on $\mathrm{Jac}(\mathfrak{R})$, respectively. Clearly, $\mathbf{V} \subset \{v(t)\}_{t\in\mathbb{R}}$. The structure of the sets $\mathbf{V}$ and $\{v(t)\}_{t\in\mathbb{R}}$ on $\mathrm{Jac}(\mathfrak{R})$ for an elliptic Riemann surface $\mathfrak{R}$ was analyzed in [61, Sec. 5], depending on arithmetic properties of the numbers $\omega_\Delta(\Delta_k)$, $k \in \{1,2,3\}$. More precisely, it was shown that $\mathbf{V} = \mathrm{Jac}(\mathfrak{R})$ when the numbers $\omega_\Delta(\Delta_k)$ are rationally independent; that $\mathbf{V}$ is a finite set of points when the numbers $\omega_\Delta(\Delta_k)$ are rational; and that $\mathbf{V}$ is the union of a finite number of pairwise disjoint arcs when the numbers $\omega_\Delta(\Delta_k)$ are rationally dependent but at least one of them is irrational.

The second conclusion of the proposition easily follows from (6.9). Indeed, if $\mathbf{z}_n = \mathbf{z}_{n+m}$ for some $n, m \in \mathbb{N}$, then

$$m\left(\omega_\Delta(\Delta_3) - \omega_\Delta(\Delta_2)\frac{\beta_3}{\beta_2}\right) \equiv 0 \quad (\text{mod periods}).$$

By comparing first imaginary and then real parts of the equation above, we see that $m\omega_\Delta(\Delta_k) \in \mathbb{N}$. Hence, whenever at least one of the numbers $\omega_\Delta(\Delta_k)$ is irrational all the points $\mathbf{z}_n$ are mutually distinct. However, if all $\omega_\Delta(\Delta_k)$ are rational, then $\mathbf{z}_n = \mathbf{z}_{n+m}$ for all $n \in \mathbb{N}$, where m is the least common multiple of their denominators.

It remains for us to show that the set of non-collinear triples $(a_1, a_2, a_3) \in \mathbb{C}^3$ whose corresponding $\omega_\Delta(\Delta_k)$ are rationally dependent has Lebesgue measure zero. Our point of departure is a characterization, given in [30], of the Chebotarëv center a_0 for triples of the form $(a_1, a_2, a_3) = (0, e^{i\alpha}, \rho^2 e^{-i\alpha})$ with $\alpha \in (0, \pi/2)$ and $0 < \rho < 1$. For such triples we indicate the dependence on (α, ρ) by writing $\Delta_k(\alpha, \rho)$ for the arcs constituting the Chebotarëv continuum, and we set $\omega_\Delta(\Delta_j(\alpha, \rho))$, $j = 1, 2, 3$, for the mass of the equilibrium measure on $\Delta_j(\alpha, \rho)$. Note that $\omega_\Delta(\Delta_2(\alpha, \beta)) > \omega_\Delta(\Delta_3(\alpha, \beta))$ since $\rho < 1$ [30, Thm. 1.4]. We also define

$$(6.10) \qquad \left\{ \begin{array}{rcl} \lambda_1 & := & \omega_\Delta(\Delta_2(\alpha, \rho)) + \omega_\Delta(\Delta_3(\alpha, \rho)), \\ \lambda_2 & := & \omega_\Delta(\Delta_2(\alpha, \rho)) - \omega_\Delta(\Delta_3(\alpha, \rho)). \end{array} \right.$$

Let $\mathbb{C}^*/\pm$ be the quotient of $\mathbb{C} \setminus \{0\}$ by the equivalence relation $(z_1 \sim z_2) \implies z_1 = \pm z_2$. Denote by $U \subset (\mathbb{R}_+)^2$ the set of pairs (t_1, t_2) with $0 < t_2 < t_1$ and $t_1 + t_2 < 2$. From [30, Thm. 1.6], and its proof, we deduce that λ_1, λ_2, and a_0 are the last components of a unique 6-tuple $(p, k, \mu, \lambda_1, \lambda_2, a_0) \in (\mathbb{C} \setminus \{0, \pm 1\}) \times (\mathbb{C}^*/\pm) \times \mathbb{C} \times U \times \mathbb{C}$ satisfying[7]

$$(6.11) \qquad \left\{ \begin{array}{l} p - \left(1 + (e^{-i\alpha} + \rho^{-2}e^{i\alpha})a_0 + \rho^{-2}a_0^2\right)^{1/2} = 0, \\[2mm] k - \left(\frac{p + 1 - (e^{-i\alpha} + \rho^{-2}e^{i\alpha})a_0/2}{2p}\right)^{1/2} = 0, \\[2mm] \mathrm{cn}(\mu, k) - \frac{1-p}{1+p} = 0, \\[2mm] \mu - \lambda_1 \mathbf{K}(k) - i\lambda_2 \mathbf{K}(k') = 0, \\[2mm] 2\mathbf{K}(k)\left(\frac{a_0 p^{1/2}}{\rho(1+p)} - \frac{\Theta_4'\left(\frac{\mu}{2\mathbf{K}(k)}|\tau\right)}{2\mathbf{K}(k)\Theta_4\left(\frac{\mu}{2\mathbf{K}(k)}|\tau\right)}\right) + i\pi\lambda_2 = 0, \end{array} \right.$$

---

[7]The square root in the second equation of (6.11) is apparently missing in the statement just quoted; the need for it can be checked from the proof, *cf.* equation (1.36) of that reference.

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind with (complex) modulus k and $k' = (1-k^2)^{1/2}$ is the complementary modulus, $\Theta_4(z\,|\,\tau)$ is the Jacobi theta function with period 1 and quasi-period $\tau := i\mathbf{K}(k')/\mathbf{K}(k)$, while $\mathbf{cn}(\cdot, k)$ is the Jacobi elliptic function with periods $4\mathbf{K}(k)$ and $2\mathbf{K}(k) + i2\mathbf{K}(k')$. The principal branch of the square root is used in the first equation, where the quantity under square root there cannot be negative because a_0 necessarily lies in the open triangle $K(0, e^{i\alpha}, \rho^2 e^{-i\alpha})$. Consequently p actually lies in $\mathscr{H}_+ \setminus \{1\}$, where $\mathscr{H}_+$ indicates the open right half plane. The branch used in the second equation is immaterial. The third equation addresses the equivalence class of μ modulo periods, but the fourth selects a unique representative for μ because $(t_1, t_2) \in U$. Of necessity, it holds that $k^2 \neq 1$, that $\mathbf{K}(k), \mathbf{K}(k') \neq 0, \infty$, and that $\mathrm{Im}(i\mathbf{K}(k')/\mathbf{K}(k)) \neq 0, \infty$ [30, eqns. (1.33)&(1.60)]. In particular, the fourth equation in (6.11) yields that

$$(6.12) \qquad \lambda_1 = \frac{\mathrm{Re}\left(\overline{\mu}\,\mathbf{K}(k')\right)}{\mathrm{Re}\left(\overline{\mathbf{K}(k)}\mathbf{K}(k')\right)} \quad \text{and} \quad \lambda_2 = \frac{\mathrm{Im}\left(\overline{\mu}\,\mathbf{K}(k)\right)}{\mathrm{Re}\left(\overline{\mathbf{K}(k)}\mathbf{K}(k')\right)}.$$

Set $Z := \rho e^{-i\alpha} + (\rho e^{-i\alpha})^{-1}$, and note that the map $(\alpha, \rho) \mapsto Z$ is a real analytic homeomorphism from $(0, \pi/2) \times (0, 1)$ onto the positive quadrant $\mathscr{Q} := \{z = x + iy : x > 0, y > 0\}$. If we further let $A_0 := a_0/\rho$, then (6.11) and (6.12) provide us with four relations:

$$(6.13) \qquad \begin{cases} 0 &= p^2 - 1 - ZA_0 - A_0^2, \\ 0 &= 2k^2 p - p - 1 + ZA_0/2, \\ 0 &= \mathbf{cn}(\mu, k) - (1-p)/(1+p), \\ 0 &= 2\mathbf{K}(k)\left(\dfrac{A_0 p^{1/2}}{1+p} - \dfrac{\Theta_4'\left(\frac{\mu}{2\mathbf{K}(k)}\Big|\tau\right)}{2\mathbf{K}(k)\Theta_4\left(\frac{\mu}{2\mathbf{K}(k)}\Big|\tau\right)}\right) + i\pi\dfrac{\mathrm{Im}\left(\overline{\mu}\,\mathbf{K}(k)\right)}{\mathrm{Re}\left(\overline{\mathbf{K}(k)}\mathbf{K}(k')\right)}. \end{cases}$$

From the first two equations in (6.13) we obtain

$$(6.14) \qquad A_0 = (p^2 - 3 - 2p + 4k^2 p)^{1/2}$$

where the principal branch of the square root is used (again the quantity under square root cannot be negative when (6.13) holds for $a_0 \in K(0, e^{i\alpha}, \rho^2 e^{-i\alpha})$), and we are left with the following system of equations

$$(6.15) \qquad H_1(p, k, Z) = H_2(p, k, \mu) = H_3(p, k, \mu) = 0$$

where

$$\begin{cases} H_1(p, k, Z) &:= 2k^2 p - p - 1 + Z(p^2 - 3 - 2p + 4k^2 p)^{1/2}/2, \\ H_2(p, k, \mu) &:= \mathbf{cn}(\mu, k) - (1-p)/(1+p), \\ H_3(p, k, \mu) &:= 2\mathbf{K}(k)\left(\dfrac{(p^2 - 3 - 2p + 4k^2 p)^{1/2} p^{1/2}}{1+p} - \dfrac{\Theta_4'\left(\frac{\mu}{2\mathbf{K}(k)}\Big|\tau\right)}{2\mathbf{K}(k)\Theta_4\left(\frac{\mu}{2\mathbf{K}(k)}\Big|\tau\right)}\right) + i\pi\dfrac{\mathrm{Im}\left(\overline{\mu}\,\mathbf{K}(k)\right)}{\mathrm{Re}\left(\overline{\mathbf{K}(k)}\mathbf{K}(k')\right)}. \end{cases}$$

Set $\mathscr{M}$ to be the open subset of $\mathbb{C}^*/\pm$ comprised of $k \neq \pm 1$ for which $\mathrm{Im}(i\mathbf{K}(k')/\mathbf{K}(k)) \neq 0, \infty$ ($\mathbb{C}^*/\pm$ being endowed with the quotient topology), and let $\mathscr{U}$ be the open subset of the analytic manifold $\left(\mathscr{H}_+ \setminus \{1\}\right) \times \mathscr{M} \times \mathbb{C}$ consisting of triples (p, k, μ) for which the quantity $-(4k^2 p - 2p - 2)/(4k^2 p - 2p - 3 + p^2)$ (i.e. the value of Z when $H_1(p, k, Z) = 0$) lies in $\mathscr{Q}$ and such that the right hand sides of equations (6.12) define a member of U.

Then, solutions $(p, k, \mu, \lambda_1, \lambda_2, a_0)$ to (6.11) as described above project injectively onto the real analytic[8] variety $\mathscr{V} \subset \mathscr{U}$ of those (p, k, μ) such that $H_2(p, k, \mu) = H_3(p, k, \mu) = 0$. Note that $\mathscr{V}$ is distinct

---

[8]Note that H_3 is not complex analytic

from $\mathcal{U}$ since for fixed p,k the set of μ with $H_2(p,k,\mu)=0$ is discrete. By a theorem of Lojaciewicz [28, Thm. 5.2.3], as $\mathcal{U}$ has real dimension 6, we get that $\mathcal{V}$ decomposes into a disjoint union $\cup_{j=0}^{5}\mathcal{V}_j$ where $\mathcal{V}_j$ is a real analytic submanifold of $\mathcal{U}$ of dimension j. Pick $j\in\{0,\ldots,5\}$ and consider the map $\Phi_j:\mathcal{V}_j\times\mathcal{Q}\to\mathbb{C}$ given by $\Phi_j(p,k,\mu,Z)=H_1(p,k,Z)$. From (6.15) and (6.14), we see that the partial derivative[9] $\mathbf{D}_Z\Phi_j$ is multiplication by $A_0/2$, which is bijective at any (p,k,μ,Z) where $\Phi_j(p,k,\mu,Z)=0$ since $A_0\neq 0$. Hence, by the transversality theorem [25, Ch. 2], it holds for almost every $Z\in\mathcal{Q}$ that the partial map $\Phi_{j,Z}(p,k,\mu):=\Phi_j(p,k,\mu,Z)$, defined on $\mathcal{V}_j$ with values in $\mathbb{C}$, is transverse to the submanifold $\{0\}\subset\mathbb{C}$ which has codimension 2. This means that, for a.e. Z, the set $\Phi_{j,Z}^{-1}(0)$ is either empty or a submanifold of $\mathcal{V}_j$ of codimension 2 whose tangent space at $(p,k,\mu)\in\Phi_{j,Z}^{-1}(0)$ is the kernel of the derivative $\mathbf{D}\Phi_{j,Z}(p,k,\mu):\mathbf{T}_{(p,k,\mu)}\mathcal{V}_j\to\mathbb{R}^2$, where $\mathbf{T}_{(p,k,\mu)}\mathcal{V}_j$ indicates the tangent space to $\mathcal{V}_j$ at (p,k,μ). However, we know from existence and uniqueness of a solution to (6.11) that $\Phi_{j,Z}^{-1}(0)$ consists of a single point, say $(p_Z,k_Z,\mu_Z)\in\mathcal{U}$. Therefore, we get for a.e. Z that $\Phi_{j,Z}^{-1}(0)=\emptyset$ if $j\neq 2$, and that $(p_Z,k_Z,\mu_Z)\in\mathcal{V}_2$ is such that $\mathbf{D}\Phi_{2,Z}(p_Z,k_Z,\mu_Z)$ is an isomorphism from $\mathbf{T}_{(p_Z,k_Z,\mu_Z)}\mathcal{V}_2$ onto $\mathbb{R}^2$. Subsequently, by the implicit function theorem, such Z form an open set $\mathcal{Z}\subset\mathcal{Q}$ over which p_Z,k_Z,μ_Z are real analytic functions of Z.

We claim that λ_1,λ_2 cannot both be constant on some nonempty open set $B\subset\mathcal{Z}$. Suppose indeed this is the case. Then, we see from (6.11) that $\mu_Z=F(k_Z)$ for $Z\in B$, where F is a globally defined holomorphic function on $(\mathbb{C}^*/\pm)\setminus\{\pm 1\}$. In turn, the third equation in (6.13) entails that $p_Z=G(k_Z)$ where G is again a globally defined holomorphic function on $(\mathbb{C}^*/\pm)\setminus\{\pm 1\}$. Thus, by uniqueness of a solution, the correspondence $Z\mapsto k_Z$ must be injective from B onto some set $E\subset\mathbb{C}^*/\pm$, and since this correspondence is continuous (in fact: real analytic) E is open [38, Thm. 36.5]. This shows that the open subset of $\mathcal{V}_2$ consisting of triples (p_Z,k_Z,μ_Z) with $Z\in B$ is holomorphically parametrized by $k_Z\in E$ and is in fact a Riemann surface. In particular, the third equation in (6.15) tells us that

$$2\mathbf{K}(k)\left(\frac{(G^2(k)-3-2G(k)+4k^2G(k))^{1/2}G^{1/2}(k)}{1+G(k)}-\frac{\Theta_4'\left(\frac{F(k)}{2\mathbf{K}(k)}|\tau\right)}{2\mathbf{K}(k)\Theta_4\left(\frac{F(k)}{2\mathbf{K}(k)}|\tau\right)}\right)+i\pi\lambda_2=0$$

for all $k\in E$. As the left hand side is a globally defined holomorphic function on $(\mathbb{C}^*/\pm)\setminus\{\pm 1\}$ (remember λ_2 is assigned to some constant value) it must be the zero function. Hence, for *any* $Z\in\mathcal{H}_+\setminus\{1\}$, the solution to (6.15), which is known to exists and to be unique, is obtained from k by plugging $\mu=F(k)$ and $p=G(k)$ while fixing λ_1,λ_2 to the constant values they assume on B, because then all the equations will be satisfied. In particular λ_1,λ_2 are constant functions of $Z\in\mathcal{H}_+\setminus\{1\}$. Back to the original variables, we get that the $\omega_\Delta(\Delta_j(\alpha,\rho))$ are constant (remember $\sum_j\omega_\Delta(\Delta_j(\alpha,\rho))=1$). However, this cannot be because $\omega_\Delta(\Delta_1(\alpha,\rho))=\omega_\Delta(\Delta_2(\alpha,\rho))$ when $\rho^2=1/(2\cos(2\alpha))$ for some $\alpha<\pi/6$ (*i.e.* a_3 lies on the perpendicular bisector of $[a_1,a_2]$) whereas $\omega_\Delta(\Delta_1(\alpha,\rho))<\omega_\Delta(\Delta_2(\alpha,\rho))$ when $2\rho^2\cos(2\alpha)<1$ (*i.e.* if $|a_3-a_1|<|a_3-a_2|$), see [30, Thm. 1.4]. *This proves the claim.*

Recaping what we did in terms of variables (α,ρ), we find in view of (6.10) and (6.14) that there is an open subset $\mathcal{W}\subset(0,\pi/2)\times(0,1)$, whose complement has zero measure, over which a_0 and $\omega_\Delta(\Delta_j(\alpha,\rho))$, $j=1,2,3$, are real analytic functions of (α,ρ). Moreover, the correspondence

$$(\alpha,\rho)\mapsto\left(\omega_\Delta(\Delta_1(\alpha,\rho)),\omega_\Delta(\Delta_2(\alpha,\rho))\right)$$

cannot be constant on a nonempty open subset of $\mathcal{W}$.

---

[9]We understand by $\mathbf{D}_Z\Phi_j$ the derivative of $Z\mapsto\Phi_j(p,k,\mu,Z)$ as a map from an open subset of $\mathbb{R}^2$ (*i.e.* $\mathcal{Q}$) into $\mathbb{R}^2\sim\mathbb{C}$. As H_1 is holomorphic in Z, this derivative is just multiplication by the complex number $\partial\Phi_j/\partial Z(p,k,\mu,Z)$ viewed as a real linear map on $\mathbb{R}^2$.

We now consider triples of the form $(a_1, a_2, a_3) = (0, i, a)$ where $a \in \mathbb{D}_+ := \{z; |z| < 1, \mathrm{Re}\, z > 0\}$. Accordingly, we put $\Delta_k(a)$, $k \in \{1, 2, 3\}$, for the analytic arcs constitutive of the corresponding Chebotarëv continuum, and write $c(a)$ for the Chebotarëv center. Observe that the map $a(\alpha, \rho) := i\rho^2 e^{-2i\alpha}$ is a real analytic diffeomorphism from $(0, \pi/2) \times (0, 1)$ onto $\mathbb{D}_+$, and that the triples $(0, e^{i\alpha}, \rho^2 e^{-i\alpha})$ and $(0, i, a(\alpha, \rho))$ differ by a rotation of angle $\pi/2 - \alpha$. Thus $c(a(\alpha, \rho)) = ie^{-i\alpha} a_0$ and $\omega_\Delta(\Delta_j(a(\alpha, \rho))) = \omega_\Delta(\Delta_j(\alpha, \rho))$, $1 \leq j \leq 3$. Moreover, by what precedes, there is an open subset $\mathscr{T} \subset \mathbb{D}_+$, with $\mathbb{D}_+ \setminus \mathscr{T}$ of zero measure, such that the maps $a \mapsto c(a)$ and $a \mapsto \omega_\Delta(\Delta_j(a))$, from $\mathscr{T}$ into $\mathbb{C}$ and $\mathbb{R}$ respectively, are real analytic. Moreover, $a \mapsto (\omega_\Delta(\Delta_1(a)), \omega_\Delta(\Delta_2(a))$ is not constant over a nonempty open subset of $\mathscr{T}$.

Consider the open subset $\mathscr{A} \subset \mathscr{T} \times \mathbb{D}_+$ of those (a, b) such that b lies in the triangle $K(0, i, a)$. For $(a, b) \in \mathscr{A}$, consider the quadratic differential

$$(6.16) \qquad Q_{a,b}(z) := -\frac{1}{\pi} \frac{z - b}{z(z - i)(z - a)} dz^2.$$

Note that $Q_{a,c(a)}$ is minus the quadratic differential (2.1) where $a_1 = 0$, $a_2 = i$, and $a_3 = a$, so that $Q_{a,c(a}(z)dz^2 > 0$ on $\Delta_j^\circ(a)$, $j = 1, 2, 3$. Define further

$$(6.17) \qquad \Psi_1(a, b) := \int_0^b Q_{a,b}^{1/2}(z)\, dz, \qquad \Psi_2(a, b) := \int_i^b Q_{a,b}^{1/2}(z)\, dz$$

where, by Cauchy's theorem, the integrals may be taken over any smooth path joining 0 (resp. i) to b whose interior lies in $K(0, i, a)$, and where the branch of the square root is positive if $b = c(a)$ and if the path $\Delta_1(a)$ (resp. $\Delta_2(a)$) is used. By (2.13), we have that

$$(6.18) \qquad \Psi_1(a, c(a)) = \omega_\Delta(\Delta_1(a)), \qquad \Psi_2(a, c(a)) = \omega_\Delta(\Delta_2(a)).$$

Writing $a = x_a + iy_a$ to single out real and imaginary parts, we introduce differential operators $\partial_a := (\partial_{x_a} - i\partial_{y_a})/2$ and $\overline{\partial}_a := (\partial_{x_a} + i\partial_{y_a})/2$. We define ∂_b and $\overline{\partial}_b$ similarly. Those a for which $\overline{\partial}_a c(a) = 0$ form a real analytic variety, say $\mathscr{X} \subset \mathscr{T}$. We claim that $\mathscr{X}$ has measure zero. To prove this, it is enough by Lojaciewicz's theorem to show that $\mathscr{X}$ has no interior. Assume for a contradiction that it contains an open set $V \neq \emptyset$, so that $c(a)$ is a holomorphic function of $a \in V$. Then, by inspection of (6.16)-(6.17), the function $\Psi_j(a, c(a))$ is in turn holomorphic on V for $j = 1, 2$. However, it is real valued by (6.18) hence it must be constant. But then, $a \mapsto (\omega_\Delta(\Delta_1(a)), \omega_\Delta(\Delta_2(a)))$ is constant over V which is impossible, as pointed out earlier. *This proves the claim.* Thus, $\mathscr{A} := \mathscr{T} \setminus \mathscr{X}$ is open and $\mathbb{D}_+ \setminus \mathscr{A}$ has zero measure.

Next, since $\Psi_j(a, b)$ is holomorphic in a, b, it holds that $\overline{\partial}_a \Psi_j(a, b) = \overline{\partial}_b \Psi_j(a, b) = 0$, so by (6.18) and the chain rule

$$(6.19) \qquad \overline{\partial}_a \omega_\Delta(\Delta_1(a)) = \left(\partial_b \Psi_1(a, b)_{|b=c(a)} \right) \overline{\partial}_a c(a) = \frac{\overline{\partial}_a c(a)}{2} \int_0^{c(a)} \frac{Q_{a,c(a)}^{1/2}(z)}{z - c(a)}\, dz$$

and likewise

$$(6.20) \qquad \overline{\partial}_a \omega_\Delta(\Delta_2(a)) = \frac{\overline{\partial}_a c(a)}{2} \int_i^{c(a)} \frac{Q_{a,c(a)}^{1/2}(z)}{z - C(a)}\, dz.$$

Fix $\Delta_1(a)$ to be the integration path in (6.19), and let z_3 be the intersection of the straight line through a, $c(a)$ with the segment $[0, i]$. Since a is strictly closer to 0 than i, it follows from [30, Thm. 4.1.] that $\Delta_1(a)$ is included in the closure of the triangle $K(0, z_3, c(a))$ but not in its boundary. Therefore, since $Q_{a,c(a)}^{1/2}dz > 0$ on $\Delta_1^\circ(a)$, the integral in (6.19) lies in $\overline{C(-c(a), 0, z_3 - c(a))}$, the complex conjugate of the open positive cone $C(-c(a), 0, z_3 - c(a))$ with vertex 0 generated by the triangle $K(-c(a), 0, z_3 - c(a))$. Similarly, if we let z_1 be the intersection of the straight line through 0, $c(a)$ with the segment $[i, a]$,

we get from [30, Thm. 4.1.] that $\Delta_2(a)$ is contained[10] in the closure of $K(i, c(a), z_1) \cup K(i, c(a), z_3)$, hence the integral in (6.20) lies in the closure of $\overline{C(z_1 - c(a), 0, z_3 - c(a))}$. Since the latter is disjoint from $\overline{C(-c(a), 0, z_3 - c(a))} \cup \left(-\overline{C(-c(a), 0, z_3 - c(a))}\right)$, we deduce that the integrals in (6.19) and (6.20) define two complex numbers that are linearly independent over $\mathbb{R}$. If in addition $a \in \mathscr{A}$, we conclude since $\overline{\partial}_a c(a) \neq 0$ that $\overline{\partial}_a \omega_\Delta(\Delta_1(a))$ and $\overline{\partial}_a \omega_\Delta(\Delta_2(a))$ are in turn linearly independent over $\mathbb{R}$. By definition of $\overline{\partial}_a$, this means that the map $\Lambda(a) := (\omega_\Delta(\Delta_1(a)), \omega_\Delta(\Delta_1(a)))$ from $\mathbb{D}_+$ into $\mathbb{R}^2$ has nonsingular derivative at each point of $\mathscr{A}$, in particular its restriction to $\mathscr{A}$ is locally a bi-Lipschitz homeomorphism. From this, as $\mathscr{A}$ is open of full measure in $\mathbb{D}_+$, it is elementary to check that if $E \subset \Lambda(\mathbb{D}_+))$ has measure zero (resp. is dense in $\Lambda(\mathbb{D}_+)$) then its inverse image $\Lambda^{-1}(E)$ has measure zero (resp. is dense in $\mathbb{D}_+$).

Now, since $\omega_\Delta(\Delta_1(a)) + \omega_\Delta(\Delta_2(a)) + \omega_\Delta(\Delta_3(a)) = 1$, the $\omega_\Delta(\Delta_j(a))$ are rationally dependent if and only if there exist integers n_1, n_2, not both zero, for which $n_1 \omega_\Delta(\Delta_1) + n_2 \omega_\Delta(\Delta_2) \in \mathbb{Q}$. To each nonzero pair of integers (n_1, n_2), we can pick real numbers t_1, t_2 with $n_1 t_2 - n_2 t_1 \neq 0$, hence the subset of $\mathbb{R}^2$ comprised of (x, y) such that $n_1 x + n_2 y \in \mathbb{Q}$ has measure zero, being the inverse image under $(x, y) \mapsto (n_1 x + n_2 y, t_1 x + t_2 y)$ of those points whose first coordinate is rational. Consequently the set $\mathscr{Y} \subset \mathbb{R}^2$ of those (x, y) for which there exists a nonzero pair of integers (n_1, n_2) such that $n_1 x + n_2 y \in \mathbb{Q}$ has measure zero as countable union of sets of measure zero. Note that $\mathscr{Y}$ contains the dense subset of rational pairs. Thus, by properties of the map Λ we just proved, the set of $a \in \mathbb{D}_+$ for which the triple $(0, i, a)$ has $\mathbb{Q}$-linearly independent (resp dependent, rational) $\omega_\Delta(\Delta_j(a))$ has full measure (resp. is dense) in $\mathbb{D}_+$. Because the $\omega_\Delta(\Delta_k)$ are invariant under nonsingular affine transformations of the triple (a_1, a_2, a_3) and their conjugates [48, Thm. 5.1.2], it follows easily that, for *any* pair $(a_1, a_2) \in \mathbb{C}^2$, the set of non-collinear a_3 for which the triple (a_1, a_2, a_3) has $\mathbb{Q}$-linearly independent (resp. $\mathbb{Q}$-linearly dependent, rational) $\omega_\Delta(\Delta_j)$ has complement of measure zero (resp. is dense) in $\mathbb{C}$. Proposition 3 is now a consequence of Fubini's theorem.   □

7. ASYMPTOTICS OF PADÉ APPROXIMANTS

7.1. **Integral Representation of the Error.** Let f_h be given by (2.20) and $\pi_n = p_n/q_n$ be the n-th Padé approximant to f_h. Then we deduce from (2.18) that

$$\oint_\Gamma z^k (q_n f_h - p_n)(z) dz = 0, \quad k \in \{0, \ldots, n-1\},$$

by Cauchy integral formula applied in the exterior of Γ, where Γ is any positively oriented Jordan curve encompassing Δ. Applying Cauchy integral formula once more, this time in the interior of Γ, we get that

$$0 = \oint_\Gamma z^k (q_n f_h)(z) dz = \oint_\Gamma z^k q_n(z) \frac{1}{\pi i} \int_\Delta \frac{h(t)}{t - z} \frac{dt}{w^+(t)} dz.$$

Further, using the Fubini-Tonelli and Cauchy integral theorems, we obtain that

$$(7.1) \qquad 0 = \frac{1}{\pi i} \int_\Delta h(t) \oint_\Gamma \frac{z^k q_n(z)}{t - z} dz \frac{dt}{w^+(t)} = 2 \int_\Delta t^k q_n(t) \frac{h(t) dt}{w^+(t)}, \quad k \in \{0, \ldots, n-1\}.$$

Thus, polynomials q_n, the denominators of Padé approximants π_n, satisfy non-Hermitian orthogonality relations on Δ with respect to the weight h/w^+.

For each polynomial q_n we define its function of the second kind by the rule

$$(7.2) \qquad\qquad R_n(z) := \frac{1}{\pi i} \int_\Delta \frac{q_n(t) h(t)}{t - z} \frac{dt}{w^+(t)}, \quad z \in D.$$

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[10] $\Delta_2(a)$ lies in the closure of $K(i, c(a), z_1$ or of $K(i, c(a), z_3)$ according whether 0 is closer to i than a or not; if $|i - a| = |i| = 1$, then $\Delta_2(a)$ is the segment $[c(a), i]$.

It can be seen by developing $1/(t-z)$ in powers of z at infinity and using (7.1) that

$$(7.3) \qquad (wR_n)(z) = O(z^{-n+1}) \quad \text{as} \quad z \to \infty.$$

Moreover, it also easily follows from (7.1) that

$$\int_\Delta \frac{q_n(t) - q_n(z)}{t - z} q_n(t) h(t) \frac{dt}{w^+(t)} = 0$$

and therefore

$$R_n(z) = \frac{1}{q_n(z)} \frac{1}{\pi i} \int_\Delta \frac{q_n^2(t) h(t)}{t - z} \frac{dt}{w^+(t)}.$$

Applying Cauchy integral theorem to $q_n(q_n f_h - p_n)$ on the bases of (2.18), we get that

$$e_n(z) := (f_h - \pi_n)(z) = \frac{q_n(z)(q_n f_h - p_n)(z)}{q_n^2(z)} = \frac{1}{q_n^2(z)} \frac{1}{2\pi i} \oint_\Gamma \frac{q_n(\tau)(q_n f_h - p_n)(\tau)}{z - \tau} d\tau$$

for z in the exterior of Γ. Hence, we derive from Cauchy integral formula and Fubini-Tonelli theorem that

$$e_n(z) = \frac{1}{q_n^2(z)} \frac{1}{\pi i} \int_\Delta h(t) \frac{1}{2\pi i} \oint_\Gamma \frac{q_n^2(\tau)}{(z - \tau)(t - \tau)} d\tau \frac{dt}{w^+(t)}$$

and hence

$$(7.4) \qquad e_n(z) = \frac{1}{q_n^2(z)} \frac{1}{\pi i} \int_\Delta \frac{q_n^2(t) h(t)}{t - z} \frac{dt}{w^+(t)} = \frac{R_n(z)}{q_n(z)}.$$

Thus, to describe the behavior of the error of approximation, we need to analyze the asymptotic behavior of q_n and R_n.

7.2. **Boundary Value Problem.** According to (3.3), it holds that

$$R_\Delta(q_n h; z) = \frac{(wR_n)(z)}{2}, \quad z \in D.$$

Since $q_n h$ is Dini-continuous on Δ, wR_n has unrestricted continuous boundary values on ∂D. Furthermore, it follows from (3.12) that

$$(7.5) \qquad (wR_n)^+ + (wR_n)^- = 2q_n h \quad \text{on} \quad \Delta.$$

Below, we turn this boundary value problem on Δ into a boundary value problem on L.

Firstly, set

$$A_n(\mathbf{z}) := \frac{(wR_n)(z)}{S_n(\mathbf{z})}, \quad \mathbf{z} \in D^{(2)},$$

where S_n is the function granted by Proposition 1. Since S_n vanishes at $\infty^{(2)}$ with order $n-1$ when $\mathbf{z}_n \neq \infty^{(2)}$ and with order n otherwise, and wR_n vanishes at infinity with order at least $n-1$ by (7.3), A_n is a holomorphic function in $D^{(2)}$ except for a single simple pole at $\mathbf{z}_n$ when $\mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}$ and a possible simple pole at $\infty^{(2)}$ when $\mathbf{z}_n = \infty^{(2)}$. Moreover, A_n has continuous trace on $L^- \setminus \{\mathbf{z}_n\}$.

Secondly, put

$$B_n(\mathbf{z}) := \frac{2q_n(z)}{S_n(\mathbf{z})}, \quad \mathbf{z} \in \mathfrak{R} \setminus L.$$

Then B_n is a holomorphic function in $\mathfrak{R} \setminus (L \cup \{\mathbf{z}_n\})$ with a simple pole at $\mathbf{z}_n$ when $\mathbf{z}_n \notin L \cup \{\infty^{(1)}\}$ and a possible simple pole at $\infty^{(1)}$ when $\mathbf{z}_n = \infty^{(1)}$ and $\deg(q_n) = n$. Moreover, B_n has continuous traces on both sides of $L \setminus \{\mathbf{z}_n\}$.

Thus, by the very definition of functions A_n and B_n, we have that

$$(7.6) \qquad A_n^-(\mathbf{t}) - B_n^+(\mathbf{t}) = \frac{(wR_n)^\pm(t)}{S_n^-(\mathbf{t})} - \frac{2q_n(t)}{S_n^+(\mathbf{t})} = \frac{(wR_n)^\pm(t)}{S_n^-(\mathbf{t})} - \frac{2(q_n h)(t)}{S_n^-(\mathbf{t})} = -\frac{(wR_n)^\mp(t)}{S_n^-(\mathbf{t})},$$

where, as usual, $t = \pi(\mathbf{t})$ and we used (2.11) and (7.5).

Thirdly, define $A_n(\mathbf{z}) = A_n(\mathbf{z}^*)$, $\mathbf{z} \in D^{(1)}$, where $\mathbf{z}^*$ is the point conjugate to $\mathbf{z}$. Clearly, A_n enjoys in $D^{(1)}$ the same properties as in $D^{(2)}$. As to the boundary values on L, it holds that $A_n^+(\mathbf{t}) = A_n^-(\mathbf{t}^*)$ and therefore

$$A_n^-(\mathbf{t}) = \frac{(wR_n)^\pm(t)}{S_n^-(\mathbf{t})} \quad \text{and} \quad A_n^+(\mathbf{t}) = \frac{(wR_n)^\mp(t)}{S_n^-(\mathbf{t}^*)},$$

where $t \in \Delta^\pm$. Hence, boundary value problem (7.6) can be rewritten as

$$(7.7) \qquad A_n^-(\mathbf{t}) - B_n^+(\mathbf{t}) = -A_n^+(\mathbf{t}) \frac{S_n^-(\mathbf{t}^*)}{S_n^-(\mathbf{t})}, \quad \mathbf{t} \in L.$$

Finally, define

$$X_n(\mathbf{z}) = \frac{S_n(\mathbf{z}^*)}{S_n(\mathbf{z})}, \quad \mathbf{z} \in \mathfrak{R} \setminus L.$$

Then X_n is a sectionally holomorphic function in $\mathfrak{R} \setminus (L \cup \{\mathbf{z}_n\} \cup \{\infty^{(2)}\})$, it vanishes at $\infty^{(1)}$ with order at least $2(n-1)$ and has continuous traces on $L \setminus \{\mathbf{z}_n\}$. Moreover, it holds that $X_n^+(\mathbf{t}) = S_n^-(\mathbf{t}^*)/S_n^+(\mathbf{t})$. Thus, we get from (7.7) and (2.11) that

$$(7.8) \qquad A_n^- = B_n^+ - \frac{(A_n X_n)^+}{h \circ \pi}, \quad \text{on} \quad L,$$

which is our final boundary value problem.

7.3. Proof of Theorem 4. Since $h \equiv 1/p$, (7.8) becomes

$$A_n^- = (B_n - PA_n X_n)^+,$$

where $P := p \circ \pi$ is the lift of p onto $\mathfrak{R}$. Observe that the left-hand side of the equality above is given by a function meromorphic in $D^{(2)}$ and the right-hand side is given by a function meromorphic in $D^{(1)}$ and holomorphic at $\infty^{(1)}$ for all $2(n-1) > \deg(p)$ (recall that X_n vanishes at $\infty^{(1)}$ with order at least $2(n-1)$). As they have continuous boundary values on $L \setminus \{\mathbf{z}_n\}$ from within the respective domains and at $\mathbf{z}_n$ they have a polar singularity, the function

$$\Phi_n := \begin{cases} B_n - PA_n X_n, & \text{in} \quad D^{(1)}, \\ A_n, & \text{in} \quad D^{(2)}, \end{cases}$$

is rational over $\mathfrak{R}$. Observe now that

$$(PA_n X_n)(\mathbf{z}) = \frac{(pwR_n)(z)}{S_n(\mathbf{z})}, \quad \mathbf{z} \in D^{(1)},$$

and hence Φ_n has at most one simple pole at $\mathbf{z}_n$. However, there are no rational functions over $\mathfrak{R}$ with one pole and therefore Φ_n is a constant. As pX_n vanishes at $\infty^{(1)}$, $\Phi_n \equiv 2$ by the normalization of S_n and the definition of B_n. Summarizing, we derived that

$$(7.9) \qquad 2 \equiv A_n(z^{(2)}) = \frac{(wR_n)(z)}{S_n(z^{(2)})} \quad \text{and} \quad 2 \equiv \frac{2q_n(z)}{S_n(z^{(1)})} - 2\frac{p(z)S_n(z^{(2)})}{S_n(z^{(1)})}.$$

Hence, Theorem 4 follows from (7.4). □

7.4. **Proof of Theorem 5.** Let p_n be the best uniform approximant to $1/h$ on Δ among all polynomial of degree at most n. Then the norms $\|p_n\|_\Delta$ are uniformly bounded and

$$(7.10) \qquad \left| p_n(z)\varphi^{-n}(z) \right| \leq \text{const.}, \qquad z \in D,$$

by the Bernstein-Walsh inequality. Set $P_n := p_n \circ \pi$ and define

$$(7.11) \qquad \Phi_n := \begin{cases} B_n - P_n A_n X_n, & \text{in} \quad D^{(1)}, \\ A_n, & \text{in} \quad D^{(2)}. \end{cases}$$

Analyzing Φ_n as in the proof of Theorem 4, we see that Φ_n is a sectionally meromorphic function on $\mathfrak{R} \setminus L$ with at most one pole, necessarily at $\mathbf{z}_n$, and continuous traces on both sides of $L \setminus \{\mathbf{z}_n\}$ that satisfy

$$(7.12) \qquad \Phi_n^+ - \Phi_n^- = \epsilon_n (A_n X_n)^+,$$

where $\epsilon_n := (1/h - p_n) \circ \pi$. Observe that $\max_L |\epsilon_n| = \omega_n$ by the definition of ω_n, see (2.22).

Let O be a neighborhood of L in $\mathfrak{R}$ as in Section 6.3. Consider first those indices n for which $\mathbf{z}_n \notin O$. Assume in addition that $\mathbf{z}_n \neq \infty^{(1)}$. Let T_n be the constant 0 if $\mathbf{z}_n = \infty^{(2)}$, and if $\mathbf{z}_n \neq \infty^{(2)}$ the unique rational function on $\mathfrak{R}$ vanishing at $\infty^{(1)}$ and having two simple poles at $\mathbf{z}_n$ and $\infty^{(2)}$, normalized so that $T_n(\mathbf{z})/z \to 1$ as $\mathbf{z} \to \infty^{(2)}$. Put

$$(7.13) \qquad 2E_n := F_n - \ell_n + 2u_n T_n,$$

where $F_n(\mathbf{z}) := F_{\epsilon_n(A_n X_n)^+}(\mathbf{z})$, $\ell_n := \ell_{\epsilon_n(A_n X_n)^+}$, and $u_n := u_{\epsilon_n(A_n X_n)^+}$, see (4.45)-(4.47). Then E_n is a sectionally meromorphic function on $\mathfrak{R} \setminus L$ with at most one pole at $\mathbf{z}_n$ and well-defined boundary values on both sides of L that satisfy a.e.

$$(7.14) \qquad 2E_n^+ - 2E_n^- = \epsilon_n (A_n X_n)^+.$$

Hence, we derive from (7.12) and (7.14) that $\Phi_n = B_n(\infty^{(1)}) + 2E_n$ by the principle of meromorphic continuation and since there are no rational functions over $\mathfrak{R}$ with one pole. Moreover, it holds that $B_n(\infty^{(1)}) = 2$ when $\deg(q_n) = n$ since q_n is monic and $S_n(\mathbf{z}) = z^n + \dots$ at $\infty^{(1)}$ and $B_n(\infty^{(1)}) = 0$ otherwise.

If $\deg(q_n) < n$, then

$$(7.15) \qquad \|A_n^-\|_{2,L} = \|\Phi_n^-\|_{2,L} = \|2E_n^-\|_{2,L}.$$

Clearly, for $\mathbf{z}_n \notin O$ the values of T_n on L form a uniformly bounded family of continuous functions. Hence, it follows from (4.48), (4.47) and (7.13) that

$$(7.16) \qquad \|E_n^-\|_{2,L} \leq \text{const.} \omega_n \|(A_n X_n)^+\|_{2,L}.$$

Moreover, since for such $\mathbf{z}_n$ the traces X_n^+ form a normal family on L by (2.16) and $A_n^+(\mathbf{t}) = A_n^-(\mathbf{t}^*)$ on L by the definition of A_n, (7.15) and (7.16) yield that

$$(7.17) \qquad \|A_n^+\|_{2,L} \leq \text{const.} \omega_n \|A_n^+\|_{2,L}.$$

However $\omega_n \to 0$ and therefore (7.17) cannot be true for large n. That is, for all n large enough and $\mathbf{z}_n \in \mathfrak{R} \setminus O$, it holds that $\mathbf{z}_n \neq \infty^{(1)}$ and $\deg(q_n) = n$. Thus, for such n, we have that $B_n(\infty^{(2)}) = 2$. Then, we get by repeating the steps (7.15)-(7.17) that

$$(7.18) \qquad \|A_n^+ - 2\|_{2,L} \leq \text{const.} \omega_n \|A_n^+\|_{2,L},$$

which yields that $\|A_n^+\|_{2,L} \to 2$ as $n \to \infty$ for all admissible n. In particular, it holds that

$$(7.19) \qquad \|E_n^\pm\|_{2,L} \leq \text{const.} \omega_n.$$

In another connection, we deduce as in (7.9) that

$$(7.20) \quad \begin{cases} q_n(z)/S_n(z^{(1)}) & = \quad 1 + \dfrac{p_n(z)S_n(z^{(2)})(1+E_n(z^{(2)}))}{S_n(z^{(1)})} + E_n(z^{(1)}), \\[2ex] (wR_n)(z)/S_n(z^{(2)}) & = \quad 2\left(1+E_n(z^{(2)})\right), \end{cases}$$

which implies (2.23) by (7.4), (7.19) and (2.16) (note that $S_n E_n$ is holomorphic in $D^{(2)}$).

Suppose now that $\mathbf{z}_n = \infty^{(1)}$. If $\deg(q_n) = n - 1$ then $q_{n-1} = q_n$ and $R_{n-1} = R_n$. As $S_n = S_{n-1}$ and $\mathbf{z}_{n-1} = \infty^{(2)}$ by Proposition 1, the asymptotics of q_n and R_n is described by what precedes. Hence, in what follows we can assume that n belongs to an infinite subsequence such that $\mathbf{z}_n = \infty^{(1)}$ and $\deg(q_n) = n$. As before, one can check that

$$(7.21) \quad \frac{\Phi_n(\mathbf{z})}{\eta_n} = 2 + \frac{2z + F_n(\mathbf{z}) - \ell_n(z)}{\eta_n}, \quad \mathbf{z} \in \mathfrak{R} \setminus L,$$

where $B_n(z) = 2z + 2\eta_n + \mathcal{O}(1/z)$. Recall that $\ell_n(z) = u_n z + v_n$ by the very definition (4.47) and $F_n(\mathbf{z}) = -\ell_n(z) + \mathcal{O}(1/z)$ near $\infty^{(2)}$. Thus, $u_n = 1$ and therefore

$$(7.22) \quad 1 \leq \mathrm{const.}\omega_n \|A_n^+\|_{2,L}$$

as in (7.16). On the other hand, we get from (7.21) as in (7.18) that

$$(7.23) \quad \|\eta_n^{-1}A_n^+ - 2\|_{2,L} = \|\eta_n^{-1}(F_n^- - \ell_n)\|_{2,L} \leq \mathrm{const.}\omega_n\|\eta_n^{-1}A_n^+\|_{2,L}.$$

Hence, we have by (7.22) that

$$(7.24) \quad \|\eta_n^{-1}A_n^+\|_{2,L} \to 2 \quad \text{and} \quad |\eta_n^{-1}| \leq \mathrm{const.}\omega_n.$$

Set

$$E_n(\mathbf{z}) := \frac{2z + F_n(\mathbf{z}) - \ell_n(z)}{2\eta_n}, \quad \mathbf{z} \in \mathfrak{R} \setminus L.$$

Then $\|E_n^\pm\|_{2,L} \leq \mathrm{const.}\omega_n$ by (7.23) and (7.24). Moreover, it holds that

$$\begin{cases} q_n(z)/S_n(z^{(1)}) & = \quad \eta_n\left(1 + \dfrac{p_n(z)S_n(z^{(2)})(1+E_n(z^{(2)}))}{S_n(z^{(1)})} + E_n(z^{(1)})\right), \\[2ex] (wR_n)(z)/S_n(z^{(2)}) & = \quad 2\eta_n\left(1+E_n(z^{(2)})\right), \end{cases}$$

which yields (2.23) by (7.4).

Assume next that n ranges over an infinite subsequence with $\mathbf{z}_n \in O$. Define $\widetilde{\Phi}_n := \Phi_n U_n$, where the functions U_n were constructed in Section 6.3. Recall that $U_n(\infty^{(1)}) = 1$, $(U_n) = \mathbf{z}_n + \mathbf{w}_n - \mathbf{t}_{n1} - \mathbf{t}_{n2}$, and $\mathbf{w}_n, \mathbf{t}_{n1}, \mathbf{t}_{n2} \notin O$. Therefore, $\widetilde{\Phi}_n$ is a meromorphic function on $\mathfrak{R} \setminus L$, with a zero at $\mathbf{w}_n$, two poles at $\mathbf{t}_{n1}$ and $\mathbf{t}_{n2}$, and $\widetilde{\Phi}_n(\infty^{(1)}) = B_n(\infty^{(1)})$. Furthermore, since U_n is holomorphic across L, it holds that

$$(7.25) \quad \widetilde{\Phi}_n^+ - \widetilde{\Phi}_n^- = \epsilon_n(\widetilde{A}_n\widetilde{X}_n)^+$$

by (7.12), where we set

$$\begin{array}{llll} \widetilde{A}_n(z^{(2)}) & := & A_n(z^{(2)})U_n(z^{(2)}), & \\ \widetilde{A}_n(z^{(1)}) & := & \widetilde{A}_n(z^{(2)}), & \end{array} \quad \text{and} \quad \begin{array}{llll} \widetilde{B}_n(z^{(1)}) & := & B_n(z^{(1)})U_n(z^{(1)}), \\ \widetilde{X}_n(z^{(1)}) & := & X_n(z^{(1)})U_n(z^{(1)})/U_n(z^{(2)}), \end{array}$$

for $z \in D$. As in (7.13), define

$$2\widetilde{E}_n := \widetilde{F}_n - \widetilde{\ell}_n + 2\widetilde{u}_n\widetilde{T}_n,$$

where $\widetilde{F}_n := F_{\epsilon_n(\widetilde{A}_n\widetilde{X}_n)^+}$, $\widetilde{\ell}_n := \ell_{\epsilon_n(\widetilde{A}_n\widetilde{X}_n)^+}$, $\widetilde{u}_n := u_{\epsilon_n(\widetilde{A}_n\widetilde{X}_n)^+}$, and $\widetilde{T}_n$ is a rational function over $\mathfrak{R}$ with a zero at $\infty^{(1)}$, two poles at $\infty^{(2)}$ and $\mathbf{t}_{n2}$, and normalized so $\widetilde{T}_n(z^{(2)})/z \to$ as $z \to \infty$. Note from (4.46) that across L

$$(7.26) \qquad \widetilde{E}_n^+ - \widetilde{E}_n^- = \epsilon_n(\widetilde{A}_n\widetilde{X}_n)^+.$$

As Ω_1 is Lipschitz on $\widetilde{\mathfrak{R}}_2$ with respect to any fixed Riemannian metric on $\mathfrak{R}$, it follows from (6.5) and (6.4) (where $\mathbf{z} = \mathbf{z}_n$) that $\widetilde{T}_n(\mathbf{w}_n)$ is bounded independently of n (along the considered subsequence), see Figure 6. Moreover, $|w_n|$ remains bounded in D because $\mathbf{w}_n \in \partial O_{\infty^{(1)}}$, see (6.3). Hence, from (4.45), (4.47), and since the traces $\widetilde{X}_n^+$ are uniformly bounded on L by (6.8), we deduce that

$$(7.27) \qquad |\widetilde{E}_n(\mathbf{w}_n)| \le \mathrm{const.}\omega_n\|\widetilde{A}_n^+\|_{2,L}.$$

Likewise, as $\mathbf{t}_{n2} \notin O$, it holds that $|\widetilde{T}_n|$ is bounded on L, therefore by (4.48) and (4.47) again

$$(7.28) \qquad \|\widetilde{E}_n^+\|_{2,L} \le \mathrm{const.}\omega_n\|\widetilde{A}_n^+\|_{2,L}.$$

Now, from (7.25) and (7.26) we get that $\widetilde{\Phi}_n - 2\widetilde{E}_n$ has no jump across L, and since it can only have poles at $\mathbf{t}_{n1}$, $\mathbf{t}_{n2}$ it must be a scalar multiple of U_n. Checking values at $\infty^{(1)}$ and $\mathbf{w}_n$ (remember $\widetilde{\Phi}_n(\mathbf{w}_n) = U_n(\mathbf{w}_n) = 0$ and $\widetilde{\Phi}_n(\infty^{(1)}) = B_n(\infty^{(1)})$), we conclude that

$$(7.29) \qquad \widetilde{\Phi}_n = B_n(\infty^{(1)}) + 2\widetilde{E}_n + c_n(1 - U_n),$$

where $c_n = 2\widetilde{E}_n(\mathbf{w}_n) - B_n(\infty^{(1)})$. If we had $B_n(\infty^{(1)}) = 0$ for all n large enough (within the considered subsequence), it would hold that $c_n = 2\widetilde{E}_n(\mathbf{w}_n)$ and, by (7.11) and (7.29), that $\widetilde{A}_n^- = 2\widetilde{E}_n^- + c_n(1 - U_{n|L})$. Because U_n is bounded on L independently of n as $\mathbf{t}_{n2} \notin O$, and since $\|\widetilde{A}_n^-\|_{2,L} = \|\widetilde{A}_n^+\|_{2,L}$ by construction, this would entail with (7.27),(7.28), (7.26) that

$$(7.30) \qquad \|\widetilde{A}_n^+\|_{2,L} \le \mathrm{const.}\omega_n\|\widetilde{A}_n^+\|_{2,L}.$$

But $\omega_n \to 0$, thus (7.30) is impossible for n large enough, therefore $B_n(\infty^{(1)}) = 2$ and $\deg(q_n) = n$ for all such n. Repeating the arguments leading to (7.30), this time with $B_n(\infty^{(1)}) = 2$, we get that

$$(7.31) \qquad |2 + c_n| \le \mathrm{const.}\omega_n\|\widetilde{A}_n^+\|_{2,L} \quad \text{and} \quad \|\widetilde{A}_n^+ - 2U_n\|_{2,L} \le \mathrm{const.}\omega_n\|\widetilde{A}_n^+\|_{2,L}.$$

The last inequality implies that $\|\widetilde{A}_n^+\|_{2,L}$ is uniformly bounded. Rewrite (7.29) as

$$\widetilde{\Phi}_n = 2U_n + 2\widetilde{E}_n + (2 + c_n)(1 - U_n)$$

or equivalently

$$\Phi_n = 2 + 2\left(\widetilde{E}_n + (1 + c_n/2)(1 - U_n)\right)U_n^{-1} =: 2\left(1 + E_n\right).$$

By its very definition, E_n has a pole at $\mathbf{z}_n$ but is holomorphic at $\mathbf{w}_n$ (even though U_n^{-1} is not). Formula (2.23) now follows exactly as (7.20). To show (2.24), observe from the definition of E_n and since $|U_n^{-1}(l_n \circ \pi)|$ is bounded on L that

$$\|2E_n^\pm(l_n \circ \pi)\|_{2,L} \le \mathrm{const.}\|\widetilde{E}_n^\pm + (1 + c_n/2)(1 - U_n)\|_{2,L} \le \mathrm{const.}\omega_n,$$

where the last estimate follows from (7.28), (7.31), and the boundedness of U_n on L. $\qquad \square$

7.5. **Proof of Corollary 6.** Assume first that $\mathbf{z}_n \in D^{(2)} \setminus \{\infty^{(2)}\}$ for n large enough, $n \in \mathbb{N}_1$, so that $\mathbf{z} \in D^{(2)} \cup L$. For such n, the function

$$\frac{(\xi - z_n) E_n^*(\xi)}{w(\xi)|\varphi(\mathbf{z}_n)|}, \quad \xi \in D,$$

is holomorphic and vanishes at ∞. Hence, it converges locally uniformly to zero there by (2.24) (to handle the case where $\mathbf{z}_n \to \infty^{(2)}$, observe that $(\xi - z_n)/|\varphi(z_n)|$ is bounded on Δ independently of z_n). In turn, E_n is also holomorphic in D and converges to zero locally uniformly there. Indeed, if $\mathbf{z} \notin L$, this follows directly from (2.24) (where we can choose $l_n \equiv 1$) and the Cauchy representation formula. On the other hand, if $\mathbf{z} \in L$, observe that

$$\frac{(\xi - z_n) E_n(\xi)}{\varphi(\xi)}, \quad \xi \in D,$$

is also holomorphic. Moreover, its $L^1(\partial D)$-norm does not exceed const.ω_n by (2.24), the Schwarz inequality, and since $\|w^{1/2}/\varphi\|_{L^2(\partial D)} \leq$ const. The latter immediately entails that this function tends to zero locally uniformly in D by the Cauchy representation formula. However, as $|(\xi - z_n)/\varphi(\xi)|$ is bounded away from zero on compact subsets of D uniformly with respect to z_n (remember $z_n \to z \in \Delta$), E_n converges to zero locally uniformly in D as well.

Gathering what we did, it can be concluded from (2.16) and (2.23) that $f_h - \pi_n$ converges locally uniformly to zero on D and in fact geometrically fast because $|\varphi| > 1 + \varepsilon_K$ on any compact $K \subset D$. If $\mathbf{z}_n = \infty^{(2)}$ for each n, then E_n^* may have a pole at ∞ but E_n^*/w is holomorphic in D and vanishes at ∞. Thus, it converges locally uniformly to zero in D by (2.24) and the Cauchy formula. As before E_n also converges locally uniformly to zero in D and, from (2.16) and (2.23) again, we get the desired conclusion.

Assume now that $\mathbf{z}_n \in D^{(1)}$ for n large enough, $n \in \mathbb{N}_1$, so that $\mathbf{z} \in D^{(1)} \cup L$. If $\mathbf{z} \in L$ and K is compact in D, then $|\xi - z_n| \geq c_k > 0$ for $\xi \in K$ and n large. Moreover, the functions

$$\frac{(\xi - z_n) E_n(\xi)}{\varphi(\xi)} \quad \text{and} \quad \frac{(\xi - z_n) E_n^*(\xi)}{\varphi(\xi)}, \quad \xi \in D,$$

are holomorphic in D, and using (2.24) and the Schwarz inequality as before we see that they go to zero locally uniformly in D. In particular E_n and E_n^* tend to zero uniformly on K. Thus, we conclude again from (2.16) and (2.23) that $f_h - \pi_n$ converges to zero geometrically fast on K. The argument when $\mathbf{z} \in D^{(1)}$ and $K \subset D \setminus \{z_n\}$ is similar.

Finally, observe from (2.23) that

$$(7.32) \qquad \left| (f_h - \pi_n) - \frac{2S_n^*}{w S_n} \right| = \left| \frac{2 S_n^*}{w S_n} \right| \left| \frac{E_n^* - E_n - \mathcal{O}(|\varphi|^{-n})}{1 + E_n + \mathcal{O}(|\varphi|^{-n})} \right|.$$

Now if $\mathbf{z} \in D^{(1)}$ and $\mathbb{D}(z, r) \subset D$ is a disk of radius r centered at z (the set $\{z : |z| > r\}$ if $z = \infty$) with boundary circle $\mathbb{T}(z, r)$, it follows from what precedes that E_n and E_n^* tend to zero on $\mathbb{T}(z, r)$. Hence, the second factor on the right-hand side of (7.32) also converges to zero on $\mathbb{T}(z, r)$ as $\mathbb{N}_1 \ni n \to \infty$. Thus, by Rouché's theorem, $f_h - \pi_n$ has exactly one pole in $\mathbb{D}(z, r)$ for n large enough because $2S_n/(w S_n^*)$ has exactly one pole there (namely z_n) and none of these two functions can have a zero in $\mathbb{D}(z, r)$ by (2.16), (2.23), and the fact that E_n and E_n^* converge to zero on $\mathbb{T}(z, r)$. This achieves the proof of the corollary. $\qquad \square$

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