CONVERGENT INTERPOLATION TO CAUCHY INTEGRALS OVER ANALYTIC ARCS WITH JACOBI-TYPE WEIGHTS

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ABSTRACT. We design convergent multipoint Padé interpolation schemes to Cauchy transforms of non-vanishing complex densities with respect to Jacobi-type weights on analytic arcs, under mild smoothness assumptions on the density. We rely on the work [10] for the choice of the interpolation points, and dwell on the Riemann-Hilbert approach to asymptotics of orthogonal polynomials introduced in [33] in the case of a segment. We also elaborate on the $\bar{\partial}$-extension of the Riemann-Hilbert technique, initiated in [37] on the line to relax analyticity assumptions. This yields strong asymptotics for the denominator polynomials of the multipoint Padé interpolants, from which convergence follows.

1. INTRODUCTION

Classical Padé approximants (or interpolants) and their multipoint generalization are probably the oldest and simplest candidate rational-approximants to a holomorphic function of one complex variable. They are simply those rational functions of type[1] (m,n) that interpolate the function in $m+n+1$ points of the domain of analyticity, counting multiplicity. *Classical* Padé approximants refer to the case where interpolation takes place in a single point with multiplicity $m+n+1$ [38].

Besides their everlasting number-theoretic success [43, 32, 41], they are common tools in modeling and numerical analysis of various fields, ranging from boundary value problems and convergence acceleration [16, 29, 13, 26, 20] to continuous mechanics [6, 51], quantum mechanics [8, 52], condensed matter physics [42], fluid mechanics [40], system and circuits theory [12, 31, 17], and even page ranking the *Web* [14].

In spite of this, the convergence properties of Padé or multipoint Padé approximants are still far from being understood. For particular classes of functions like Markov functions, some elliptic functions, and certain entire functions such as Pólya frequencies or functions with smooth and fast decaying Taylor coefficients, classical Padé approximants at infinity are known to converge, locally uniformly on the domain of analyticity [36, 50, 4, 34]. But when applied to more general cases they seldom accomplish the same, due to the occurrence of "spurious poles" that may wander about the domain of analyticity. Further distinction should be made here between diagonal approximants (*i.e.* interpolants of type (m,m)) and row approximants (*i.e.* interpolants of type (m,n) where n is kept fixed), and we refer the reader to the comprehensive monograph [7] for a detailed account of many works on the subject. Let us simply mention that, for the case of diagonal approximants which is the most interesting as it treats poles and zeros on equal footing, the disproof of the Padé conjecture [35] and of the Stahl conjecture [15] have only added to the picture that classical Padé approximants are not seen best through the spectacles of uniform convergence.

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[1] A rational function is said to be of type (m,n) if it can be written as the ratio of a polynomial of degree at most m and a polynomial of degree at most n.

The case of multipoint Padé approximants is somewhat different, since choosing the interpolation points offers new possibilities to help convergence. However, it is not immediately clear how to use these additional parameters. The theory was initially developed for Markov functions (*i.e.* Cauchy transforms of positive measures compactly supported on the real line) showing that multipoint Padé approximants converge locally uniformly on the complement of the smallest segment containing the support of the defining measure, provided the interpolation points are conjugate symmetric [25]. The crux of the proof is the remarkable connection between rational interpolants and orthogonal polynomials: the denominator of the n-th diagonal multipoint Padé approximant is the n-th orthogonal polynomial of the measure defining the Markov function, weighted by the inverse of the polynomial whose zeros are the interpolation points (this polynomial is identically 1 for classical Padé approximants). The conjugate symmetric distribution of the interpolation points is to the effect that the weight is positive, so one can apply the asymptotic theory of orthogonal polynomials with varying weights [49].

When trying to generalize this approach to more general Cauchy integrals than Markov functions, one is led to consider non-Hermitian orthogonal polynomials with respect to complex-valued measures on more general arcs than segments, and for a while it was unclear what could be hoped for. In the pathbreaking papers [44, 46, 47, 48], devoted to the convergence *in capacity* of classical Padé approximants to functions with branchpoints, it was shown that such orthogonal polynomials lend themselves to analysis when the measure is supported on a system of arcs of minimal logarithmic capacity linking the branchpoints, in the complement of which the function is single-valued. Shortly after, the same type of convergence was established for multipoint Padé approximants to Cauchy integrals of continuous (quasi-everywhere) non-vanishing densities over arcs of minimal *weighted capacity*, provided that the interpolation points asymptotically distribute like a measure whose potential is the logarithm of the weight [27]. Such an extremal system of arcs is called a *symmetric contour*, or *S-contour*, and is characterized by a symmetry property of the (two-sided) normal derivatives of its equilibrium potential. The corresponding condition on the distribution of the interpolation points may be viewed as a far-reaching generalization of the conjugate-symmetry with respect to the real line that was required to interpolate Markov functions in a convergent way.

After these works it became apparent that the appropriate class of Cauchy integrals for Padé approximation should consist of those taken over S-contours, and that the interpolation points should distribute according to the weight that defines the symmetry property. However, it is not so easy to decide which systems of arcs are S-contours, since finding a weight making the arcs of smallest weighted capacity is a nontrivial inverse problem, and in any case convergence in capacity is much weaker than locally uniform convergence.

For the class of Jordan arcs, new ground was recently broken in [10] where it is shown that such an arc, if rectifiable and Ahlfors regular at the endpoints, is an S-contour if and only if it is analytic. The proof recasts the S-property for Jordan arcs as the existence of a sequence of "pseudo-rational" functions, holomorphic and tending to zero off the arc, whose boundary values from each side of the latter remain bounded, and whose zeros remain at positive distance from the arc. There are in fact many such sequences that can be computed explicitly from an analytic parameterization of the arc. Then, translating the non-Hermitian orthogonality equation for the denominator into an integral equation involving Hankel operators and using compactness properties of the latter, the reference just quoted establishes that multipoint Padé approximants to Cauchy transforms of Dini-continuous (essentially) non-vanishing densities with respect to the equilibrium distribution of the arc converge locally uniformly in its complement when the interpolation points are the zeros of these pseudo-rational functions.

Still the above result remains unsatisfactory, for the hypotheses entail that the density with respect to arclength in the integral goes to infinity towards the endpoints of the arc, since so does the equilibrium distribution. In particular, ultra-smooth situations like the one of Cauchy integrals of smooth functions over analytic arcs are not covered. The present paper develops a new technique to handle *any* non-vanishing integrable Jacobi-type density under mild smoothness assumptions, thereby settling more or less the issue of convergence in multipoint Padé interpolation to functions defined as Cauchy integrals over analytic Jordan arcs.

We dwell on the Riemann-Hilbert approach to asymptotics of orthogonal polynomials with analytic weights, pioneered on the line in [11, 19] and carried over to the segment in [33]. We also elaborate on the $\bar{\partial}$-extension thereof, initiated on the line in [37] to relax the analyticity requirement. This will provide us with strong (*i.e.* Plancherel-Rotach type) asymptotics for the denominator polynomials of the multipoint Padé interpolants we construct and for their associated functions of the second kind, from which the local uniform convergence we seek follows easily. It is interesting to note that the Riemann-Hilbert approach, which is typically a tool to obtain sharp quantitative asymptotics, is here used as a means to solve a qualitative question namely the convergence of the interpolants.

The interpolation points shall be the same as in [10], namely the zeros of a sequence of pseudo-rational functions adapted to the arc. Such an interpolation scheme will prove convergent for all Cauchy integrals with sufficiently smooth density with respect to a Jacobi weight on the arc at the same time. This provides us with a varying weight which is not of power type, nor in general converging sufficiently fast to a weight of power type to take advantage of the results of [3], where the Riemann-Hilbert approach is adapted to non-Hermitian orthogonality with analytic weights on smooth S-arcs. Instead, when "opening the lens", we set up a sequence of Riemann-Hilbert problems with *varying contours* whose solutions converge to the desired one by properties of the pseudo rational functions.

We pay special attention to keep smoothness requirements low, in order to obtain as general a result as the method permits. Roughly speaking, the higher the Jacobi exponents the smoother the density should be, see the precise assumptions (2.9). When the Jacobi exponents are negative, only a fraction of a derivative is needed, which compares not too badly with the Dini-continuity assumption in [10]. In the present setting, however, the density cannot vanish whereas some weak vanishing is still allowed in [10]. We are of course rewarded not only with a larger range of Jacobi exponents than could be handled in [10], but also with stronger asymptotics even for those exponents already treated there.

As the varying part of our weight is analytic, the extension inside the lens with controlled $\bar{\partial}$-estimates, introduced in [37] for power weights, needs only deal with the density defining the Cauchy integral we interpolate. This step is treated using either tools from real analysis, *e.g.* Muckenhoupt weights and Sobolev traces, or else classical Hölder estimates for singular integrals, whichever yields the best results granted the Jacobi exponents.

Since we consider analytic arcs only, it is natural to ask how general our results with respect to the general class of Cauchy integrals over rectifiable Jordan arcs. It turns out that they are as general as can be, because if the Cauchy integral of a nontrivial Jacobi weight can be interpolated in a convergent way with a triangular scheme of interpolation points that stay away from the arc, then the arc is in fact *analytic* [9].

The paper is organized as follows. Section 2 fixes notation and defines pseudo rational functions as well as multipoint Padé approximants before stating the main results. In Section 3, the contours that will later be instrumental for the solution of the Riemann-Hilbert problem are introduced. Section 4 contains preliminaries on smooth extensions from boundary data in domains

with polygonal boundaries. Section 5 is devoted to key estimates of certain singular integral operators that play a main role in the extension of the weight. Section 6 and 7 deal with the analytic Riemann-Hilbert problem, while Section 8 solves the $\overline{\partial}$ version thereof. Finally, in Section 9, we gather the material developed so far to establish the asymptotics and the convergence of multipoint Padé approximants stated in Section 2.

2. Statements of Results

Let Δ be a closed analytic Jordan *arc* with endpoints -1 and 1. That is to say, there exists a holomorphic univalent function Ξ, defined in some domain $D_\Xi \supset [-1,1]$, such that

$$\Delta = \Xi([-1,1]), \quad \Xi(\pm 1) = \pm 1.$$

We call Ξ an analytic parameterization of Δ. We orient Δ from -1 to 1 and, according to this orientation, we distinguish the left and the right sides of Δ denoted by Δ^+ and Δ^-, respectively. It will be convenient to introduce two unbounded arcs, say, Δ_l and Δ_r, that respectively connect $-\infty$ to -1 and 1 to $+\infty$, in such a manner that $\Delta_l \cup \Delta \cup \Delta_r$ is a smooth unbounded Jordan arc that coincides with the real line in some neighborhood of infinity. Define on Δ the Jacobi weight

$$w(z) = w(\alpha,\beta;z) := (1-z)^\alpha (1+z)^\beta, \quad \alpha,\beta > -1, \tag{2.1}$$

where we choose branches of $(1-z)^\alpha$ and $(1+z)^\beta$ that are holomorphic outside of Δ_r and Δ_l, respectively, and assume value 1 at the origin. In particular, w is analytic across $\Delta^\circ := \Delta \setminus \{\pm 1\}$. Further, set

$$\mathfrak{w}(z) := \sqrt{z^2-1}, \quad \mathfrak{w}(z)/z \to 1, \quad \text{as} \quad z \to \infty, \tag{2.2}$$

to be a holomorphic branch of the square root outside of Δ. Then

$$\varphi(z) := z + \mathfrak{w}(z), \quad z \in D := \overline{\mathbb{C}} \setminus \Delta, \tag{2.3}$$

is holomorphic in $D \setminus \{\infty\}$, has continuous boundary values $\varphi^\pm$ on $\Delta^\pm$, respectively, and satisfies

$$\varphi^+\varphi^- = 1 \quad \text{on} \quad \Delta \quad \text{and} \quad \varphi(z)/2z \to 1 \quad \text{as} \quad z \to \infty. \tag{2.4}$$

It is immediate that φ is inverse of the Joukovski transformation $J(z) := (z^2+1)/2z$, i.e., $J(\varphi(z)) = z$, $z \in D$. Moreover, φ maps D conformally onto an unbounded domain whose boundary is an analytic Jordan curve [10, Sec. 3.1] which is symmetric with respect to the transformation $z \mapsto 1/z$. In particular, φ does not vanish.

2.1. Symmetric Contours.

Multipoint Padé approximants to a given function f are defined to be rational interpolants to f. In this paper we are interested in those functions f that can be expressed as Cauchy integrals of Jacobi-type complex densities defined on Δ (see the smoothness assumptions in (2.8) and (2.9)). In order for multipoint Padé approximants to converge to such a function, it is necessary to choose the interpolation schemes appropriately with respect to Δ. We presently characterize these schemes in terms of the associated monic polynomials vanishing at the interpolation points.

Let $\{v_n\}$ be a sequence polynomials such that $\deg(v_n) \le 2n$ and each v_n has no zeros on Δ. To this sequence we associate a sequence of "pseudo-rational" functions, say $\{r_n\}$, given by

$$r_n(z) := \left(\frac{1}{\varphi(z)}\right)^{2n-\deg(v_n)} \prod_{\{e:v_n(e)=0\}} \frac{\varphi(z)-\varphi(e)}{1-\varphi(e)\varphi(z)}, \quad z \in D, \tag{2.5}$$

where the product is taken over all zeros of v_n according to their multiplicities. It is easy to see that each function r_n is holomorphic in D, has the same zeros as v_n counting multiplicities, and vanishes at infinity with order $2n - \deg(v_n)$. Hence, each r_n has exactly 2n zeros counting

multiplicities. Moreover, the unrestricted boundary values $r_n^{\pm}$ exist continuously from each side of Δ and satisfy $r_n^+ r_n^- \equiv 1$ by the first part of (2.4).

Hereafter, the normalized counting measure of a finite set is the probability measure that has equal mass at each point counting multiplicities. Below, the weak* topology refers to the duality between complex measures and continuous functions with compact support in $\overline{\mathbb{C}}$.

Definition 1. *We say that a sequence of polynomials $\{v_n\}$ with no zeros on Δ belongs to the class* $\mathrm{S}(\Delta)$ *if the following conditions hold:*

(1) *the associated functions r_n via (2.5) satisfy $|r_n^{\pm}| = O(1)$ uniformly on Δ and $r_n = o(1)$ locally uniformly in D;*
(2) *there exists a neighborhood of Δ that contains no zeros of r_n for all n large enough;*
(3) *the normalized counting measures of zeros of r_n form a weak* convergent sequence.*

The third requirement in the definition of $\mathrm{S}(\Delta)$ is purely technical and is placed only to simplify the forthcoming considerations since one can always proceed with subsequences as far as convergence is concerned.

Regarding the nature of the class $\mathrm{S}(\Delta)$, the following result was obtained in [10, Thm. 1]. For a closed analytic Jordan arc Δ, there always exist sequences $\{v_n\}$ belonging to $\mathrm{S}(\Delta)$ and they can be constructed explicitly granted the parameterization Ξ. A partial converse is also true. Namely, let Δ be a rectifiable Jordan arc with endpoint ± 1 such that for $x = \pm 1$ and all $t \in \Delta$ sufficiently close to x it holds that $|\Delta_{t,x}| \leq \text{const.}|x-t|^{\beta}$, $\beta > 1/2$, where $|\Delta_{t,x}|$ is the length of the subarc of Δ joining t and x and "const." is an absolute constant. If there exists a sequence of polynomials $\{v_n\}$ meeting the first two requirements of Definition 1, then Δ is necessarily analytic. The class $\mathrm{S}(\Delta)$ is also intimately related to the so-called *symmetry property* of the contour Δ [44, 45, 10].

For our investigation we need to detail further the properties of the just defined interpolation schemes. We gather them in the following theorem. We agree that the arcs involved have endpoints ± 1. Moreover, we say that two holomorphic functions are analytic continuations of each other if they are defined on domains that have nonempty intersection on which the functions coincide.

Theorem 1. *Let Δ be a closed analytic Jordan arc and $\{v_n\} \in \mathrm{S}(\Delta)$. Then there exists a sequence of closed analytic Jordan arcs $\{\Delta_n\}$ such that:*

(i) *there exist analytic parametrizations Ξ_n of Δ_n and Ξ of Δ such that the functions Ξ_n converge to Ξ uniformly in some neighborhood of $[-1,1]$ as $n \to \infty$;*
(ii) *for each function r_n, associated to v_n via (2.5), there exists an analytic continuation r_n^*, holomorphic in $D_n := \overline{\mathbb{C}} \setminus \Delta_n$, such that $|(r_n^*)^{\pm}| \equiv 1$ on Δ_n.*

Let $\mathfrak{w}_n$ and φ_n be defined relative to Δ_n as $\mathfrak{w}$ and φ were defined in (2.2) and (2.3) relative to Δ. Clearly, $\mathfrak{w}_n$ and φ_n are analytic continuations of $\mathfrak{w}$ and φ to D_n. In fact, r_n^* is simply the function associated to v_n via (2.5) with φ replaced by φ_n. It is apparent that r_n^* is nothing but the Blaschke product with respect to D_n that has the same zeros as r_n.

2.2. **Multipoint Padé Approximation.** Let μ be a complex Borel measure with compact support. We define the Cauchy transform of μ as

$$f_\mu(z) := \int \frac{d\mu(t)}{z-t}, \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu). \tag{2.6}$$

Clearly, f_μ is a holomorphic function in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$ that vanishes at infinity.

Classically, diagonal (multipoint) Padé approximants to f_μ are rational functions of type (n,n) that interpolate f_μ at a prescribed system of $2n+1$ points. However, when the approximated

function is of the from (2.6), it is customary to place at least one interpolation point at infinity so as to let the approximants vanish at infinity as well by construction.

Definition 2. *Let f_μ be given by (2.6) and $\{v_n\}$ be a sequence of monic polynomials, $\deg(v_n) \le 2n$, with zeros in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$. The n-th diagonal Padé approximant to f_μ associated with $\{v_n\}$ is the unique rational function $\Pi_n = p_n/q_n$ satisfying:*

- $\deg p_n \le n$, $\deg q_n \le n$, *and* $q_n \not\equiv 0$;
- $\left(q_n(z)f_\mu(z) - p_n(z)\right)/v_n(z)$ *is analytic in* $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$;
- $\left(q_n(z)f_\mu(z) - p_n(z)\right)/v_n(z) = O\left(1/z^{n+1}\right)$ *as* $z \to \infty$.

A multipoint Padé approximant always exists since the conditions for p_n and q_n amount to solving a system of $2n+1$ homogeneous linear equations with $2n+2$ unknown coefficients, no solution of which can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic); note that the required interpolation at infinity is entailed by the last condition and therefore Π_n is, in fact, of type $(n-1, n)$.

We consider only absolutely continuous measures that are supported on Δ and whose densities are Jacobi weights (2.1) multiplied by suitably smooth non-vanishing functions. This leads us to define smoothness classes $\mathrm{C}^{m,\varsigma}$.

Definition 3. *Let K be an infinitely smooth closed Jordan arc or curve. We say that $\theta \in \mathrm{C}^{m,\varsigma}(K)$ if θ is m-times continuously differentiable on K with respect to the arclength and its m-th derivative is uniformly Hölder continuous with exponent ς, i.e.,*

$$|\theta^{(m)}(t_1) - \theta^{(m)}(t_2)| \le \text{const.}|t_1 - t_2|^{\varsigma}, \quad t_1, t_2 \in K.$$

When $K = \Delta$, we simply write $\mathrm{C}^{m,\varsigma}$ instead of $\mathrm{C}^{m,\varsigma}(\Delta)$. We also write $\mathrm{C}^{\infty}(K)$ for the space of infinitely differentiable functions on K.

Together with $\mathrm{C}^{m,\varsigma}$, we also consider fractional Sobolev spaces.

Definition 4. *Let K be an infinitely smooth Jordan arc or curve. We say that $\theta \in \mathrm{W}_p^{1-1/p}(K)$, $p \in (1,\infty)$, if*

$$\iint_{K\times K} \left| \frac{\theta(x) - \theta(y)}{x - y} \right|^p |dx||dy| < \infty.$$

When $K = \Delta$, we simply write $\mathrm{W}_p^{1-1/p}$ instead of $\mathrm{W}_p^{1-1/p}(K)$.

We shall be interested only in the case $p \in (2,\infty)$ since in this range it holds that

$$\mathrm{W}_p^{1-1/p} \subset \mathrm{C}^{0,\varsigma}, \quad \varsigma = 1 - \frac{2}{p}, \quad p \in (2,\infty). \tag{2.7}$$

by Sobolev imbedding theorem (see Section 4.1).

In what follows, we assume that the measure μ in (2.6) is of the form

$$d\mu(t) = (wh)(t)dt, \quad h(t) = e^{\theta(t)}, \quad t \in \Delta, \tag{2.8}$$

where the Jacobi weight $w = w(\alpha, \beta; \cdot)$ and the complex function θ are such that

$$\alpha, \beta \in (-s, s) \cap (-1, \infty) \tag{2.9}$$

with

$$s := \begin{cases} 1-\frac{2}{p}, & \text{if} \quad \theta \in \mathrm{W}_p^{1-1/p}, \quad p \in (2,\infty), \\ 2\varsigma - 1, & \text{if} \quad \theta \in \mathrm{C}^{0,\varsigma}, \quad \varsigma \in \left(\frac{1}{2}, 1\right], \\ m+\varsigma, & \text{if} \quad \theta \in \mathrm{C}^{m,\varsigma}, \quad m \in \mathbb{N}, \quad \varsigma \in (0,1]. \end{cases}$$

To describe the asymptotic behavior of the approximation error to functions f_μ by the multipoint Padé approximants, we need to introduce complex geometric means and Szegő functions. The *geometric mean* of $h = e^\theta$ is given by

$$G_h := \exp\left\{\int \theta d\omega\right\}, \quad d\omega(t) := \frac{i\,dt}{\pi \mathfrak{w}^+(t)}, \quad t \in \Delta. \tag{2.10}$$

The measure ω is, in some sense, natural for the considered problem as suggested by the forthcoming Theorem 2. Observe also that ω simply becomes the normalized arcsine distribution on Δ when $\Delta = [-1,1]$. As $\int d\omega = 1$, G_h depends only on h and is non-zero when θ is Hölder continuous (see Section 5.3). Moreover, in this case the *Szegő function* of h, given by

$$S_h(z) := \exp\left\{\frac{\mathfrak{w}(z)}{2}\int \frac{\theta(t)}{z-t} d\omega(t) - \frac{1}{2}\int \theta d\omega\right\}, \quad z \in D, \tag{2.11}$$

is the unique non-vanishing holomorphic function in D that has continuous unrestricted boundary values on Δ from each side and satisfies

$$h = G_h S_h^+ S_h^- \quad \text{on} \quad \Delta \quad \text{and} \quad S_h(\infty) = 1. \tag{2.12}$$

The main result of the paper is the following theorem.

Theorem 2. *Let Δ be a closed analytic Jordan arc connecting ± 1 and $\{v_n\} \in \mathrm{S}(\Delta)$. Let also f_μ a Cauchy integral (2.6) with μ given by (2.8) and (2.9). Then $\{\Pi_n\}$, the sequence of diagonal multipoint Padé approximants to f_μ associated with $\{v_n\}$, is such that*

$$(f_\mu - \Pi_n)\mathfrak{w} = \left[2G_{\dot\mu} + o(1)\right] S_{\dot\mu}^2\, r_n,$$

with $o(1)$ satisfying

$$|o(1)| \le \frac{\text{const.}}{n^a}, \quad a \in \begin{cases} \left(0, \frac{s-\max\{|\alpha|,|\beta|\}}{2}\right), & s - \max\{|\alpha|,|\beta|\} \le 1, \\ \left(0, \frac{1}{2}\right), & s - \max\{|\alpha|,|\beta|\} > 1, \end{cases} \tag{2.13}$$

locally uniformly in D, where the constant const. *depends on a, $d\mu = \dot\mu d\omega$, i.e., $\dot\mu = -i\pi w h \mathfrak{w}^+$, and the functions r_n are associated to the polynomials v_n via (2.5) and hence converge to zero geometrically fast in D.*

Theorem 2 develops further the theory of convergent multipoint Padé interpolation started by the authors in [10, Thm. 4]. In the present work we consider a different class of approximated functions relaxing the assumptions on w ($\alpha, \beta \in [-1/2, c_\Delta)$ with $c_\Delta < 0$ in [10], while in the present work $\alpha, \beta > -1$), but at the same time putting more stringent assumptions on h (merely Dini-continuous in [10], while here we require it to be as smooth as $\max\{|\alpha|,|\beta|\}$). Moreover, the presently used method of proof yields more detailed information on the speed of convergence and the behavior of the approximants on Δ (see Theorem 3 below) as compared to the technique used in [10].

The convergence theory of Padé approximants to Cauchy integrals is strongly interwoven with asymptotic behavior of underlying orthogonal polynomials that are the denominators of Π_n. In

fact, it is easy to show that $f_\mu - \Pi_n = R_n/q_n$, where R_n is a function associated to q_n via (2.15), and q_n satisfies the orthogonality relations of the form (2.14), (2.16) (see, for example, [10, Thm. 4]). Hence, Theorem 2 follows from Theorem 3 below.

2.3. **Strong Asymptotics for non-Hermitian Orthogonal Polynomials.** In this section we investigate the asymptotic behavior of polynomials satisfying non-Hermitian orthogonality relations of the form

$$\int_\Delta t^j q_n(t) w_n(t) dt = 0, \quad j \in \{0, \dots, n-1\}, \tag{2.14}$$

together with the asymptotic behavior of their functions of the second kind, i.e.,

$$R_n(z) := R_n(q_n; z) = \int_\Delta \frac{q_n(t) w_n(t)}{t-z} \frac{dt}{\pi i}, \quad z \in D, \tag{2.15}$$

where $\{w_n\}$ is the sequence of varying weights specified in (2.16).

Theorem 3. *Let $\{q_n\}$, $\deg(q_n) \le n$, be a sequence of polynomials satisfying orthogonality relations (2.14) with weights given by*

$$w_n := \frac{w h_n h}{v_n}, \quad h = e^\theta, \quad h_n = e^{\theta_n}, \tag{2.16}$$

where θ and $w = w(\alpha, \beta; \cdot)$ are as in (2.9), $\{\theta_n\}$ is a normal family in some neighborhood of Δ, and $\{v_n\} \in \mathrm{S}(\Delta)$. Then, for all n large enough, the polynomials q_n have exact degree n and therefore can be normalized to be monic. Under such a normalization, we have that

$$\begin{cases} q_n &= [1+o(1)]/S_n \\ R_n \mathfrak{w} &= [1+o(1)]\gamma_n S_n \end{cases} \tag{2.17}$$

with $o(1)$ satsfying (2.13) locally uniformly in D, where

$$S_n := (2/\varphi)^n S_{w_n \mathfrak{w}^+}, \quad \gamma_n := 2^{1-2n} G_{w_n \mathfrak{w}^+}, \tag{2.18}$$

and R_n was defined in (2.15). Moreover, it holds that

$$\begin{cases} q_n &= [1+o(1)]/S_n^+ + [1+o(1)]/S_n^- \\ (R_n \mathfrak{w})^\pm &= [1+o(1)]\, \gamma_n S_n^\pm \end{cases}, \tag{2.19}$$

where $o(1)$ satisfies (2.13) locally uniformly in Δ°.

The method of proof can also be used to derive the asymptotics of q_n and R_n around ± 1 as was done in [33]. However, the corresponding calculations are lengthy and do not impinge on the convergence of Padé interpolants proper, which is why the authors decided to omit them here.

The appearance of the normal family $\{\theta_n\}$ in (2.16) is not necessitated by Theorem 2 but is included for possible application to meromorphic approximation [53].

3. Proof of Theorem 1 and g-Functions

In this section we prove Theorem 1. The notion of g-function, which we introduce along the way, will be needed later on for the proof of Theorem 3.

3.1. **Parameterization Ξ and functions g and $\widetilde{g}$.** Let $\{v_n\} \in \mathrm{S}(\Delta)$ and r_n be associated to v_n by (2.5). As required by Definition 1-(2) and (3), the normalized counting measures of the zeros of r_n converge weak* to a Borel measure ν, $\mathrm{supp}(\nu) \subset D$. Denote by V_D^ν the *Green potential* of this measure with respect to D. It was shown in the course of the proof of [10, Thm. 1, see (4.34)] that Definition 1-(1) yields

$$V_D^\nu(z) = -\int \log\left|\frac{\varphi(z)-\varphi(t)}{1-\varphi(z)\varphi(t)}\right| d\nu(t). \tag{3.1}$$

In other words, the Green kernel $-\log\left|\frac{\phi(z)-\phi(t)}{1-\overline{\phi(t)}\phi(z)}\right|$, where ϕ is the conformal map of D onto $\{|z|>1\}$ such that $\phi(\infty)=\infty$ and $\phi'(\infty)>0$, can be replaced by the one in (3.1) for this special measure ν.

The Green potential V_D^ν is a positive harmonic function in $D\setminus\mathrm{supp}(\nu)$ whose boundary values vanish everywhere on Δ. Let $L_\rho := \{z:\ V_D^\nu(z)=\log\rho\}$, $\rho>1$, be a level line of V_D^ν in $\Xi(D_\Xi)$, the range of Ξ. Without loss of generality we may assume that ρ is a regular value and therefore L_ρ is a smooth Jordan curve encompassing $\Delta = L_1$. Denote by O the domain bounded by L_ρ and Δ. Set

$$\partial f := \frac{1}{2}\left(\partial_x f - i\partial_y f\right) \quad \text{and} \quad \bar\partial f := \frac{1}{2}\left(\partial_x f + i\partial_y f\right). \tag{3.2}$$

Since ν is a probability measure, it can be verified as in the proof of [10, Thm. 1, see (4.39) and after] that the function

$$\Phi(z) := \exp\left\{2\int_1^z \frac{\partial V_D^\nu}{\partial z}(t)dt\right\} = \exp\left\{-\int \log\frac{\varphi(z)-\varphi(t)}{1-\varphi(z)\varphi(t)}d\nu(t)\right\} \tag{3.3}$$

is well-defined in O and maps it conformally onto the annulus $\{z:\ 1<|z|<\rho\}$ while $\Phi(\pm1)=\pm1$, where we take any path from 1 to z contained in $O\setminus\Delta$. Moreover, by direct examination of the kernel in (3.3), we get that

$$\Phi^+ = \overline{\Phi^-} = 1/\Phi^- \quad \text{on} \quad \Delta. \tag{3.4}$$

This, in particular, yields that $J\circ\Phi$ is holomorphic across Δ, where $J(z)=(z+1/z)/2$ is the Joukovski transformation. Consequently, the inverse $(J\circ\Phi)^{-1}$ is a holomorphic univalent map in some neighborhood of $[-1,1]$ that analytically parametrizes Δ. In what follows, we assume that $\Xi = (J\circ\Phi)^{-1}$.

Based on the conformal map Φ, we define two more functions, g and $\widetilde{g}$ as follows. Set $L := \Phi^{-1}([-\rho,-1])$, $\widetilde{L} := \Phi^{-1}([1,\rho])$ (see Fig. 1), and define

$$\begin{aligned} &g := \log\Phi, && \lim_{z\to1} g(z)=0, && g\in \mathrm{H}(O\setminus L),\\ &\widetilde{g} := \log\Phi - \pi i, && \lim_{z\to-1} \widetilde{g}(z)=0, && \widetilde{g}\in \mathrm{H}(O\setminus \widetilde{L}). \end{aligned} \tag{3.5}$$

It follows immediately from (3.4) that

$$g^+ = -g^- \quad \text{and} \quad \widetilde{g}^+ = -\widetilde{g}^- \quad \text{on} \quad \Delta. \tag{3.6}$$

Hence, g^2 and $\widetilde{g}^2$ are analytic in $O_g := (O\cup\Delta)\setminus L$ and $O_{\widetilde{g}} := (O\cup\Delta)\setminus\widetilde{L}$, respectively. Moreover, it holds that $g^2(\Delta)=\widetilde{g}^2(\Delta)=[-\pi^2,0]$ and $g^2(1)=\widetilde{g}^2(-1)=0$. It is also true that g^2 and $\widetilde{g}^2$ are univalent in O_g and $O_{\widetilde{g}}$, respectively. Indeed, suppose that $g^2(z_1)=g^2(z_2)$, $z_1,z_2\in O_g$. Then either $\Phi(z_1)=\Phi(z_2)$ and therefore $z_1=z_2$ by conformality of Φ or $\Phi(z_1)=1/\Phi(z_2)$, which is possible only if $\Phi^+(z_1)=\overline{\Phi^-(z_2)}$, i.e., if $z_1=z_2\in\Delta$. The case of $\widetilde{g}^2$ is no different.

3.2. **Jordan arcs Δ_n and functions g_n and $\widetilde{g}_n$.** By Definition 1-(2) and upon taking ρ smaller if necessary, we may assume that functions r_n have no zeros in $\overline{O}$. Moreover, as r_n is holomorphic in $D\setminus\Delta$ and has $2n$ zeros in $D\setminus\overline{O}$, its winding number is equal to $-2n$ on any positively oriented curve homologous to L_ρ and contained in O. In other words, r_n has a continuous argument that decreases by $4n\pi$ as Δ is encompassed once in the positive direction. Thus, the functions

$$\Phi_n := r_n^{-1/2n} = \exp\left\{-\int \log\frac{\varphi(z)-\varphi(t)}{1-\varphi(z)\varphi(t)}d\nu_n(t)\right\},$$

are well-defined and analytic in O, where ν_n is the normalized counting measure of the zeros of r_n. Moreover, as the counting measures of zeros of r_n converge weak* to ν by assumption, the functions Φ_n converge to Φ uniformly in $\overline{O}$, distinguishing the one-sided values on $\Delta^\pm$.

Hence, we can define

$$\begin{aligned} g_n &:= \log\Phi_n, & \lim_{z\to 1} g_n(z) &= 0, & g_n &\in \mathrm{H}(O\setminus L),\\ \widetilde{g}_n &:= \log\Phi_n - \pi i, & \lim_{z\to -1} \widetilde{g}_n(z) &= 0, & \widetilde{g}_n &\in \mathrm{H}(O\setminus \widetilde{L}). \end{aligned} \tag{3.7}$$

By (2.4), it is straightforward to see that $\Phi_n^+\Phi_n^- \equiv 1$ on Δ, and therefore

$$g_n^+ = -g_n^- \quad \text{and} \quad \widetilde{g}_n^+ = -\widetilde{g}_n^- \quad \text{on} \quad \Delta. \tag{3.8}$$

Thus, g_n^2 and $\widetilde{g}_n^2$ are analytic in O_g and $O_{\widetilde{g}}$, respectively. We choose domains $O_L \subset O_g$ and $O_{\widetilde{L}} \subset O_{\widetilde{g}}$ in such a manner that $O_L \supset \widetilde{L}$, $O_{\widetilde{L}} \supset L$, and $O_L \cup O_{\widetilde{L}}$ is simply connected and contains Δ (see Fig. 1). Then it is an easy consequence of the convergence of Φ_n to Φ that g_n^2 and $\widetilde{g}_n^2$ converge uniformly to g^2 and $\widetilde{g}^2$ on $\overline{O}_L$ and $\overline{O}_{\widetilde{L}}$, respectively.

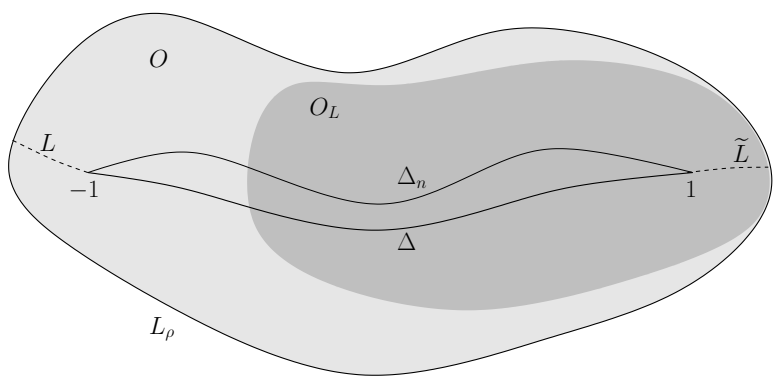

FIGURE 1. The domain O bounded by Δ and L_ρ(light grey), the domain $O_L \subset O\cup\Delta$ (dark grey), the cuts L and $\widetilde{L}$ (dashed arcs).

Next, we claim that g_n^2 and $\widetilde{g}_n^2$ are univalent for all n large enough in $\overline{O}_L$ and $\overline{O}_{\widetilde{L}}$, respectively. Assume to the contrary that there exist two sequences of points $\{z_{1,n}\},\{z_{2,n}\} \subset \overline{O}_L$ such that $g_n^2(z_{1,n}) = g_n^2(z_{2,n})$. As $\overline{O}_L$ is compact, we can assume that $z_{j,n} \to z_j \in \overline{O}_L$, $j = 1,2$. Since g_n^2 converges to g^2 uniformly on $\overline{O}_L$, we have that $g^2(z_1) = g^2(z_2)$ and therefore $z_1 = z_2$. Set $d_n(z) := (g_n^2(z) - g_n^2(z_{1,n}))/(z - z_{1,n})$. Then d_n are analytic functions on $\overline{O}_L$ that converge uniformly to $d(z) := (g^2(z) - g^2(z_1))/(z - z_1)$. Moreover, the values $d_n(z_{2,n})$ are equal to 0 and converge to

$d(z_1)$. Thus, $(g^2)'(z_1) = d(z_1) = 0$, which is impossible since g^2 is univalent. This proves the claim as the case of $\widetilde{g}_n^2$ is no different.

From the above we see that each g_n^2 maps O_L conformally onto a neighborhood of zero as $g_n^2(1) = 0$. Set $\Delta_{n,1}$ to be the preimage of the intersection of this neighborhood with $\Sigma_2 := \{\zeta : \operatorname{Arg}(\zeta) = \pi\}$. Then $\Delta_{n,1}$ is an analytic arc with one endpoint being 1. Analogously, $\widetilde{g}_n^2$ maps $O_{\widetilde{L}}$ conformally into another neighborhood of zero. Thus, we can define $\Delta_{n,-1}$ to be again the preimage of the intersection of this neighborhood with Σ_2. Clearly, $\Delta_{n,-1}$ is an analytic arc with one endpoint being -1. Noticing that g_n^2 assumes negative values if and only if $\widetilde{g}_n^2$ assumes negative values on the common set of definition $O_L \cap O_{\widetilde{L}}$, we derive that $\Delta_n := \Delta_{n,1} \cup \Delta_{n,-1}$ is an analytic arc with endpoint ± 1.

3.3. **Parameterizations Ξ_n and functions g_n^* and $\widetilde{g}_n^*$.** Now, define $\mathfrak{w}_n$ and φ_n with respect to Δ_n like $\mathfrak{w}$ and φ were defined in (2.2) and (2.3) with respect to Δ. Clearly, φ_n is an analytic continuation of φ from D onto $D_n = \overline{\mathbb{C}} \setminus \Delta_n$. Further, let r_n^* be defined by (2.5) with φ replaced by φ_n while keeping the same zeros as r_n. Hence, r_n^* and r_n are analytic continuations of each other defined in D and D_n, respectively. Finally, set Φ_n^*, g_n^*, and $\widetilde{g}_n^*$ to be the analytic continuations of Φ_n, g_n, and $\widetilde{g}_n$ from O, $O \setminus L$, and $O \setminus \widetilde{L}$, onto $O_n := (O \cup \Delta) \setminus \Delta_n$, $O_n \setminus L$, and $O_n \setminus \widetilde{L}$, defined in, by now, obvious manner. Hence

$$g_n^2 = (g_n^*)^2 \quad \text{and} \quad \widetilde{g}_n^2 = (\widetilde{g}_n^*)^2,$$

while $(g_n^*)^2$ is negative on Δ_n. Since, in addition, $(g_n^*)^+ = -(g_n^*)^-$ on Δ_n, $(g_n^*)^\pm$ are pure imaginary on Δ_n and $|\Phi_n^*| \equiv 1$ there. Therefore, $(\Phi_n^*)^+ = \overline{(\Phi_n^*)^-}$ and $(r_n^*)^+ = \overline{(r_n^*)^-}$. Furthermore, Φ_n^* maps O_n onto some annular domain having the unit circle as a component of its boundary. Arguing as was done after (3.4), we derive that $J \circ \Phi_n^*$ is holomorphic across Δ_n and that $\Xi_n := (J \circ \Phi_n^*)^{-1}$ is a holomorphic parameterization of Δ_n, $\Xi_n([-1,1]) = \Delta_n$. Moreover, as the functions Φ_n converge to Φ uniformly in some annular domain encompassing Δ, we see that the functions $J \circ \Phi_n^*$ converge locally uniformly to $J \circ \Phi$ in some neighborhood of Δ. Hence, the sequence of analytic parameterizations $\{\Xi_n\}$ of Δ_n converges uniformly to the analytic parameterization Ξ of Δ in some neighborhood of $[-1,1]$. This finishes the proof of Theorem 1.

4. Trace Theorems and Extensions

As is usual in the Riemann-Hilbert approach to asymptotics of orthogonal polynomials, we shall need to extend the weights of orthogonality from Δ into subsets of the complex plane. As the weights are not analytic, this extension will require a special construction that we carry out in this section.

4.1. **Domains with Smooth Boundaries.** In this section we suppose that Ω is a bounded simply connected domain with boundary Γ which is infinitely smooth and contains Δ, i.e., $\Delta \subset \Gamma$.

Definition 5. *Set $\mathrm{L}^p(\Omega)$, $p \in [1,\infty)$, to be the space of all measurable functions f such that $|f|^p$ is integrable over Ω. The Sobolev space $\mathrm{W}_p^1(\Omega)$, $p \in [1,\infty)$, is the subspace of $\mathrm{L}^p(\Omega)$ that comprises of functions with weak partial derivatives also in $\mathrm{L}^p(\Omega)$.*

Then the following theorem takes place [30, Thm. 1.5.1.2].

Theorem T1. *For each $f \in \mathrm{W}_p^{1-1/p}(\Gamma)$, $p \in (1,\infty)$, there exists $F \in \mathrm{W}_p^1(\Omega)$ such that $F_{|\Gamma} = f$. Moreover, the extension operator can be made independent of p. Conversely, for every $F \in \mathrm{W}_p^1(\Omega)$ it holds that $F_{|\Gamma} \in \mathrm{W}_p^{1-1/p}(\Gamma)$.*

Together with the Sobolev spaces $\mathrm{W}^1_p(\Omega)$, we consider smoothness classes $\mathrm{C}^{m,\varsigma}(\overline{\Omega})$.

Definition 6. *By $\mathrm{C}^{m,\varsigma}(\overline{\Omega})$, $m \in \mathbb{Z}_+$, $\varsigma \in (0,1]$, we denote the space of all functions on $\overline{\Omega}$ whose partial derivatives up to the order m are continuous on $\overline{\Omega}$ and whose partial derivatives of order m are uniformly Hölder continuous on $\overline{\Omega}$ with exponent ς. Moreover, $\mathrm{C}^{m,\varsigma}_0(\overline{\Omega})$ will stand for the subset of $\mathrm{C}^{m,\varsigma}(\overline{\Omega})$ consisting of functions whose partial derivatives up to order m, including the function itself, vanish on Γ. Finally, $\mathrm{C}^{\infty}(\overline{\Omega})$ will denote the space of functions on Ω whose partial derivatives of any order exist and are continuous on $\overline{\Omega}$.*

It is known from Sobolev's imbedding theorem [2, Thm. 5.4] that

$$\mathrm{W}^1_p(\Omega) \subset \begin{cases} \mathrm{L}^{2p/(2-p)}(\Omega), & p \in [1,2), \\ \mathrm{C}^{0,1-2/p}(\overline{\Omega}), & p \in (2,\infty). \end{cases} \tag{4.1}$$

Hence, for $f \in \mathrm{W}^{1-1/p}_p(\Gamma)$, $p \in (2,\infty)$, the function F granted by Theorem T1 belongs to $\mathrm{C}^{0,1-2/p}(\overline{\Omega})$ and therefore $f \in \mathrm{C}^{0,1-2/p}(\Gamma)$, which is exactly what was stated in (2.7).

Later on, we shall need the following proposition.

Proposition 4. *Let f be a continuous function on Δ such that $f(\pm 1) = 0$. If $f \in \mathrm{W}^{1-1/p}_p$, $p \in (2,\infty)$, then there exists $F \in \mathrm{W}^1_p(\Omega)$ such that $F_{|\Delta} = f$. Moreover, if $f \in \mathrm{C}^{0,\varsigma}$, $\varsigma \in (1/2,1]$, then there exists $F \in \mathrm{W}^1_q(\Omega)$ for any $q \in (2, \frac{1}{1-\varsigma})$ such that $F_{|\Delta} := f$.*

Proof. In both cases set $\tilde{f} = f$ on Δ and $\tilde{f} \equiv 0$ on $\Gamma \setminus \Delta$. When $f \in \mathrm{W}^{1-1/p}_p$, it is immediate to check that $\tilde{f} \in \mathrm{W}^{1-1/p}_p(\Gamma)$ and therefore the first claim follows from Theorem T1. When $f \in \mathrm{C}^{0,\varsigma}$, it holds that $\tilde{f} \in \mathrm{C}^{0,\varsigma}(\Gamma)$ and $\widetilde{f} \in \mathrm{W}^{1-1/q}_q(\Gamma)$ for any $q \in (1, \frac{1}{1-\varsigma})$ by an easy estimate (see Definition 4). Hence, the second claim of the proposition again follows from Theorem T1. □

To state a trace theorem for classes $\mathrm{C}^{m,\varsigma}(\overline{\Omega})$, we need to introduce the notion of a directional derivative. Namely, let ξ be a continuous function on $\overline{\Omega}$ and $f \in \mathrm{W}^1_p(\Omega)$. With the slight abuse of notation, we define the derivative of f in the direction of the field ξ, denoted by ∂_ξ, as

$$\partial_\xi f := \bar{\xi}\,\bar{\partial} f + \xi\,\partial f = \mathrm{Re}(\xi)\partial_x f + \mathrm{Im}(\xi)\partial_y f, \tag{4.2}$$

where $\vec{\xi}$ is the vector field with values in $\mathbb{R}^2$ corresponding to ξ.

As Γ is infinitely smooth, any conformal map ϕ of Ω onto the unit disk belongs to $\mathrm{C}^{\infty}(\overline{\Omega})$ [39, Thm. 3.6]. Moreover, it holds that $\phi' \neq 0$ in $\overline{\Omega}$. Thus, we may set

$$n(z) := \frac{\phi(z)}{\phi'(z)} \quad \text{and} \quad \mathbf{n}(z) := \phi(z)\frac{|\phi'(z)|}{\phi'(z)}, \quad z \in \overline{\Omega}. \tag{4.3}$$

Then $n, \mathbf{n} \in \mathrm{C}^{\infty}(\overline{\Omega})$ and n is holomorphic in Ω. Moreover, for any $z \in \Gamma$, $\mathbf{n}(z) = n(z)/|n(z)|$ represents the complex number corresponding to the outer unit normal to Γ at z. Then the following theorem takes place [30, Thm. 6.2.6].

Theorem T2(1). *Let $\{f_k\}_{k=0}^m$ be such that $f_k \in \mathrm{C}^{m-k,\varsigma}(\Gamma)$, $m \in \mathbb{N}$, $\varsigma \in (0,1]$, $k \in \{0,\dots,m\}$. Then there exists $F \in \mathrm{C}^{m,\varsigma}(\overline{\Omega})$ such that $(\partial^k_{\mathbf{n}} F)_{|\Gamma} = f_k$, $k \in \{0,\dots,m\}$.*

Now, observe that

$$\partial_{\mathbf{n}}^k = |\phi'|^k \partial_n^k + \sum_{j=1}^{k-1} c_{k,j} \partial_n^j,$$

where the functions $c_{k,j}$ involve sums and products of the powers of the iterated directional derivatives of $|\phi'|$ with respect to the field n and therefore belong to $\mathrm{C}^\infty(\overline{\Omega})$. Set $c_{k,k} := |\phi'|^k$ and $c_{k,j} \equiv 0$, $j \in \{k+1,\dots,m\}$, $k \in \{1,\dots,m\}$. Then the matrix $\mathscr{C}_\phi = [c_{k,j}]_{k,j=1}^m$ is such that $\det(\mathscr{C}_\phi) = |\phi'|^{m(m+1)/2}$, which is non-vanishing at any $z \in \overline{\mathbb{C}}$, and

$$(\partial_{\mathbf{n}},\dots,\partial_{\mathbf{n}}^m)^{\mathrm{T}} = \mathscr{C}_\phi (\partial_n,\dots,\partial_n^m)^{\mathrm{T}}.$$

Thus, to every family of functions $\{f_k\}_{k=0}^m$, $f_k \in \mathrm{C}^{m-k,\varsigma}(\Gamma)$, there corresponds another family, say $\{\tilde{f}_k\}_{k=0}^m$, $\tilde{f}_k \in \mathrm{C}^{m-k,\varsigma}(\Gamma)$, such that there exists $F \in \mathrm{C}^{m,\varsigma}(\overline{\Omega})$ satisfying $(\partial_{\mathbf{n}}^k F)_{|\Gamma} = f_k$ and $(\partial_n^k F)_{|\Gamma} = \tilde{f}_k$. Moreover, this correspondence is one-to-one and onto. Hence, Theorem T2(1) can be equivalently reformulated as follows.

Theorem T2(2). *Let $\{f_k\}_{k=0}^m$ be such that $f_k \in \mathrm{C}^{m-k,\varsigma}(\Gamma)$, $m \in \mathbb{N}$, $\varsigma \in (0,1]$, $k \in \{0,\dots,m\}$. Then there exists $F \in \mathrm{C}^{m,\varsigma}(\overline{\Omega})$ such that $(\partial_n^k F)_{|\Gamma} = f_k$, $k \in \{0,\dots,m\}$.*

Finally, we define $\tau := in$. Clearly, $\tau(z)/|\tau(z)|$, $z \in \Gamma$, is the complex number corresponding to the positively oriented unit tangent to Γ at z. Since n and τ are holomorphic functions such that $\tau = in$, it is a simple computation to verify that $\partial_n \partial_\tau F = \partial_\tau \partial_n F$ in $\overline{\Omega}$. Then the following proposition holds.

Proposition 5. *Let $f \in \mathrm{C}^{m,\varsigma}(\Delta)$, $m \in \mathbb{N}$, $\varsigma \in (0,1]$, $f^{(k)}(\pm 1) = 0$, $k \in \{0,\dots,m\}$. Then there exists $F \in \mathrm{C}^{m,\varsigma}(\overline{\Omega})$ such that $F_{|\Delta} = f$ and $\bar{\partial} F \in \mathrm{C}_0^{m-1,\varsigma}(\overline{\Omega})$.*

Proof. Set $f_0 = f$ on Δ and $f_0 \equiv 0$ on $\Gamma \setminus \Delta$. It is clear that $f_0 \in \mathrm{C}^{m,\varsigma}(\Gamma)$. Further, set $f_k := (-i)^k \partial_\tau^k f_0$, $k \in \{1,\dots,m\}$. As $f_k \in \mathrm{C}^{m-k,\varsigma}(\Gamma)$, $k \in \{0,\dots,m\}$, Theorem T2(2) yields that there exists $F \in \mathrm{C}^{m,\varsigma}(\overline{\Omega})$ such that $(\partial_n^k F)_{|\Gamma} = f_k$. In particular, $F_{|\Delta} = f$.

It remains only to show that $\bar{\partial} F \in \mathrm{C}_0^{m-1,\varsigma}(\overline{\Omega})$. It can be easily checked that

$$2\bar{n}\bar{\partial} F = \partial_n F + i\partial_\tau F =: H \quad \text{in} \quad \Omega', \tag{4.4}$$

where Ω' is an annular domain such that $\Gamma \subset \partial\Omega'$ and n is non-vanishing on this domain. As $n \in \mathrm{C}^\infty(\overline{\Omega})$ and is zero free in Ω', it holds that $\bar{\partial} F \in \mathrm{C}_0^{m-1,\varsigma}(\overline{\Omega})$ if and only if $H \in \mathrm{C}_0^{m-1,\varsigma}(\overline{\Omega})$. Moreover, it is immediate from the construction of F that $H \in \mathrm{C}^{m-1,\varsigma}(\overline{\Omega})$. Thus, it is only necessary to verify that all the partial derivatives of H of order k, for any $k \in \{0,\dots,m-1\}$, vanish on Γ. Since partial derivatives with respect to n and τ commute, and these fields are non-vanishing and non-collinear in Ω', it is enough to show that

$$0 \equiv (\partial_\tau^{k_1} \partial_n^{k_2} H)_{|\Gamma} = \left(\partial_\tau^{k_1} \partial_n^{k_2+1} F\right)_{|\Gamma} + i\left(\partial_\tau^{k_1+1} \partial_n^{k_2} F\right)_{|\Gamma}$$

for all $k_1 + k_2 \in \{0,\dots,m-1\}$. The latter holds since

$$\left(\partial_\tau^{k_1} \partial_n^{k_2+1} F\right)_{|\Gamma} = \partial_\tau^{k_1} f_{k_2+1} = (-i)^{k_2+1} \partial_\tau^{k_1+k_2+1} f_0 = -i\partial_\tau^{k_1+1} f_{k_2} = -i\left(\partial_\tau^{k_1+1} \partial_n^{k_2} F\right)_{|\Gamma},$$

by the choice of $\{f_k\}$. □

4.2. **Domains with Polygonal Boundary.** The previous results also hold, with some modifications, for domains with polygonal boundaries. Namely, let Ω be a domain whose boundary is a curvilinear polygon consisting of two pieces, say Δ_1 and Δ_2, such that they might form corners at the joints. As we do not strive for generality at this point, we assume that each Δ_j is an analytic arc connecting -1 and 1.

The first trace theorem of this section states the following [30, Thm. 1.5.2.3].

Theorem T3. *Given $f_j \in \mathrm{W}_p^{1-1/p}(\Delta_j)$, $j=1,2$, satisfying $f_1(\pm 1) = f_2(\pm 1)$, there exists $F \in \mathrm{W}_p^1(\Omega)$ such that $F_{|\Delta_j} = f_j$, $j=1,2$. The choice of F can be made in such a way that it depends only on f_j and not on p.*

To state an analogous theorem for the classes $\mathrm{C}^{m,\varsigma}(\Delta_j)$, we again need to define normal fields on Δ_j, $j=1,2$. Let Γ_j be an infinitely smooth Jordan curve such that $\Delta_j \subset \Gamma_j$. Moreover, assume that the interior domain of Γ_j, say Ω_j, contains Ω, $j=1,2$. Define n_j for Ω_j as it was done in (4.3). Composing with a self-map of the disk if necessary, we can choose conformal maps in (4.3) so n_j does not vanish in Ω. Further, set $\tau_j := i n_j$ if Ω lies on the left side of Δ_j and $\tau_j := -i n_j$ otherwise. In particular, the fields τ_j and n_j commute, are infinitely smooth, non-vanishing and non-collinear. Finally, observe that (4.4) holds with n, τ, and Ω' replaced by n_j, τ_j, Ω, and the plus sign replaced by the minus sign in the right-hand side of (4.4) when $\tau_j = -i n_j$.

With all the necessary material at hand, we can state a special case of the trace theorem for smoothness classes on domains with polygonal boundary[2] [30, Cor. 6.2.8].

Theorem T4. *Given $\{f_{jk}\}_{k=0}^m$, $f_{jk} \in \mathrm{C}^{m-k,\varsigma}(\Delta_j)$, $m \in \mathbb{N}$, $\varsigma \in (0,1]$, $j=1,2$, satisfying $f_{jk_1}^{(k_2)}(\pm 1) = 0$, $k_1 + k_2 \in \{0,\dots,m\}$, there exists $F \in \mathrm{C}^{m,\varsigma}(\overline{\Omega})$ such that $(\partial_{n_j}^k F)_{|\Delta_j} = f_{jk}$, $k \in \{0,\dots,m\}$, $j=1,2$.*

Now, as in Section 4.1, we shall make Theorems T3 and T4 suit our needs. Let Δ be a closed analytic Jordan arc and $\Delta_\pm$ be two closed analytic Jordan arcs with endpoints ± 1 such that the interior domain of $\Delta \cup \Delta_+$, say Ω_+, is simply connected and lies to the left of Δ while the interior domain of $\Delta \cup \Delta_-$, say Ω_-, is again simply connected and lies to the right of Δ. Then the following proposition holds.

Proposition 6. *Let f be a continuous function on Δ such that $f(\pm 1) = 0$. If $f \in \mathrm{W}_p^{1-1/p}$, $p \in (2,\infty)$, then there exist $F_\pm \in \mathrm{W}_p^1(\Omega_\pm)$ such that*

$$F_{\pm|\Delta} = \pm f \quad \text{and} \quad F_{\pm|\Delta_\pm} \equiv 0. \tag{4.5}$$

Moreover, if $f \in \mathrm{C}^{0,\varsigma}$, $\varsigma \in (1/2,1]$, then there exist $F_\pm \in \mathrm{W}_q^1(\Omega_\pm)$ for any $q \in (2, \frac{1}{1-\varsigma})$ satisfying (4.5).

Proof. This proposition follows from Theorem T3 in the same fashion as Proposition 4 followed from Theorem T1. □

Finally, we state the counterpart of Proposition 5 for domains with corners.

Proposition 7. *Let $f \in \mathrm{C}^{m,\varsigma}(\Delta)$, $m \in \mathbb{N}$, $\varsigma \in (0,1]$, $f^{(k)}(\pm 1) = 0$, $k \in \{0,\dots,m\}$. Then there exist functions $F_\pm \in \mathrm{C}^{m,\varsigma}(\overline{\Omega}_\pm)$ such that*

$$F_{\pm|\Delta} = \pm f, \quad F_{\pm|\Delta_\pm} \equiv 0, \quad \text{and} \quad \bar\partial F_\pm \in \mathrm{C}_0^{m-1,\varsigma}(\overline{\Omega}_\pm). \tag{4.6}$$

[2] In Theorem T4 we use non-unit normal fields n_j rather than fields that are unit on Δ_j as it was done in the original reference. However, we have already explained after Theorem T2(1) that these formulations are equivalent.

Proof. First, we consider the case of Ω_+. By setting $f_{1k} := (-i)^k f^{(k)}$, $k \in \{0,\dots,m\}$, we see that $f_{1k} \in C^{s-k}(\Delta)$, $k \in \{0,\dots,m\}$. Moreover, after putting $f_{2k} \equiv 0$, we observe that

$$f_{jk_1}^{(k_2)}(\pm 1) = 0, \quad k_1 + k_2 \in \{0,\dots,m\}, \quad j = 1,2.$$

Then the existence of F_+ in $C^{m,\varsigma}(\overline{\Omega}_+)$ follows from Theorem T4. The fact that $\bar\partial F_+ \in C_0^{m-1,\varsigma}(\overline{\Omega}_+)$ can be shown exactly as in Proposition 5. In the case of Ω_- the only difference is that we need to set $f_{1k} := -i^k f^{(k)}$ since this time the normal and tangent on Δ satisfy $\tau = -in$. □

5. Scalar Boundary Value Problems

In this section we dwell on smoothness properties of certain integral operators.

5.1. Integral Operators.

Below we introduce contour and area integral operators and explain the solution of a certain $\bar\partial$-problem.

Let ϕ be an L^p, $p > 1$, function on Δ, where $L^p = L^p(\Delta)$ stands for the space of functions with p-summable modulus on Δ with respect to arclength differential $|dt|$. The Cauchy integral operator on Δ is defined by

$$\mathscr{C}\phi(z) := \mathscr{C}_\Delta\phi(z) = \frac{1}{2\pi i}\int_\Delta \frac{\phi(t)}{t-z}dt, \quad z \in D. \tag{5.1}$$

It is known that $\mathscr{C}\phi$ is a holomorphic function in D with L^p traces on Δ, i.e., non-tangential limits a.e. on Δ, from above and below denoted by $\mathscr{C}^\pm\phi$. These traces are connected by the Sokhotski-Pemelj formulae [24, Sec. I.4.2], i.e.,

$$\mathscr{C}^+\phi - \mathscr{C}^-\phi = \phi \quad \text{and} \quad \mathscr{C}^+\phi + \mathscr{C}^-\phi = \mathscr{S}\phi, \quad \text{a.e. on} \quad \Delta, \tag{5.2}$$

where $\mathscr{S}$ is the singular integral operator on Δ given by

$$\mathscr{S}\phi(\tau) := \mathscr{S}_\Delta\phi(\tau) = \frac{1}{\pi i}\int_\Delta \frac{\phi(t)}{t-\tau}dt, \quad \tau \in \Delta^\circ, \tag{5.3}$$

with the integral being understood in the sense of the principal value.

Let now Ω be a simply connected bounded domain with smooth boundary Γ. We define $\mathscr{C}_\Gamma$ and $\mathscr{S}_\Gamma$ by (5.1) and (5.3) integrating this time over Γ rather than Δ. The Sokhotski-Plemelj formulae (5.2) still hold for $\phi \in L^p(\Gamma)$, $p > 1$, with the only difference that now $\mathscr{C}_\Gamma\phi$ is a sectionally holomorphic function and therefore $\mathscr{C}_\Gamma^+\phi$ is the trace of $\mathscr{C}_\Gamma\phi$ from within Ω and $\mathscr{C}_\Gamma^-\phi$ is the trace of $\mathscr{C}_\Gamma\phi$ from within $\overline{\mathbb{C}} \setminus \overline{\Omega}$.

Concerning the smoothness of $\mathscr{C}_\Gamma\phi$ the following is known. If $\phi \in C^{0,\varsigma}(\Gamma)$, $\varsigma \in (0,1)$, then $\mathscr{C}_\Gamma\phi \in C^{0,\varsigma}(\overline{\Omega})$ [24, Sec. 5.5.1]. In particular, this means that $\mathscr{C}_\Gamma\phi$ extends continuously from Ω to Γ. Further, if ϕ is continuously differentiable on Γ, then $\mathscr{C}_\Gamma'\phi = \mathscr{C}_\Gamma\phi'$ [24, Sec. 4.4.4]. Thus, we may conclude that when $\phi \in C^{m,\varsigma}(\Gamma)$, $m \in \mathbb{Z}_+$, $\varsigma \in (0,1)$, then $\mathscr{C}_\Gamma\phi \in C^{m,\varsigma}(\overline{\Omega})$.

Let now $\phi \in L^p(\Omega)$. The Cauchy area integral on Ω is defined as

$$\mathscr{K}\phi(z) := \frac{1}{2\pi i}\iint_\Omega \frac{\phi(\zeta)}{\zeta - z}d\zeta \wedge d\bar\zeta, \quad z \in \Omega. \tag{5.4}$$

Then it is well-known [5, Sec. 4.9] that

$$\bar\partial\mathscr{K}\phi = \phi \quad \text{and} \quad \partial\mathscr{K}\phi = \mathscr{B}\phi, \tag{5.5}$$

in the distributional sense, where $\mathscr{B}$ is the Beurling transform, i.e.,

$$\mathscr{B}\phi(z) := \frac{1}{2\pi i}\iint_{\Omega}\frac{\phi(\zeta)}{(\zeta-z)^2}d\zeta\wedge d\bar{\zeta}, \quad z\in\Omega, \tag{5.6}$$

and the integral is understood in the sense of the principal value.

The transformation $\mathscr{K}$ defines a bounded operator from $\mathrm{L}^p(\Omega)$ into $\mathrm{L}^{2p/(2-p)}(\Omega)$ for $p\in(1,2)$, [5, Thm. 4.3.8], and into $\mathrm{C}^{1-2/p}(\overline{\Omega})$ for $p\in(2,\infty)$, [5, Thm. 4.3.13]. Since nothing prevents us from taking z outside of $\overline{\Omega}$, $\mathscr{K}\phi$ is, in fact, defined throughout $\overline{\mathbb{C}}$ and is clearly holomorphic outside of $\overline{\Omega}$ and vanishes at infinity. Moreover, $\mathscr{K}\phi$ is continuous across Γ when $p\in(2,\infty)$. The latter can be easily seen if we continue ϕ by zero to a larger domain, say $\widetilde{\Omega}$, and observe that this extension is in $\mathrm{L}^p(\widetilde{\Omega})$.

The Beurling transform $\mathscr{B}$ defines a bounded operator from a weighted space $\mathrm{L}^p_v(\mathbb{C}) := \{f : f^p v\in \mathrm{L}^p(\mathbb{C})\}$ into itself when the non-negative function v is an A_p*-weight (Muckenhoupt weight)*, $p\in(1,\infty)$ [5, Thm. 4.9.6]. Let $\phi\in\mathrm{L}^p(\Omega)$. We can suppose that $\phi\in\mathrm{L}^p(\mathbb{C})$ with $\phi\equiv 0$ outside of Ω and therefore $\phi/\mathfrak{w}\in\mathrm{L}^p_{|\mathfrak{w}|^p}(\mathbb{C})$. It holds that $|\mathfrak{w}|^p$ is an A_p-weight for $p>2$ [28, Sec. 9.1.b]. Thus, $\mathscr{B}(\phi/\mathfrak{w})\in\mathrm{L}^p_{|\mathfrak{w}|^p}(\mathbb{C})$ and therefore

$$\phi\in\mathrm{L}^p(\Omega) \quad \text{implies} \quad \mathfrak{w}\mathscr{B}(\phi/\mathfrak{w})\in\mathrm{L}^p(\Omega), \quad p>2. \tag{5.7}$$

Finally, we point out that $\phi\in\mathrm{W}^1_p(\Omega)$ can be recovered by means $\mathscr{C}_\Gamma$ and $\mathscr{K}$ in the following fashion:

$$\phi = \mathscr{C}_\Gamma\phi + \mathscr{K}\bar{\partial}\phi \quad \text{a.e. in} \quad \Omega, \tag{5.8}$$

which is the Cauchy-Green formula for Sobolev functions.

5.2. **Functions of the Second Kind.** Let R_n be given by (2.15) with q_n satisfying (2.14) and w_n defined as in (2.16). Clearly, R_n is holomorphic in D, and it vanishes at infinity with order at least $n+1$, i.e., $R_n = O(z^{-n-1})$ as $z\to\infty$, on account of (2.14). It is also clear that $R_n = 2\mathscr{C}(q_n w_n)$. Thus, it holds by (5.2) that

$$R_n^+ - R_n^- = 2q_n w_n.$$

Further, since $q_n w_n/w = q_n h_n h/v_n$ is Hölder continuous by the conditions of Theorem 3, we have that

$$R_n = \begin{cases} O(|1-z|^\alpha), & \text{if} \quad \alpha<0,\\ O(\log|1-z|), & \text{if} \quad \alpha=0,\\ O(1), & \text{if} \quad \alpha>0, \end{cases} \tag{5.9}$$

and analogous asymptotics holds near -1. Indeed, the case $\alpha<0$ follows from [24, Sec. I.8.3 and I.8.4]. (Observe that we defined $(1-t)^\alpha$, $t\in\Delta^\circ$, as the values on Δ of $(1-z)^\alpha$, where the latter is holomorphic outside of the branch cut taken along Δ_r. However, $(1-t)^\alpha$ equivalently can be regared as the boundary values of $(1-z)^\alpha$ on Δ^+, where the latter is holomorphic outside of the branch cut taken along $\Delta_l\cup\Delta$. Hence, the analysis in [24, Sec. I.8.3] indeed applies to the present situation.) The case $\alpha=0$ follows from [24, Sec. I.8.1 and I.8.4]. Finally, the case $\alpha>0$ holds since $R_n(1)$ exists for such α as $w_n(t)/(t-1)$ is integrable near 1 in this situation.

5.3. **Szegő Functions.** Let $\theta \in C^{m,\varsigma}$, $m \in \mathbb{Z}_+$, $\varsigma(0,1]$, and $h := e^{\theta}$. The definition of the Szegő function given in (2.11) can be rewritten as

$$S_h = \exp\left\{ \mathfrak{w}\mathscr{C}\left(\frac{\theta}{\mathfrak{w}^+}\right) - \frac{1}{2}\int \theta d\omega \right\}.$$

Note that decomposition (2.12) easily follows from the Sokhotski-Plemelj formulae (5.2). Moreover, as the lemma in the next section shows, the traces $S_h^{\pm}$ belong to $C^{m,\varsigma'}$, $0 < \varsigma' < \varsigma$, and $S_h^+(\pm 1) = S_h^-(\pm 1)$. In particular, the functions[3] $c_h^+ := S_h^+/S_h^-$ and $c_h^- := S_h^-/S_h^+$ are continuous on Δ and assume value 1 at ± 1. It also follows from the Sokhotski-Plemelj formulae that

$$c_h^{\pm} = \exp\left\{ \mathfrak{w}^{\pm}\mathscr{S}\left(\frac{\theta}{\mathfrak{w}^+}\right) \right\}. \tag{5.10}$$

The following facts are explained in detail in [10, Sec. 3.2 and 3.3]. First, if $\theta_1, \theta_2 \in C^{m,\varsigma}$, then $S_{h_1 h_2} = S_{h_1} S_{h_2}$. Second, if $\{\theta_n\}$ is a normal family in some neighborhood of Δ then $\{S_{h_n}\}$ is a normal family in D. If, in addition, $\{\theta_n\}$ converges then $\{S_{h_n}\}$ converges as well and the convergence is *uniform* on the closure of D, that is, including the boundary values from each side.

Third, the uniqueness of decomposition (2.12), which was shown, for instance, in [53, eq. (2.7) and after], implies the following formula for the Szegő function of the polynomial v_n, $\deg(v_n) \le 2n$, with zeros in D:

$$S_{v_n^2} = S_{v_n}^2 = \frac{1}{G_{v_n}} \frac{v_n}{r_n \varphi^{2n}}, \tag{5.11}$$

where r_n was defined in (2.5).

Fourth, observe that it is possible to define continuous arguments of $(z+1)/\varphi(z)$ and $(z-1)/\varphi(z)$ that vanish on the real axis in some neighborhood of infinity. Therefore it holds that

$$S_w(z) = \left(2\frac{z-1}{\varphi(z)}\right)^{\alpha/2} \left(2\frac{z+1}{\varphi(z)}\right)^{\beta/2} \quad \text{and} \quad G_w = 2^{-(\alpha+\beta)}, \tag{5.12}$$

where w was defined in (2.1) and the branches of the power functions are taken such that the positive reals are mapped into the positive reals.

Finally, using (5.12) with $w = w(1/2, 1/2; \cdot)$ we have that

$$S_w^+(t) S_w^-(t) = 2\sqrt{1-t^2} = -2i\mathfrak{w}^+(t) \quad t \in \Delta.$$

Hence, we get that

$$S_{\mathfrak{w}^+} = \sqrt{2\mathfrak{w}/\varphi} \quad \text{and} \quad G_{\mathfrak{w}^+} = i/2, \tag{5.13}$$

where, as usual, the branch of the square root is chosen so that $S_{\mathfrak{w}^+}$ is positive for large positive reals. It will be useful for us later to note that

$$(\varphi S_{\mathfrak{w}^+})^{\pm} = (S_{\mathfrak{w}^+}^{\pm})^2 \frac{\varphi^{\pm}}{S_{\mathfrak{w}^+}^{\pm} S_{\mathfrak{w}^+}^{\mp}} S_{\mathfrak{w}^+}^{\mp} = (S_{\mathfrak{w}^+}^{\pm})^2 \frac{i\varphi^{\pm}}{\pm 2\mathfrak{w}^{\pm}} S_{\mathfrak{w}^+}^{\mp} = \pm i S_{\mathfrak{w}^+}^{\mp}, \tag{5.14}$$

where we used (2.12) and (5.13).

[3] Here we slightly abuse the notation and use superscripts $+$ and $-$ as a part of the symbol for the function. However, in Lemma 16 we shall construct a function c_h whose traces on Δ will coincide with $c_h^{\pm}$.

5.4. **Smoothness of a Singular Integral Operator.** In this section we show that the boundary values on Δ of the Szegő function of e^{θ} have essentially the same Sobolev or Hölder smoothness as θ.

We start with the case of functions in $\mathrm{W}_p^{1-1/p}$, $p\in(2,\infty)$. Observe that $\mathrm{W}_q^{1-1/q}\supset \mathrm{W}_p^{1-1/p}$ when $q<p$, which is immediate from Definition 4.

Proposition 8. *Let $\theta\in \mathrm{W}_p^{1-1/p}$, $p\in(2,\infty)$. Then*

$$\mathfrak{w}^{\pm}\mathscr{S}(\theta/\mathfrak{w}^{+})=\pm d+\mathfrak{w}^{\pm}\ell,\quad d(\pm 1)=0,$$

where $d\in \mathrm{W}_q^{1-1/q}$ for any $q\in(2,p)$ and ℓ is a polynomial, $\deg(\ell)\le 1$.

Proof. It follows immediately from Cauchy integral formula and the Sokhotski-Plemelj formulae (5.2) that

$$\mathscr{C}\left(\frac{1}{\mathfrak{w}^{+}}\right)=\frac{1}{2\mathfrak{w}^{+}}\quad\text{and}\quad \mathscr{S}\left(\frac{1}{\mathfrak{w}^{+}}\right)=0. \tag{5.15}$$

Hence, for any polynomial ℓ_0 and $\tau\in\Delta^{\circ}$ it holds that

$$\mathscr{S}\left(\frac{\ell_0}{\mathfrak{w}^{+}}\right)(\tau)=\frac{1}{\pi i}\int_{\Delta}\frac{\ell_0(t)}{t-\tau}\frac{dt}{\mathfrak{w}^{+}(t)}=\frac{1}{\pi i}\int_{\Delta}\frac{\ell_0(t)-\ell_0(\tau)}{t-\tau}\frac{dt}{\mathfrak{w}^{+}(t)}=\ell_1(\tau),$$

where ℓ_1 is a polynomial, $\deg(\ell_1)<\deg(\ell_0)$, since $\frac{\ell_0(\cdot)-\ell_0(\tau)}{\cdot-\tau}$ is a polynomial in τ. Choose ℓ_0 to be the polynomial interpolating θ at ± 1, $\deg(\ell_0)\le 1$. Then

$$\mathfrak{w}^{\pm}\mathscr{S}\left(\frac{\theta}{\mathfrak{w}^{+}}\right)=\mathfrak{w}^{\pm}\mathscr{S}\left(\frac{\theta-\ell_0}{\mathfrak{w}^{+}}\right)+\mathfrak{w}^{\pm}\ell_1.$$

Thus, it holds by (5.2) that

$$\mathfrak{w}^{\pm}\mathscr{S}\left(\frac{\theta}{\mathfrak{w}^{+}}\right)=2\mathfrak{w}^{\pm}\left(\mathscr{C}^{+}\left(\frac{\theta-\ell_0}{\mathfrak{w}^{+}}\right)-\ell_2\right)\mp(\theta-\ell_0)+\mathfrak{w}^{\pm}(2\ell_2+\ell_1),$$

where ℓ_2, $\deg(\ell_2)\le 1$ will be chosen later. Set $\ell:=\ell_1+2\ell_2$ and

$$d:=2\mathfrak{w}^{+}\left(\mathscr{C}^{+}\left(\frac{\theta-\ell_0}{\mathfrak{w}^{+}}\right)-\ell_2\right)-(\theta-\ell_0)=:2d_1-(\theta-\ell_0).$$

Clearly, to prove the proposition, we need to show that $d_1\in \mathrm{W}_q^{1-1/q}$ and $d_1(\pm 1)=0$.

Let Γ be any infinitely smooth curve containing Δ. Assume also that the inner domain of Γ, say Ω, lies to the left of Δ, i.e., Δ^{+} is accessible from within Ω. Define $\theta_{e|\Delta}:=\theta-\ell_0$ and $\theta_{e|\Gamma\setminus\Delta}\equiv 0$. It is clear that $\theta_e\in \mathrm{W}_p^{1-1/p}(\Gamma)$. Moreover, since θ_e is identically zero on $\Gamma\setminus\Delta$, it holds by (5.2) that

$$d_1=\mathfrak{w}^{+}\left(\mathscr{C}^{+}\left(\frac{\theta_e}{\mathfrak{w}^{+}}\right)-\ell_2\right)=\mathfrak{w}\left(\mathscr{C}_{\Gamma}^{+}\left(\frac{\theta_e}{\mathfrak{w}}\right)-\ell_2\right),$$

where from now on we agree that $\mathfrak{w}_{|\Gamma}$ is the trace of $\mathfrak{w}$ from within Ω, i.e. it is equal to $\mathfrak{w}^{+}$ on Δ. Furthermore, we can regard d_1 as a function holomorphic in Ω. Thus, by Theorem T1, to show that $d_{1|\Delta}\in \mathrm{W}_q^{1-1/q}$ it is enough to prove that $d_1\in \mathrm{W}_q^{1}(\Omega)$, $q\in(2,p)$. As d_1 is holomorphic in Ω, it is, in fact, sufficient to get that $\partial d_1\in \mathrm{L}^{q}(\Omega)$.

Now, Proposition 4 insures that there exists $\Theta \in \mathrm{W}^1_p(\Omega)$ such that $\Theta_{|\Gamma} = \theta_e$. Observe that $\Theta/\mathfrak{w} \in \mathrm{W}^1_s(\Omega)$ for any $s \in [1, \frac{4p}{p+4})$ since $\partial\Theta/\mathfrak{w}, \bar\partial\Theta/\mathfrak{w} \in \mathrm{L}^s(\Omega)$ by Hölder inequality and $\Theta/\mathfrak{w}^3 \in \mathrm{L}^s(\Omega)$ by the estimate

$$\left|\frac{z\Theta(z)}{\mathfrak{w}^3(z)}\right| \le \text{const.}|z^2-1|^{-2/p-1/2}, \quad z \in \overline{\Omega},$$

where we used the definition of Θ and (4.1). Thus, Cauchy-Green formula (5.8) applied to $\Theta/\mathfrak{w}$ implies that

$$\partial d_1 = \partial\left(\Theta - \mathfrak{w}\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right) - \mathfrak{w}\ell_2\right) = \partial\Theta - \partial\left(\mathfrak{w}\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right)\right) - \mathfrak{w}\ell_2' - \frac{z\ell_2}{\mathfrak{w}}$$

a.e. in Ω, where we used that $\bar\partial\mathfrak{w} = 0$. Since $\partial\Theta \in \mathrm{L}^p(\Omega)$ and $\mathfrak{w}\ell_2'$ is bounded, it is only necessary to show that

$$\begin{aligned}\partial\left(\mathfrak{w}\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right)\right) - \frac{z\ell_2}{\mathfrak{w}} &= \frac{z}{\mathfrak{w}}\left(\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right) - \ell_2\right) + \mathfrak{w}\partial\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right)\\ &= \frac{z}{\mathfrak{w}}\left(\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right) - \ell_2\right) + \mathfrak{w}\mathscr{B}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right)\end{aligned}$$

belongs to $\mathrm{L}^q(\Omega)$, $q \in (2,p)$, where we used (5.5). The fact that $\mathfrak{w}\mathscr{B}(\bar\partial\Theta/\mathfrak{w}) \in \mathrm{L}^p(\Omega)$ follows from (5.7). Now, to show that $(1/\mathfrak{w})\left(\mathscr{K}(\bar\partial\Theta/\mathfrak{w}) - \ell_2\right) \in \mathrm{L}^q(\Omega)$, $q \in (2,p)$, recall that $\bar\partial\Theta/\mathfrak{w} \in \mathrm{L}^s(\Omega)$ for any $s \in [1, \frac{4p}{p+4})$. So, as mentioned after (5.6), $\mathscr{K}(\bar\partial\Theta/\mathfrak{w}) \in \mathrm{L}^{2s/2-s}(\Omega)$ when $p \in (2,4]$, i.e., $s \in (\frac{4}{3}, 2)$; and $\mathscr{K}(\bar\partial\Theta/\mathfrak{w}) \in \mathrm{C}^{0,1-2/s}(\Omega)$ when $p \in (4,\infty)$, i.e., s can be chosen to lie in $\left(2, \frac{4p}{p+4}\right)$. In the first case, we get that $(1/\mathfrak{w})\mathscr{K}(\bar\partial\Theta/\mathfrak{w}) \in \mathrm{L}^q(\Omega)$, $q \in (2,p)$, simply by applying Hölder inequality once more. This shows that $\partial d_1 \in \mathrm{L}^q(\Omega)$, $q \in (2,p)$, when $p \in (2,4]$ with $\ell_2 \equiv 0$. In the second case, let ℓ_2 be the polynomial interpolating $\mathscr{K}(\bar\partial\Theta/\mathfrak{w})$ at ± 1, $\deg(\ell_2) \le 1$. Then

$$\left|\frac{z}{\mathfrak{w}(z)}\left(\mathscr{K}\left(\frac{\bar\partial\Theta}{\mathfrak{w}}\right) - \ell_2\right)(z)\right| \le \text{const.}|z^2-1|^{(s-4)/2s}, \quad z \in \overline{\Omega}.$$

Since $\frac{s-4}{2s} \in \left(-\frac{1}{2}, -\frac{2}{p}\right)$, it holds that $(1/\mathfrak{w})\left(\mathscr{K}(\bar\partial\Theta/\mathfrak{w}) - \ell_2\right) \in \mathrm{L}^q(\Omega)$, $q \in (2,p)$, which shows that $\partial d_1 \in \mathrm{L}^q(\Omega)$, $q \in (2,p)$, when $p \in (4,\infty)$.

It only remains to show that $d_1(\pm 1) = 0$. As $\theta_e \in \mathrm{C}^{1-2/p}(\Gamma)$ by (2.7) and $\theta_e(\pm 1) = 0$, the function $\mathscr{C}^+(\theta_e/\mathfrak{w}^+)$ is either bounded near ± 1 or blows up there with the order strictly less than $1/2$ [24, Sec. I.8.4], see also (5.9). Thus, $\mathfrak{w}^+\mathscr{C}^+(\theta_e/\mathfrak{w}^+)$ vanishes at ± 1 and so does d_1. □

We continue with the case of functions in $\mathrm{C}^{m,\varsigma}$.

Proposition 9. *Let $\theta \in \mathrm{C}^{m,\varsigma}$, $m \in \mathbb{Z}_+$, $\varsigma \in (0,1]$. Then*

$$\mathfrak{w}^\pm\mathscr{S}(\theta/\mathfrak{w}^+) = \pm d + \mathfrak{w}^\pm\ell, \quad d^{(k)}(\pm 1) = 0, \quad k \in \{0,\dots,m\},$$

where ℓ is a polynomial, $\deg(\ell) \le 2m+1$, and $d \in \mathrm{C}^{m,\varsigma}$ when $\varsigma \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, while $d \in \mathrm{C}^{m,\varsigma-\epsilon}$ for arbitrarily small $\epsilon > 0$ when $\varsigma = \frac{1}{2}, 1$.

When $\varsigma \in \left(0, \frac{1}{2}\right)$, the conclusion of the proposition follows from [21, Thm. 3]. Therefore, we are required[4] to prove Proposition 9 only for $\varsigma \in \left[\frac{1}{2}, 1\right]$. To this end, we shall need several geometrical lemmas. In all of them we assume that Γ is as in Proposition 8 and we omit superscript $+$ for $\mathfrak{w}$ when dealing with the values of $\mathfrak{w}$ on Δ. By C_Γ we shall denote a constant such that $|\tau| \le C_\Gamma$, $\tau \in \Gamma$. Moreover, τ_1 and τ_2 will stand for two different points on Γ satisfying

$$|1-\tau_1^2| \le |1-\tau_2^2|. \tag{5.16}$$

Lemma 10. *Let $N \in \mathbb{N}$ or $N = -1$. Then*

$$\left|\frac{1}{\mathfrak{w}^N(\tau_1)} - \frac{1}{\mathfrak{w}^N(\tau_2)}\right| \le C_1|N| \max\left\{\frac{1}{|1-\tau_1^2|^{1+N/2}}, \frac{1}{|1-\tau_2^2|^{1+N/2}}\right\}|\tau_1-\tau_2|, \tag{5.17}$$

where C_1 is a constant depending only on Γ.

Proof. If N is an even integer, then

$$\begin{aligned}\left|\frac{1}{\mathfrak{w}^N(\tau_1)} - \frac{1}{\mathfrak{w}^N(\tau_2)}\right| &= \frac{1}{|1-\tau_1^2|^{N/2}}\left|1-\left(\frac{1-\tau_1^2}{1-\tau_2^2}\right)^{N/2}\right| \\ &= \frac{|\tau_1^2-\tau_2^2|}{|1-\tau_1^2|^{N/2}|1-\tau_2^2|}\left|\sum_{j=0}^{N/2-1}\left(\frac{1-\tau_1^2}{1-\tau_2^2}\right)^j\right| \le \frac{NC_\Gamma|\tau_1-\tau_2|}{|1-\tau_1^2|^{1+N/2}}\end{aligned}$$

by (5.16). If N is an odd integer, then

$$\begin{aligned}\left|\frac{1}{\mathfrak{w}^N(\tau_1)} - \frac{1}{\mathfrak{w}^N(\tau_2)}\right| &= \frac{1}{|1-\tau_1^2|^{N/2}|1-\tau_2^2|^{1/2}}\left|\mathfrak{w}(\tau_2)-\mathfrak{w}(\tau_1)\left(\frac{1-\tau_1^2}{1-\tau_2^2}\right)^{(N-1)/2}\right| \\ &= \frac{|\mathfrak{w}(\tau_1)-\mathfrak{w}(\tau_2)|}{|1-\tau_1^2|^{N/2}|1-\tau_2^2|^{1/2}} + \frac{(N-1)C_\Gamma|\tau_1-\tau_2|}{|1-\tau_1^2|^{N/2}|1-\tau_2^2|}\end{aligned}$$

by the first estimate and (5.16). Clearly, it only remains to prove the lemma for $N = -1$. It can be readily verified that it is enough to consider $|\tau_1 - \tau_2|$ small enough. As $\mathfrak{w}$ is zero free in Ω, interior of Γ, an argument function, say a, is well-defined and continuous in Ω. Since $\mathfrak{w}$ extends holomorphically across Δ° and $\Gamma \setminus \Delta$, the trace of a is uniformly continuous on any compact subset of $\Gamma \setminus \{\pm 1\}$. Moreover, it also has one-sided limits at ± 1 with the jumps of magnitude $\pi/2$. Thus, there exists $\delta > 0$ such that for all $|\tau_1 - \tau_2| < \delta$ it holds that $|a(\tau_1) - a(\tau_2)| < \frac{2\pi}{3}$. Then

$$|\mathfrak{w}(\tau_1)+\mathfrak{w}(\tau_2)| \ge |\mathfrak{w}(\tau_2)| - \frac{1}{2}|\mathfrak{w}(\tau_1)| \ge \frac{1}{2}|\mathfrak{w}(\tau_2)|$$

for $|\tau_1 - \tau_2| < \delta$ by (5.16) for the last inequality and therefore

$$|\mathfrak{w}(\tau_1)-\mathfrak{w}(\tau_2)| \le 2C_\Gamma\frac{|\tau_1-\tau_2|}{|\mathfrak{w}(\tau_1)+\mathfrak{w}(\tau_2)|} \le 4C_\Gamma\frac{|\tau_1-\tau_2|}{|1-\tau_2^2|^{1/2}},$$

which finishes the proof of the lemma. □

Lemma 11. *Let $\varrho \in \mathrm{C}^{0,\varsigma}(\Gamma)$, $\varsigma \in \left(\frac{1}{2}, 1\right)$, and $\varrho(\pm 1) = 0$. Then $(\varrho/\mathfrak{w}) \in \mathrm{C}^{0,\varsigma-1/2}(\Gamma)$ and*

$$\left|\frac{\varrho(\tau_1)}{\mathfrak{w}(\tau_1)} - \frac{\varrho(\tau_2)}{\mathfrak{w}(\tau_2)}\right| \le C_2 \min\left\{\frac{1}{|1-\tau_1^2|^{1/2}}, \frac{1}{|1-\tau_2^2|^{1/2}}\right\}|\tau_1-\tau_2|^\varsigma, \tag{5.18}$$

[4]The authors were surprised not to find this case in the literature.

where C_2 is a constant depending only on Γ.

Proof. We start by proving (5.18). By the condition of the lemma it holds that

$$\begin{cases} |\varrho(\tau_1)-\varrho(\tau_2)| \le M|\tau_1-\tau_2|^{\varsigma} \\ |\varrho(\tau)| \le M|1-\tau^2|^{\varsigma}, \end{cases} \qquad \tau_1,\tau_2,\tau\in\Gamma, \tag{5.19}$$

for some finite constant M. Set, for brevity, $\varkappa := \varrho/\mathfrak{w}$. First, let $\tau_1 = 1$. Observe that

$$|\varkappa(\tau)| \le M|1-\tau^2|^{\varsigma-1/2} \le M(1+C_\Gamma)\frac{|1-\tau|^{\varsigma}}{|1-\tau^2|^{1/2}}$$

by (5.19). Thus, $\varkappa(1)=0$ by continuity and it holds that

$$|\varkappa(\tau_1)-\varkappa(\tau_2)| \le M(1+C_\Gamma)\frac{|\tau_1-\tau_2|^{\varsigma}}{|1-\tau_2^2|^{1/2}}. \tag{5.20}$$

Clearly, an analogous bound holds when $\tau_1=-1$.

Second, let $|\tau_1-\tau_2| \ge |1-\tau_1^2|$. In this case, we also have that

$$|1-\tau_2^2| \le |\tau_1^2-\tau_2^2| + |1-\tau_1^2| \le C_\Gamma^*|\tau_1-\tau_2|, \quad C_\Gamma^* := 1+2C_\Gamma. \tag{5.21}$$

Then it follows from (5.19), (5.16), and (5.21) that

$$\begin{aligned} |\varkappa(\tau_1)-\varkappa(\tau_2)| &\le |\varkappa(\tau_1)|+|\varkappa(\tau_2)| \le M\left(|1-\tau_1^2|^{\varsigma-1/2}+|1-\tau_2^2|^{\varsigma-1/2}\right) \\ &\le 2M|1-\tau_2^2|^{\varsigma-1/2} \le 2M(C_\Gamma^*)^{\varsigma}\frac{|\tau_1-\tau_2|^{\varsigma}}{|1-\tau_2^2|^{1/2}}. \end{aligned} \tag{5.22}$$

Third, let $|\tau_1-\tau_2| \le |1-\tau_1^2|$. Then, it also holds that

$$|1-\tau_2^2| \le |1-\tau_1^2| + |\tau_1^2-\tau_2^2| \le C_\Gamma^*|1-\tau_1^2|. \tag{5.23}$$

Thus, (5.19) and Lemma 10 imply that

$$\begin{aligned} |\varkappa(\tau_1)-\varkappa(\tau_2)| &\le |\varrho(\tau_1)|\left|\frac{1}{\mathfrak{w}(\tau_1)}-\frac{1}{\mathfrak{w}(\tau_2)}\right| + \frac{|\varrho(\tau_1)-\varrho(\tau_2)|}{|\mathfrak{w}(\tau_2)|} \\ &\le M|1-\tau_1^2|^{\varsigma}C_1\frac{|\tau_1-\tau_2|}{|1-\tau_1^2|^{3/2}} + M\frac{|\tau_1-\tau_2|^{\varsigma}}{|1-\tau_2^2|^{1/2}}. \end{aligned}$$

Using the conditions $|\tau_1-\tau_2| \le |1-\tau_1^2|$ and (5.23), we obtain that

$$\begin{aligned} |\varkappa(\tau_1)-\varkappa(\tau_2)| &\le M|\tau_1-\tau_2|^{\varsigma}\left(C_1\frac{|\tau_1-\tau_2|^{1-\varsigma}}{|1-\tau_1^2|^{1-\varsigma}}\frac{1}{|1-\tau_1^2|^{1/2}} + \frac{1}{|1-\tau_2^2|^{1/2}}\right) \\ &\le M(C_1\sqrt{C_\Gamma^*}+1)\frac{|\tau_1-\tau_2|^{\varsigma}}{|1-\tau_2^2|^{1/2}}. \end{aligned} \tag{5.24}$$

This finishes the proof of (5.18).

Finally, it can be readily verified that the equations leading to (5.20), (5.22), and (5.24) also yield that $\varkappa \in C^{0,\varsigma-1/2}(\Gamma)$ and hence, the lemma is proved. □

Lemma 12. *Let ϱ be as in Lemma 11. Then $\mathscr{S}_\Gamma(\varrho/\mathfrak{w}) \in C^{0,\varsigma-1/2}(\Gamma)$ and*

$$\left|\mathscr{S}_\Gamma\left(\frac{\varrho}{\mathfrak{w}}\right)(\tau_1) - \mathscr{S}_\Gamma\left(\frac{\varrho}{\mathfrak{w}}\right)(\tau_2)\right| \le C_3 \min\left\{\frac{1}{|1-\tau_1^2|^{1/2}}, \frac{1}{|1-\tau_2^2|^{1/2}}\right\}|\tau_1-\tau_2|^{\varsigma}, \tag{5.25}$$

where C_3 is a constant depending only on Γ.

Proof. Since $\varkappa := \varrho/\mathfrak{w} \in \mathrm{C}^{0,\varsigma-1/2}(\Gamma)$, $\mathscr{S}_\Gamma \varkappa \in \mathrm{C}^{0,\varsigma-1/2}(\Gamma)$ as well by [24, Sec. I.5.1]. To prove (5.25), one needs to trace the local character of the proof in [24, Sec. I.5.1]. This is a tedious job but the authors felt compelled to carry it out for the reader.

Define

$$\mathscr{S}(\tau) := \mathscr{S}_\Gamma \varkappa(\tau) - \varkappa(\tau) = \frac{1}{\pi i}\int_\Gamma \frac{\varkappa(t)-\varkappa(\tau)}{t-\tau}\,dt, \quad \tau \in \Gamma.$$

In the light of (5.18), it is enough to show (5.25) with $\mathscr{S}_\Gamma \varkappa$ replaced by $\mathscr{S}$. Observe also that the integral that defines $\mathscr{S}$ is no longer singular as $\varkappa(\pm 1) = 0$.

Denote by Γ^* the connected component of $\Gamma \cap \{\tau : |\tau_1 - \tau| \le 2|\tau_1 - \tau_2|\}$ that contains τ_1. We order τ_1 and τ_2 so that (5.16) holds. Then we can write

$$\begin{aligned}
\mathscr{S}(\tau_2) - \mathscr{S}(\tau_1) &= \frac{1}{\pi i}\int_{\Gamma^*} \frac{\varkappa(\tau)-\varkappa(\tau_2)}{\tau-\tau_2}\,d\tau - \frac{1}{\pi i}\int_{\Gamma^*} \frac{\varkappa(\tau)-\varkappa(\tau_1)}{\tau-\tau_1}\,d\tau \\
&\quad + \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma^*} \frac{\varkappa(\tau_1)-\varkappa(\tau_2)}{\tau-\tau_1}\,d\tau + \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma^*} \frac{(\tau_2-\tau_1)(\varkappa(\tau)-\varkappa(\tau_2))}{(\tau-\tau_1)(\tau-\tau_2)}\,d\tau \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Before we continue, observe that there exists a finite constant M such that

$$|\Gamma(t,\tau)| \le M|t-\tau|, \quad t,\tau \in \Gamma, \tag{5.26}$$

since Γ is a smooth Jordan curve, where $|\Gamma(t,\tau)|$ is the arclength of the smallest subarc of Γ connecting t and τ.

First, let us estimate I_1. We get from (5.18) and (5.26) that

$$\begin{aligned}
|I_1| &\le \frac{C_2}{\pi}\sup_{\tau\in\Gamma^*}\min\left\{\frac{1}{|1-\tau^2|^{1/2}}, \frac{1}{|1-\tau_2^2|^{1/2}}\right\}\int_{\Gamma^*}\frac{|d\tau|}{|\tau-\tau_2|^{1-\varsigma}} \\
&\le \frac{C_2 M^{1-\varsigma}}{\pi|1-\tau_2^2|^{1/2}}\int_0^{4M|\tau_1-\tau_2|}\frac{ds}{s^{1-\varsigma}} \le \frac{C_2 4^\varsigma M}{\varsigma\pi}\frac{|\tau_1-\tau_2|^\varsigma}{|1-\tau_2^2|^{1/2}}.
\end{aligned} \tag{5.27}$$

Clearly, an analogous estimate can be made for I_2.

Now, we shall estimate I_3. It holds that

$$|I_3| = \left|(\varkappa(\tau_1)-\varkappa(\tau_2))\log\left(\frac{\tau_a-\tau_1}{\tau_b-\tau_1}\right)\right|,$$

where τ_a and τ_b are the endpoints of Γ^*. As $|\tau_a-\tau_1| = |\tau_b-\tau_1|$, we have that

$$|I_3| \le \text{const.}\frac{|\tau_1-\tau_2|^\varsigma}{|1-\tau_2^2|^{1/2}}, \tag{5.28}$$

where const. is the product of C_2 and the maximum of the argument of $\frac{\tau_a-\tau_1}{\tau_b-\tau_1}$ for all possible choices of τ_a and τ_b.

Finally, let us estimate I_4. Observe that $|\tau-\tau_1| \le 2|\tau-\tau_2|$, $\tau \in \Gamma\setminus\Gamma^*$, since

$$|\tau-\tau_1| \le |\tau-\tau_2| + |\tau_1-\tau_2| \le |\tau-\tau_2| + \frac{1}{2}|\tau-\tau_1|.$$

Then we get from (5.18) and the bound above that

$$\begin{aligned}
|I_4| &\le \frac{C_2}{\pi}\max_{\tau\in\Gamma\setminus\Gamma^*}\min\left\{\frac{1}{|1-\tau^2|^{1/2}},\frac{1}{|1-\tau_2^2|^{1/2}}\right\}\int_{\Gamma\setminus\Gamma^*}\frac{|\tau_2-\tau_1||d\tau|}{|\tau-\tau_1||\tau-\tau_2|^{1-\varsigma}}\\
&= \frac{C_2}{\pi}\frac{|\tau_2-\tau_1|}{|1-\tau_2^2|^{1/2}}\int_{\Gamma\setminus\Gamma^*}\left|\frac{\tau-\tau_1}{\tau-\tau_2}\right|^{1-\varsigma}\frac{|d\tau|}{|\tau-\tau_1|^{2-\varsigma}}\\
&\le \frac{C_2 2^{1-\varsigma}M^{2-\varsigma}}{\pi}\frac{|\tau_2-\tau_1|}{|1-\tau_2^2|^{1/2}}\int_{2|\tau_1-\tau_2|}^{\infty}\frac{ds}{s^{2-\varsigma}}=\frac{C_2M^{2-\varsigma}}{(1-\varsigma)\pi}\frac{|\tau_2-\tau_1|^{\varsigma}}{|1-\tau_2^2|^{1/2}}. \qquad (5.29)
\end{aligned}$$

Combining (5.27), (5.28), and (5.29) with (5.18), we see that (5.25) holds. □

Lemma 13. *Let $\varrho\in C^{0,\upsilon}$, $\upsilon\in\left(0,\frac{1}{2}\right)$, $\varrho(\pm1)=0$, be such that*

$$|\varrho(\tau_1)-\varrho(\tau_2)|\le C_4\min\left\{\frac{1}{|1-\tau_1^2|^{1/2}},\frac{1}{|1-\tau_2^2|^{1/2}}\right\}|\tau_1-\tau_2|^{\upsilon+1/2},$$

$\tau_1,\tau_2\in\Gamma$, $\tau_1\ne\tau_2$, where C_4 is a constant depending only on Γ. Then $\mathfrak{w}\varrho\in C^{0,\upsilon+1/2}(\Gamma)$. Further, let $\varrho\in C^{N,\upsilon}(\Gamma)$, $N\in\mathbb{N}$, $\varrho^{(j)}(\pm1)=0$, $j\in\{0,\dots,N\}$. Then

$$\begin{cases}(\varrho/\mathfrak{w}^{2N-1})\in C^{0,\upsilon+1/2}(\Gamma) & \text{if } \upsilon\in\left(0,\frac{1}{2}\right),\\ (\varrho/\mathfrak{w}^{2N+1})\in C^{0,\upsilon-1/2}(\Gamma) & \text{if } \upsilon\in\left(\frac{1}{2},1\right).\end{cases} \qquad (5.30)$$

Proof. To verify the first claim, assume first that $|\tau_1-\tau_2|\ge|1-\tau_1^2|$. Since $\varrho\in C^{0,\upsilon}(\Gamma)$ and vanishes at ±1, it holds for some finite constant M that

$$|\varrho(\tau)|\le M|1-\tau^2|^{\upsilon}. \qquad (5.31)$$

Then we get from (5.21) and the inequality above that

$$|(\mathfrak{w}\varrho)(\tau_1)-(\mathfrak{w}\varrho)(\tau_2)|\le M\left(|1-\tau_1^2|^{\varsigma}+|1-\tau_2^2|^{\varsigma}\right)\le M(1+C_\Gamma^*)|\tau_1-\tau_2|^{\varsigma}. \qquad (5.32)$$

Assume now that $|\tau_1-\tau_2|\le|1-\tau_1^2|$. Then we get by (5.31), (5.17), and the conditions of the lemma that

$$\begin{aligned}
|(\mathfrak{w}\varrho)(\tau_1)-(\mathfrak{w}\varrho)(\tau_2)| &\le |\mathfrak{w}(\tau_2)|\,|\varrho(\tau_1)-\varrho(\tau_2)|+|\varrho(\tau_1)||\mathfrak{w}(\tau_1)-\mathfrak{w}(\tau_2)|\\
&\le |1-\tau_2^2|^{1/2}C_4\frac{|\tau_1-\tau_2|^{\upsilon+1/2}}{|1-\tau_2^2|^{1/2}}+M|1-\tau_1^2|^{\upsilon}C_1\frac{|\tau_1-\tau_2|}{|1-\tau_1^2|^{1/2}}\\
&\le (C_4+C_1M)|\tau_1-\tau_2|^{\upsilon+1/2}. \qquad (5.33)
\end{aligned}$$

Equations (5.32) and (5.33) show that $(\mathfrak{w}\varrho)\in C^{0,\upsilon+1/2}(\Gamma)$.

It remains to prove (5.30). Suppose first that $\upsilon\in\left(0,\frac{1}{2}\right)$. Then, by the assumptions on ϱ, it holds for some finite constant M that

$$\begin{cases}\left|\varrho(\tau_2)-\sum_{j=1}^{N}\varrho^{(j)}(\tau_1)(\tau_2-\tau_1)^j\right|\le M|\tau_2-\tau_1|^{N+\upsilon},\\ \left|\varrho^{(j)}(\tau)\right|\le M|1-\tau^2|^{N-j+\upsilon},\end{cases}\qquad \tau_1,\tau_2,\tau\in\Gamma. \qquad (5.34)$$

Thus, for $|\tau_1-\tau_2|\geq|1-\tau_1^2|$, we have from (5.34) and (5.21) that

$$\left|\frac{\varrho(\tau_1)}{\mathfrak{w}^{2N-1}(\tau_1)}-\frac{\varrho(\tau_2)}{\mathfrak{w}^{2N-1}(\tau_2)}\right|\leq\left|\frac{\varrho(\tau_1)}{\mathfrak{w}^{2N-1}(\tau_1)}\right|+\left|\frac{\varrho(\tau_2)}{\mathfrak{w}^{2N-1}(\tau_2)}\right|\leq M(1+(C_\Gamma^*)^\varsigma)|\tau_1-\tau_2|^{\upsilon+1/2}.$$

For $|\tau_1-\tau_2|\leq|1-\tau_1^2|$, we have from (5.34) that

$$\begin{aligned}\left|\frac{\varrho(\tau_1)}{\mathfrak{w}^{2N-1}(\tau_1)}-\frac{\varrho(\tau_2)}{\mathfrak{w}^{2N-1}(\tau_2)}\right| &= \left|\frac{\varrho(\tau_1)}{\mathfrak{w}^{2N-1}(\tau_1)}\pm\sum_{j=0}^{N}\frac{\varrho^{(j)}(\tau_1)(\tau_2-\tau_1)^j}{\mathfrak{w}^{2N-1}(\tau_2)}-\frac{\varrho(\tau_2)}{\mathfrak{w}^{2N-1}(\tau_2)}\right|\\ &\leq \left|\frac{\varrho(\tau_1)}{\mathfrak{w}^{2N-1}(\tau_1)}-\sum_{j=0}^{N}\frac{\varrho^{(j)}(\tau_1)(\tau_2-\tau_1)^j}{\mathfrak{w}^{2N-1}(\tau_2)}\right|+\frac{M|\tau_2-\tau_1|^{N+\upsilon}}{|1-\tau_2^2|^{N-1/2}}\\ &\leq I+M|\tau_2-\tau_1|^{\upsilon+1/2}.\end{aligned}$$

Furthermore, it holds by (5.17) and (5.34) that

$$\begin{aligned}I &\leq \sum_{j=0}^{N}\left|\varrho^{(j)}(\tau_1)(\tau_2-\tau_1)^j\left(\frac{1}{\mathfrak{w}^{2N-1}(\tau_1)}-\frac{1}{\mathfrak{w}^{2N-1}(\tau_2)}\right)\right|+\sum_{j=1}^{N}\left|\frac{\varrho^{(j)}(\tau_1)(\tau_2-\tau_1)^j}{\mathfrak{w}^{2N-1}(\tau_1)}\right|\\ &\leq (2N-1)C_1M\sum_{j=0}^{N}\frac{|1-\tau_1^2|^{N-j+\upsilon}|\tau_2-\tau_1|^{j+1}}{|1-\tau_1^2|^{N+1/2}}+M\sum_{j=1}^{k}\frac{|1-\tau_1^2|^{N-j+\upsilon}|\tau_2-\tau_1|^{j}}{|1-\tau_1^2|^{N-1/2}}\\ &\leq 2NC_1M\sum_{j=1}^{N+1}\left|\frac{\tau_2-\tau_1}{1-\tau_1^2}\right|^{j-\upsilon-1/2}|\tau_2-\tau_1|^{\upsilon+1/2}\leq 2N^2C_1M|\tau_2-\tau_1|^{\upsilon+1/2}.\end{aligned}$$

Clearly, the case $\upsilon\in\left(\frac{1}{2},1\right)$ can be handled in a similar fashion. □

Proof of Proposition 9. Clearly, we need to prove the proposition only for $\varsigma\neq\frac{1}{2},1$ as these two cases follow from the obvious inclusion $\mathrm{C}^{m,\varsigma-\epsilon}\subset\mathrm{C}^{m,\varsigma}$.

Let ℓ_0, $\deg(\ell_0)\leq 2m+1$, be the polynomial interpolating θ at ±1 up to and including the order m. Throughout the proof we assume that Γ is as in Proposition 8 and that θ is extended to $\Gamma\setminus\Delta$ by ℓ_0. Clearly, this implies that $\theta\in\mathrm{C}^{m,\varsigma}(\Gamma)$. As $\ell_1:=\mathscr{S}(\ell_0/\mathfrak{w}^+)$ is a polynomial of degree $2m$, we have that

$$\mathscr{S}\left(\frac{\theta}{\mathfrak{w}}\right)=\mathscr{S}\left(\frac{\theta-\ell_0}{\mathfrak{w}}\right)+\ell_1=\mathscr{S}_\Gamma\left(\frac{\theta-\ell_0}{\mathfrak{w}}\right)_{|\Delta}+\ell_1.$$

Assume first that $m=0$. Then $\theta-\ell_0$ satisfies the conditions of ϱ in Lemma 11 and therefore Lemma 12 holds with $\theta-\ell_0$ in place of ϱ. Let ℓ_2 be the polynomial interpolating $\mathscr{S}_\Gamma((\theta-\ell_0)/\mathfrak{w})$ at ±1 and set

$$d:=\mathfrak{w}\left(\mathscr{S}_\Gamma\left(\frac{\theta-\ell_0}{\mathfrak{w}}\right)-\ell_2\right)\quad\text{and}\quad\ell:=\ell_1+\ell_2. \tag{5.35}$$

Clearly, $d(\pm1)=0$. Then the conclusion of the proposition follows from Lemma 13 applied with $\varrho=\mathscr{S}_\Gamma((\theta-\ell_0)/\mathfrak{w})-\ell_2$.

Assume now that $m\in\mathbb{N}$. Since the derivative of a singular integral is the singular integral of the derivative [24, Sec. I.4.4], observe that

$$\mathscr{S}_\Gamma^{(m)}\left(\frac{\theta-\ell_0}{\mathfrak{w}}\right)=\mathscr{S}_\Gamma\left(\left(\frac{\theta-\ell_0}{\mathfrak{w}}\right)^{(m)}\right)=\sum_{j=0}^{m}\binom{m}{j}\mathscr{S}_\Gamma\left(\frac{v_j(\theta-\ell_0)^{(m-j)}}{\mathfrak{w}^{2j+1}}\right),$$

where v_j are polynomials. Then $\mathscr{S}_\Gamma((\theta-\ell_0)/\mathfrak{w}) \in C^{m,\varsigma-1/2}(\Gamma)$ by (5.30), applied with $\varrho = v_j(\theta-\ell_0)^{(m-j)}$, $N=j$, and $\upsilon=\varsigma$, and the fact that singular integrals preserve Hölder smoothness [24, Sec. I.5.1]. Thus, $\mathscr{S}_\Gamma((\theta-\ell_0)/\mathfrak{w})$ has m continuous derivatives on Γ. Let then d and ℓ be defined by (5.35), where ℓ_2 is the polynomial interpolating $\mathscr{S}_\Gamma((\theta-\ell_0)/\mathfrak{w})$ at ± 1 up to and including the order m. Once more, since singular integral commutes with differentiation, we get

$$\begin{aligned} d^{(m)} &= \sum_{k=0}^{m} \binom{m}{k} \frac{u_k}{\mathfrak{w}^{2k-1}} \left(\mathscr{S}_\Gamma^{(m-k)} \left(\frac{\theta-\ell_0}{\mathfrak{w}} \right) - \ell_2^{(m-k)} \right) \\ &= \sum_{k=0}^{m} \sum_{j=0}^{m-k} \binom{m}{k} \binom{m-k}{j} \frac{u_k}{\mathfrak{w}^{2k-1}} \left(\mathscr{S}_\Gamma \left(\frac{v_j(\theta-\ell_0)^{(m-k-j)}}{\mathfrak{w}^{2j+1}} \right) - \ell_{j,k} \right), \end{aligned}$$

where u_k are polynomials and the polynomials $\ell_{j,k}$ interpolate the corresponding term in the parenthesis. Then

$$\mathscr{S}_\Gamma^{(k)} \left(\frac{v_j(\theta-\ell_0)^{(m-k-j)}}{\mathfrak{w}^{2j+1}} \right) = \sum_{l=0}^{k} \binom{k}{l} \mathscr{S}_\Gamma \left(\frac{v_{j+l}(\theta-\ell_0)^{(m-j-l)}}{\mathfrak{w}^{2(j+l)+1}} \right) \in C^{0,\varsigma-1/2}(\Gamma),$$

by (5.30), applied with $\varrho = v_{j+l}(\theta-\ell_0)^{(m-j-l)}$, $N=j+l$, $\upsilon=\varsigma$, and the fact that singular integrals preserve Hölder smoothness. Thus, (5.30) applied once more, now with $\varrho = \mathscr{S}_\Gamma(v_j(\theta-\ell_0)^{(m-k-j)}/\mathfrak{w}^{2j+1}) - \ell_{j,k}$, $N=k$, and, $\upsilon=\varsigma-1/2$, yields that $d^{(m)} \in C^{0,\varsigma}(\Gamma)$, which finishes the proof of the lemma. □

6. Riemann-Hilbert-$\bar\partial$ Problem

In what follows, we adopt the notation $\phi^{m\sigma_3}$ for the diagonal matrix $\begin{pmatrix} \phi^m & 0 \\ 0 & \phi^{-m} \end{pmatrix}$, where ϕ is a function, m is a constant, and σ_3 is the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

6.1. **Initial Riemann-Hilbert Problem.** Let $\mathscr{Y}$ be a 2×2 matrix function and w_n be given by (2.16). Consider the following Riemann-Hilbert problem for $\mathscr{Y}$ (RHP-$\mathscr{Y}$):

(a) $\mathscr{Y}$ is analytic in $\mathbb{C}\setminus\Delta$ and $\lim_{z\to\infty} \mathscr{Y}(z)z^{-n\sigma_3} = \mathscr{I}$, where $\mathscr{I}$ is the identity matrix;

(b) $\mathscr{Y}$ has continuous traces from each side of Δ°, $\mathscr{Y}_\pm$, and $\mathscr{Y}_+ = \mathscr{Y}_- \begin{pmatrix} 1 & 2w_n \\ 0 & 1 \end{pmatrix}$;

(c) $\mathscr{Y}$ has the following behavior near $z=1$:

$$\mathscr{Y} = \begin{cases} O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix}, & \text{if} \quad \alpha<0, \\ O\begin{pmatrix} 1 & \log|1-z| \\ 1 & \log|1-z| \end{pmatrix}, & \text{if} \quad \alpha=0, \qquad \text{as} \quad D\ni z\to 1; \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if} \quad \alpha>0, \end{cases}$$

(d) $\mathscr{Y}$ has the same behavior when $D\ni z\to -1$ as in (c) only with α replaced by β and $1-z$ replaced by $1+z$.

The connection between RHP-$\mathscr{Y}$ and polynomials orthogonal with respect to w_n was first realized by Fokas, Its, and Kitaev [22, 23] and lies in the following.

Lemma 14. *Let q_n be a polynomial satisfying orthogonality relations (2.14) and R_n be the corresponding function of the second kind given by (2.15). Further, let q^*_{n-1} be a polynomial satisfying*

$$\int_\Delta t^j q^*_{n-1}(t) w_n(t)\,dt = 0, \quad j \in \{0,\dots,n-2\},$$

and $R^*_{n-1} = R_{n-1}(q^*_{n-1};\cdot)$ be the function of the second kind for q^*_{n-1}. If a solution of RHP-$\mathscr{Y}$ *exists then it is unique. Moreover, in this case* $\deg(q_n) = n$, *$R^*_{n-1}(z) = O(z^{-n})$ as $z \to \infty$, and the solution of* RHP-$\mathscr{Y}$ *is given by*

$$\mathscr{Y} = \begin{pmatrix} q_n & R_n \\ m_n q^*_{n-1} & m_n R^*_{n-1} \end{pmatrix}, \tag{6.1}$$

where m_n is a constant such that $m_n R^*_{n-1}(z) = z^{-n}[1+o(1)]$ near infinity. Conversely, if $\deg(q_n) = n$ *and $R^*_{n-1}(z) = O(z^{-n})$ as $z \to \infty$, then $\mathscr{Y}$ defined in (6.1) solves* RHP-$\mathscr{Y}$.

Proof. As only the smoothness properties of the function w_n were used in [33, Lem. 2.3], this lemma translates without change to the case of a general closed analytic arc and yields the uniqueness of the solution of RHP-$\mathscr{Y}$ whenever the latter exists.

Suppose now that the solution, say $\mathscr{Y} = [\mathscr{Y}_{jk}]_{j,k=1}^2$, of RHP-$\mathscr{Y}$ exists. Then $\mathscr{Y}_{11} = z^n +$ lower order terms by the normalization in RHP-$\mathscr{Y}$(a). Moreover, by RHP-$\mathscr{Y}$(b), $\mathscr{Y}_{11}$ has no jump on Δ and hence is holomorphic in the whole complex plane. Thus, $\mathscr{Y}_{11}$ is necessarily a polynomial of degree n by Liouville's theorem. Further, since $\mathscr{Y}_{12} = O(z^{-n-1})$ and satisfies RHP-$\mathscr{Y}$(b), it holds that $\mathscr{Y}_{12} = 2\mathscr{C}(\mathscr{Y}_{11} w_n)$. From the latter, we easily deduce that $\mathscr{Y}_{11}$ satisfies orthogonality relations (2.14). Applying the same arguments to the second row of $\mathscr{Y}$, we obtain that $\mathscr{Y}_{21} = q^*_{n-1}$ and $\mathscr{Y}_{22} = m_n R^*_{n-1}$ with m_n well-defined.

Conversely, let $\deg(q_n) = n$ and $R^*_{n-1}(z) = O(z^{-n})$ as $z \to \infty$. Then it can be easily checked by direct examination of RHP-$\mathscr{Y}$(a)-(d), using the material in Section 5.2, that $\mathscr{Y}$, given by (6.1), solves RHP-$\mathscr{Y}$. □

6.2. **Renormalized Riemann-Hilbert Problem.** Throughout, unless specified otherwise, we follow the convention $\sqrt{z} = \sqrt{|z|}\exp\{i\operatorname{Arg}(z)/2\}$, $\operatorname{Arg}(z) \in (-\pi,\pi]$. Set

$$\epsilon_n := \sqrt{G_{(v_n/hh_n)}/2} \quad \text{and} \quad E_n := \epsilon_n \varphi^n S_{(v_n/hh_n)}. \tag{6.2}$$

Then E_n has continuous boundary values on each side of Δ that satisfy

$$E_n^+ E_n^- = \frac{v_n}{2hh_n} = \frac{w}{2w_n}$$

due to (2.4), (2.12), and (2.16). Further, put

$$c^+ := S_h^+/S_h^-, \quad c_n^+ := S_{h_n}^+/S_{h_n}^-, \quad c^- := 1/c^+, \quad \text{and} \quad c_n^- := 1/c_n^+. \tag{6.3}$$

Then we get on account of (2.12), (5.11), and the multiplicativity property of the Szegő functions that

$$\frac{E_n^-}{E_n^+} = \frac{\left(S_{v_n}\varphi^n\right)^-}{\left(S_{v_n}\varphi^n\right)^+}\frac{S_{hh_n}^+}{S_{hh_n}^-} = \frac{v_n c_n^+ c^+}{G_{v_n}\left(S_{v_n}^2\varphi^{2n}\right)^+} = (r_n c_n c)^+ \quad \text{and} \quad \frac{E_n^+}{E_n^-} = (r_n c_n c)^-, \tag{6.4}$$

where we slightly abuse the notation by writing $(r_n c_n c)^\pm$ instead of $r_n^\pm c_n^\pm c^\pm$. Since any Szegő function assumes value one at infinity and $\varphi(z)/2z \to 1$ as $z \to \infty$, it holds that $E_n(z)/[\epsilon_n(2z)^n] \to$

1 as $z \to \infty$. Then it is a quick computation to check that

$$\left(E_n^-\right)^{\sigma_3} \begin{pmatrix} 1 & 2w_n \\ 0 & 1 \end{pmatrix} \left(E_n^+\right)^{-\sigma_3} = \begin{pmatrix} (r_n c_n c)^+ & w \\ 0 & (r_n c_n c)^- \end{pmatrix}$$

and

$$\lim_{z\to\infty} (2^n \epsilon_n)^{\sigma_3} \mathscr{Y} E_n^{-\sigma_3}(z) = \mathscr{I}.$$

Suppose now that RHP-$\mathscr{Y}$ is solvable and $\mathscr{Y}$ is the solution. Define

$$\mathscr{T} := (2^n \epsilon_n)^{\sigma_3} \mathscr{Y} E_n^{-\sigma_3}. \tag{6.5}$$

Then $\mathscr{T}$ solves the following Riemann-Hilbert problem (RHP-$\mathscr{T}$):

(a) $\mathscr{T}$ is analytic in D and $\mathscr{T}(\infty) = \mathscr{I}$;

(b) $\mathscr{T}$ has continuous traces, $\mathscr{T}_\pm$, on Δ° and $\mathscr{T}_+ = \mathscr{T}_- \begin{pmatrix} (r_n c_n c)^+ & w \\ 0 & (r_n c_n c)^- \end{pmatrix}$;

(c) $\mathscr{T}$ has the following behavior near $z = 1$:

$$\mathscr{T} = \begin{cases} O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix}, & \text{if} \quad \alpha < 0, \\ O\begin{pmatrix} 1 & \log|1-z| \\ 1 & \log|1-z| \end{pmatrix}, & \text{if} \quad \alpha = 0, \qquad \text{as} \quad D \ni z \to 1; \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if} \quad \alpha > 0, \end{cases}$$

(d) $\mathscr{T}$ has the same behavior when $D \ni z \to -1$ as in (c) only with α replaced by β and $1-z$ replaced by $1+z$.

Trivially, the following lemma holds.

Lemma 15. RHP-$\mathscr{T}$ *is solvable if and only if* RHP-$\mathscr{Y}$ *is solvable. When solutions of* RHP-$\mathscr{T}$ *and* RHP-$\mathscr{Y}$ *exist, they are unique and connected by* (6.5).

6.3. **Opening the Lenses, Contours Σ_{ext}, Σ_n, and Σ_n^{md}.** As is standard in the Riemann-Hilbert approach, the second transformation of RHP-$\mathscr{Y}$ is based on the following factorization of the jump matrix in RHP-$\mathscr{T}$(b):

$$\begin{pmatrix} (r_n c_n c)^+ & w \\ 0 & (r_n c_n c)^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (r_n c_n c)^-/w & 1 \end{pmatrix} \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (r_n c_n c)^+/w & 1 \end{pmatrix},$$

where we took into account that $r_n^+ r_n^- \equiv 1$ on Δ. This factorization leads us to consider a new Riemann-Hilbert problem with three jumps on a lens-shaped contour Σ_n (see Fig. 2). However,

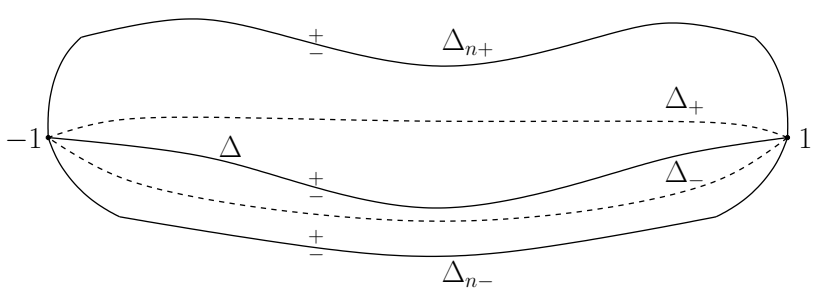

FIGURE 2. The contour $\Sigma_n := \Delta_{n+} \cup \Delta \cup \Delta_{n-} \subset \Xi(D_\Xi)$ (solid lines). The extension contour $\Sigma_{ext} := \Delta_+ \cup \Delta \cup \Delta_-$ (dashed lines and Δ).

to proceed with such a decomposition, we need to extend $c^{\pm}$ and $c_n^{\pm}$ to the complex plane. We shall do it in such a manner that the extended functions, denoted by c and c_n, are analytic outside of a fixed lens Σ_{ext} (see Fig. 2). We postpone this task until the next section and describe here the construction of the lenses Σ_{ext} and Σ_n.

We start from Σ_{ext}. When $\Delta = [-1,1]$, fix $x > 0$ and set Δ_+ to be the subarc of the circle $\{z : |z - ix| = |x+1|\}$ that lies in the upper half plane. Clearly, Δ_+ joins -1 and 1 and can be made as uniformly close to $[-1,1]$ as we want by taking x sufficiently large. We set Δ_- to be the reflection of Δ_+ across the real axis. We also denote by Ω_+ and Ω_- the upper and the lower parts of the lens Σ_{ext}, i.e., Ω_+ (resp. Ω_-) is a domain bounded by Δ_+ (resp. Δ_-) and Δ. When Δ is a general closed analytic arc parametrized by Ξ, set Σ_{ext} to be the image under Ξ of the corresponding lens for $[-1,1]$ (the latter can always be made small enough to lie in D_Ξ).

We continue by constructing the lens Σ, which will we the limit lens for the sequence $\{\Sigma_n\}$. Let g^2 and $O_g \subset \Xi(D_\Xi)$ be defined in (3.5). Then $1 \in O_g$, $g^2(1) = 0$, and g^2 is conformal in O_g. Set $U_\delta := \{z : |z-1| < \delta\}$. Choose $\delta_0 > 0$ to be so small that $U_\delta \subset O_g$ and $g^2(U_\delta)$ is convex for any $\delta < \delta_0$. We require the same conditions to be fulfilled by δ_0 and $\widetilde{U}_\delta := \{z : |z+1| < \delta\}$ with respect to $\widetilde{g}^2$ and $O_{\widetilde{g}}$ also defined in (3.5). Fix $\delta < \delta_0$. Let Jordan arcs K_j, $j = 1,3$, be the preimages of $\Sigma_1 := \{\zeta : \operatorname{Arg}(\zeta) = 2\pi/3\}$ and $\Sigma_3 := \{\zeta : \operatorname{Arg}(\zeta) = -2\pi/3\}$ under g^2 in U_δ. Let also $\widetilde{K}_j$, $j = 1,3$, be the preimages of Σ_1 and Σ_3 under $\widetilde{g}^2$ in $\widetilde{U}_\delta$. Set $K_+ := K_1 \cup K_2 \cup \widetilde{K}_3$, where K_2 is the image under Ξ of the line segment that joins $\Xi^{-1}(K_1 \cap U_\delta)$. Set also $\widetilde{\Xi}^{-1}(\widetilde{K}_3 \cap \widetilde{U}_\delta)$, and $K_- := K_3 \cup K_4 \cup \widetilde{K}_1$, where K_4 is the image under Ξ of the line segment that joins $\Xi^{-1}(K_3 \cap U_\delta)$ and $\widetilde{\Xi}^{-1}(\widetilde{K}_1 \cap \widetilde{U}_\delta)$. Then $\Delta_\pm$ are Jordan arcs that with endpoints ± 1. We define $\Sigma := \Delta \cup K_+ \cup K_-$ (see Fig. 3).

Let g_n and $\widetilde{g}_n$ be defined by (3.7). Assume that δ is small enough that $U_\delta \subset O_L$ and $\widetilde{U}_\delta \subset O_{\widetilde{L}}$. Then we construct the lens $\Sigma_n := \Delta \cup \Delta_{n+} \cup \Delta_{n-}$ exactly as we constructed Σ only with g, $\widetilde{g}$, and Ξ replaced by g_n, $\widetilde{g}_n$, and Ξ_n, where we also employ the notation $\Delta_{n\pm}$ for the upper and lower lips of the lens. It can be easily seen that the arcs $\Delta_{n\pm}$ and Δ intersect only at ± 1 for all n large enough

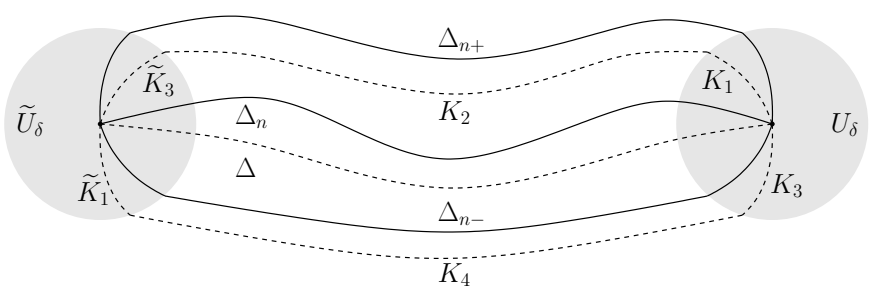

FIGURE 3. Contours Σ (dashed lines) and Σ_n^{md} (solid lines). Neighborhoods U_δ and $\widetilde{U}_\delta$ (disks around ± 1).

since Δ_n approach Δ in a uniform manner by Theorem 1 and $\Delta_{n\pm}$ and Δ_n form angles $\pi/3$ at 1 and -1 by construction.

Finally, it will be useful for us later to define one more system of contours, say Σ_n^{md}. The lens Σ_n^{md} is obtained from Σ_n simply by replacing Δ by Δ_n (see Fig. 3). We also require the lens Σ_{ext} to be contained within each lens Σ_n (see Fig. 2).

6.4. **Extension with Controlled $\bar\partial$ Derivative.** Without loss of generality we may assume that $\Sigma_n \subset D_\Xi$ and all the functions h_n are holomorphic in D_Ξ. By the very definition of $c_n^\pm$ we have that

$$c_n^\pm = G_{h_n}\left(S_{h_n}^\pm\right)^2 h_n^{-1}.$$

Thus, there is a natural holomorphic extension of each $c_n^\pm$ given by

$$c_n := G_{h_n} S_{h_n}^2 h_n^{-1} \quad \text{in} \quad D_\Xi \setminus \Delta. \tag{6.6}$$

Concerning the extension of c, we can prove the following.

Lemma 16. *Let $\theta \in \mathrm{W}_p^{1-1/p}$, $p \in (2,\infty)$, or $\theta \in \mathrm{C}^{m,\varsigma}$, $m \in \mathbb{Z}_+$, $\varsigma \in (0,1]$, $m+\varsigma > \frac{1}{2}$. Then there exists a function c, continuous in $\mathbb{C}\setminus\Delta$ and up to $\Delta^\pm$, satisfying*

$$c_{|\Delta^\pm} = c^\pm, \quad c = \exp\{\mathfrak{w}\ell\} \quad \textit{in} \quad \overline{\mathbb{C}}\setminus(\overline{\Omega_+\cup\Omega_-}), \quad \textit{and} \quad \bar\partial c = cf,$$

where ℓ is a polynomial, $\deg(\ell) \le 2m+1$, $f \in \mathrm{L}^p(\Omega_\pm)$ when $\theta \in \mathrm{W}_p^{1-1/p}$, $f \in \mathrm{L}^q(\Omega_\pm)$, $q \in \left(2, \frac{1}{1-\varsigma}\right)$ when $\theta \in \mathrm{C}^{0,\varsigma}$, and $f \in \mathrm{C}_0^{m-1,\varsigma-\epsilon}(\overline{\Omega_\pm})$ when $\theta \in \mathrm{C}^{m,\varsigma}$, $m \in \mathbb{N}$, and $\epsilon \in (0,\varsigma)$.

Proof. This lemma is a straightforward consequence of (5.10), inclusion $\mathrm{C}^{0,\varsigma} \subset \mathrm{W}_q^{1-1/q}$ for $q \in \left(2, \frac{1}{1-\varsigma}\right)$, and Propositions 8 and 9 combined with Propositions 6 and 7. □

6.5. **Formulation of Riemann-Hilbert-$\bar\partial$ Problem.** In this section we reformulate RHP-$\mathscr{T}$ as a Riemann-Hilbert-$\bar\partial$ problem. In what follows, we understand under c and c_n the extensions obtained in Section 6.4 above. Suppose that RHP-$\mathscr{T}$ is solvable and $\mathscr{T}$ is the solution. We define a matrix function $\mathscr{S}$ on $\overline{\mathbb{C}}\setminus\Sigma_n$ as follows:

$$\mathscr{S} := \begin{cases} \mathscr{T}\begin{pmatrix} 1 & 0 \\ \mp r_n c_n c/w & 1 \end{pmatrix}, & \text{in } \Omega_{n\pm}, \\ \mathscr{T}, & \text{outside the lens } \Sigma_n, \end{cases} \tag{6.7}$$

where the upper part, Ω_{n+}, (resp. lower part, Ω_{n-}) of the lens Σ_n is a domain bounded by Δ_{n+} (resp. Δ_{n-}) and Δ. This new matrix function is no longer analytic in general in the whole domain D since c may not be analytic inside the extension lens Σ_{ext}. Recall, however, that by the very construction, c coincides with a holomorphic function $\tilde c = \exp\{\mathfrak{w}\ell\}$ outside the lens Σ_{ext}. To capture the non-analytic character of $\mathscr{S}$, we introduce the following matrix function that will represent the deviation from analyticity:

$$\mathscr{W}_0 := \begin{cases} \begin{pmatrix} 0 & 0 \\ \mp r_n c_n \bar\partial c/w & 0 \end{pmatrix}, & \text{in } \Omega_\pm, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{outside the lens } \Sigma_{ext}. \end{cases} \tag{6.8}$$

Then $\mathscr{S}$ solves the following Riemann-Hilbert-$\bar\partial$ problem (RH$\bar\partial$P-$\mathscr{S}$):

(a) $\mathscr{S}$ is a continuous matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n$ and $\mathscr{S}(\infty) = \mathscr{I}$;
(b) $\mathscr{S}$ has continuous boundary values, $\mathscr{S}_\pm$, on $\Sigma_n^\circ := \Sigma_n \setminus \{\pm 1\}$ and

$$\begin{aligned} \mathscr{S}_+ &= \mathscr{S}_-\begin{pmatrix} 1 & 0 \\ r_n c_n c/w & 1 \end{pmatrix} \quad \text{on} \quad \Delta_{n+}^\circ \cup \Delta_{n-}^\circ, \\ \mathscr{S}_+ &= \mathscr{S}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \quad \text{on} \quad \Delta^\circ; \end{aligned}$$

(c) For $\alpha<0$, $\mathscr{S}$ has the following behavior near $z=1$:

$$\mathscr{S}(z)=O\begin{pmatrix} 1 & |1-z|^{\alpha} \\ 1 & |1-z|^{\alpha}\end{pmatrix}, \quad \text{as} \quad \mathbb{C}\setminus\Sigma_n\ni z\to 1.$$

For $\alpha=0$, $\mathscr{S}$ has the following behavior near $z=1$:

$$\mathscr{S}(z)=O\begin{pmatrix} \log|1-z| & \log|1-z| \\ \log|1-z| & \log|1-z|\end{pmatrix} \text{as} \quad \mathbb{C}\setminus\Sigma_n\ni z\to 1.$$

For $\alpha>0$, $\mathscr{S}$ has the following behavior near $z=1$:

$$\mathscr{S}(z)=\begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}, & \text{as } z\to 1 \text{ outside the lens } \Sigma_n, \\ O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1\end{pmatrix}, & \text{as } z\to 1 \text{ inside the lens } \Sigma_n;\end{cases}$$

(d) $\mathscr{S}$ has the same behavior when $\mathbb{C}\setminus\Sigma_n\ni z\to -1$ as in (c) only with α replaced by β and $1-z$ replaced by $1+z$;

(e) $\mathscr{S}$ deviates from an analytic matrix function according to $\bar\partial\mathscr{S}=\mathscr{S}\mathscr{W}_0$.

Then the following lemma holds.

Lemma 17. RH$\bar\partial$P-$\mathscr{S}$ *is solvable if and only if* RHP-$\mathscr{T}$ *is solvable. When solutions of* RH$\bar\partial$P-$\mathscr{S}$ *and* RHP-$\mathscr{T}$ *exist, they are unique and connected by* (6.7).

Proof. By construction, the solution of RHP-$\mathscr{T}$ yields a solution of RH$\bar\partial$P-$\mathscr{S}$. Conversely, let $\mathscr{S}^*$ be a solution of RH$\bar\partial$P-$\mathscr{S}$. It is easy to check using the Leibnitz's rule that $\bar\partial\mathscr{T}^*$ is equal to the zero matrix outside of Σ_n, where $\mathscr{T}^*$ is obtained from $\mathscr{S}^*$ by inverting (6.7). Thus, $\mathscr{T}^*$ is an analytic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n$ with continuous boundary values on each side of Σ_n°. Moreover, it can be readily verified that $\mathscr{T}^*$ has no jumps on $\Delta_{n\pm}^\circ$ and therefore is, in fact, analytic in D. It is aslo obvious that it equals to the identity matrix at infinity and has a jump on Δ described by RHP-$\mathscr{T}$(b). Thus, $\mathscr{T}^*$ complies with RHP-$\mathscr{T}$(a)–(b).

Now, if $\alpha,\beta<0$ then it follows from RH$\bar\partial$P-$\mathscr{S}$(c)–(d) and (6.7) that $\mathscr{T}^*$ has the same behavior near endpoints ±1 as $\mathscr{S}^*$. Therefore, $\mathscr{T}^*$ solves RHP-$\mathscr{T}$ in this case. When either α or β is nonnegative, it is no longer immediate that the first column of $\mathscr{T}^*$ has the behavior near ±1 required by RHP-$\mathscr{T}$(c)–(d). This difficulty was resolved in [33, Lem. 4.1] by considering $\mathscr{T}^*\mathscr{T}^{-1}$, where $\mathscr{T}$ is the unique solution of RHP-$\mathscr{T}$. However, in the present case it is not clear that such a solution exists (see Lemma 14). Thus, we are bound to consider the first column of $\mathscr{T}^*$ by itself.

Denote by $\mathscr{T}_{11}^*$ and $\mathscr{T}_{21}^*$ the 11- and 21-entries of $\mathscr{T}^*$. Then $\mathscr{T}_{11}^*$ and $\mathscr{T}_{21}^*$ are analytic functions in D with the following behavior near 1:

$$\mathscr{T}_{j1}^*(z)=\begin{cases} O(1), & \text{if} \quad \alpha<0 \\ O(\log|1-z|), & \text{if} \quad \alpha=0, \\ O(|1-z|^{-\alpha}), & \text{if} \quad \alpha>0 \text{ and } z \text{ is inside the lens}, \\ O(1), & \text{if} \quad \alpha>0 \text{ and } z \text{ is outside the lens},\end{cases} \tag{6.9}$$

for $j=1,2$. The behavior near -1 is identical only with α replaced by β and $1-z$ replaced by $1+z$. Moreover, each $\mathscr{T}_{j1}^*$ solves the following scalar boundary value problem:

$$\phi^+=\phi^-(r_nc_nc)^+ \quad \text{on} \quad \Delta, \quad \phi\in \mathrm{H}(D). \tag{6.10}$$

Now, recall that $r_n^+r_n^-\equiv 1$ on Δ and r_n has $2n$ zeros in D that lie away from the lens Σ_n. Hence, the argument of r_n^+ increases by $2\pi n$ when Δ is traversed from -1 to 1. Moreover, for c^+ and

each c_n^+ a branch of the argument can be taken continuous and vanishing at ± 1 (it is the imaginary part of $\mathfrak{w}^+\mathscr{S}(\theta/\mathfrak{w}^+)$, which is continuous and vanishing at ± 1 by Propositions 8 and 9). Define $\varrho := \log(r_n c_n c)^+$, $\varrho(-1) = 0$. This normalization is possible since $r_n^+(-1) = 1$ as r_n^+ is a product of $2n$ factors each of which is equal to -1 at -1. Furthermore, this normalization necessarily yields that $\varrho(1) = 2\pi n i$ and that the so-called canonical solution of the problem (6.10) is given by [24, Sec. 43.1]

$$\phi_c(z) := (z-1)^{-n} \exp\{\mathscr{C}(\varrho; z)\}, \quad z \in D.$$

Recall that ϕ_c is bounded above and below in the vicinities of 1 and -1, has a zero of order n at infinity, and otherwise is non-vanishing. Hence, the functions $\phi_j := \mathscr{T}_{j1}^*/\phi_c$, $j = 1, 2$, are analytic in $\mathbb{C}\setminus\{\pm 1\}$. Moreover, according to (6.9), the singularities of these functions at 1 and -1 cannot be essential, they are either removable or polar. In fact, since $\phi_j(z) = O(1)$ or $\phi_j(z) = O(\log|1 \pm z|)$ when z approaches 1 or -1 outside of the lens, ϕ_j can have only removable singularities at these points. Hence, $\phi_j(z) = O(1)$ and subsequently $\mathscr{T}_{j1}^* = O(1)$ near 1 and -1. Thus, $\mathscr{T}^*$ satisfies RHP-$\mathscr{T}$(c)–(d) for all α and β, which means that $\mathscr{T}^*$ is the solution of RHP-$\mathscr{T}$. Therefore, indeed, the problems RHP-$\mathscr{T}$ and RH$\bar\partial$P-$\mathscr{S}$ are equivalent. □

7. Analytic Approximation of RH$\bar\partial$P-$\mathscr{S}$

Elaborating on the path developed in [37], we put RH$\bar\partial$P-$\mathscr{S}$ aside for a while and consider an analytic approximation of this problem. In other words, we seek the solution of the following Riemann-Hilbert problem (RHP-$\mathscr{A}$):

(a) $\mathscr{A}$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n$ and $\mathscr{A}(\infty) = \mathscr{I}$;
(b) $\mathscr{A}$ has continuous traces, $\mathscr{A}_\pm$, on Σ_n° that satisfy the same relations as in RH$\bar\partial$P-$\mathscr{S}$(b);
(c) the behavior of $\mathscr{A}$ near 1 is described by RH$\bar\partial$P-$\mathscr{S}$(c);
(d) the behavior of $\mathscr{A}$ near -1 is described by RH$\bar\partial$P-$\mathscr{S}$(d).

Before we proceed, observe that the function c coincides on $\Delta_{n\pm}$ with the analytic function $\tilde c := \exp\{\mathfrak{w}\ell\}$, where ℓ is a polynomial, by construction. Hence, we can assume that the jump matrix in RHP-$\mathscr{A}$(b) is expressed in terms of $\tilde c$ rather than c.

7.1. **Modified** RHP-$\mathscr{A}$**.** The problem above almost falls into the scope of the classical approach to asymptotics of orthogonal polynomials. We say "almost" because it is not generally true that the functions r_n can be written as the $2n$-th power of a single function, even up to a normal family as is the case in [3, Thm. 2]. This will explain why we constructed another lens, Σ_n^{md}, in Section 6.3.

Consider the following Riemann-Hilbert problem (RHP-$\mathscr{B}$):

(a) $\mathscr{B}$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n^{md}$ and $\mathscr{B}(\infty) = \mathscr{I}$;
(b) $\mathscr{B}$ has continuous traces, $\mathscr{B}_\pm$, on $(\Sigma_n^{md})^\circ$ that satisfy

$$\begin{aligned} \mathscr{B}_+ &= \mathscr{B}_- \begin{pmatrix} 1 & 0 \\ r_n c_n \tilde c/w & 1 \end{pmatrix} \quad \text{on} \quad \Delta_{n+}^\circ \cup \Delta_{n-}^\circ, \\ \mathscr{B}_+ &= \mathscr{B}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \quad \text{on} \quad \Delta_n^\circ; \end{aligned}$$

(c) the behavior of $\mathscr{B}$ near 1 is described by RHP-$\mathscr{A}$(c) with respect to the lens Σ_n^{md};
(d) the behavior of $\mathscr{B}$ near -1 is described by RHP-$\mathscr{A}$(d), again, with respect to Σ_n^{md}.

In fact, this new problem is equivalent to RHP-$\mathscr{A}$.

Lemma 18. *The problems* RHP-$\mathscr{A}$ *and* RHP-$\mathscr{B}$ *are equivalent.*

Proof. Suppose that RHP-$\mathscr{B}$ is solvable and $\mathscr{B}$ is a solution. As before, let Ω_{n+} (resp. Ω_{n-}) be the upper (resp. lower) part of the lens Σ_n. Analogously define $\Omega_{n\pm}^{md}$ and set

$$\mathscr{A}^* := \begin{cases} \mathscr{B}\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}, & \text{in} \quad \Omega_{n+}\cap\Omega_{n-}^{md}, \\ \mathscr{B}\begin{pmatrix} 0 & -w \\ 1/w & 0 \end{pmatrix}, & \text{in} \quad \Omega_{n-}\cap\Omega_{n+}^{md}, \\ \mathscr{B}, & \text{elsewhere.} \end{cases} \tag{7.1}$$

Observe that $\Omega_{n\pm}\cap\Omega_{n\mp}^{md}$ is a finite, possibly empty, union of Jordan domains by analyticity of Δ and Δ_n. It is a routine exercise to verify that $\mathscr{A}^*$ complies with RHP-$\mathscr{A}$(a) and (b). Moreover, within $\Omega_{n+}\cap\Omega_{n-}^{md}$ and $\Omega_{n-}\cap\Omega_{n+}^{md}$ we have that for $\alpha<0$, $\mathscr{A}^*$ has the following behavior near $z=1$:

$$\mathscr{A}^*(z) = O\begin{pmatrix} 1 & |1-z|^{\alpha} \\ 1 & |1-z|^{\alpha} \end{pmatrix} O\begin{pmatrix} 0 & |1-z|^{\alpha} \\ |1-z|^{-\alpha} & 0 \end{pmatrix} = O\begin{pmatrix} 1 & |1-z|^{\alpha} \\ 1 & |1-z|^{\alpha} \end{pmatrix},$$

as $z\to 1$; for $\alpha=0$, $\mathscr{A}^*$ has the same behavior near $z=1$ as $\mathscr{B}$ since the latter is multiplied by a bounded matrix near 1; for $\alpha>0$, $\mathscr{A}^*$ has the following behavior near $z=1$:

$$\mathscr{A}^*(z) = O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix} O\begin{pmatrix} 0 & |1-z|^{\alpha} \\ |1-z|^{-\alpha} & 0 \end{pmatrix} = O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix}.$$

Hence, $\mathscr{A}^*$ has exactly the behavior near 1 required by RHP-$\mathscr{A}$(c). In the same fashion one can check that $\mathscr{A}^*$ satisfies RHP-$\mathscr{A}$(d) and therefore it is, in fact, a solution of RHP-$\mathscr{A}$. Clearly, the arguments above could be reversed and hence each solution of RHP-$\mathscr{A}$ yields a solution of RHP-$\mathscr{B}$. □

Let us now alleviate the notation by slightly abusing it. Throughout this section, we shall understand under φ, r_n, g_n, $\widetilde{g}_n$, c_n, $\widetilde{c}$, $\mathfrak{w}$, S_{h_n}, $S_{\mathfrak{w}^+}$, and S_w their holomorphic continuations that are analytic outside of Δ_n rather than Δ. Note that outside the bounded set with boundary $\Delta\cup\Delta_n$ these continued functions coincide with the original ones. Moreover, their values considered within the interior domain of $\Delta_n\cup\Delta$ can be obtained through analytic continuation of the original functions across Δ.

7.2. **Auxiliary Riemann-Hilbert Problems.** In this subsection we define the necessary objects to solve RHP-$\mathscr{B}$. This material essentially appeared in [33] for the case $\Delta=[-1,1]$.

7.2.1. *Parametrix away from the endpoints.* As r_n converges to zero geometrically fast away from Δ_n, the jump matrix in RHP-$\mathscr{B}$(b) is close to the identity on Δ_{n+}° and Δ_{n-}°. Thus, the main term of the asymptotics for $\mathscr{B}$ in $D_n=\overline{\mathbb{C}}\setminus\Delta_n$ is determined by the following Riemann-Hilbert problem (RHP-$\mathscr{N}$):

(a) $\mathscr{N}$ is a holomorphic matrix function in D_n and $\mathscr{N}(\infty)=\mathscr{I}$;

(b) $\mathscr{N}$ has continuous traces, $\mathscr{N}_{\pm}$, on Δ_n° and $\mathscr{N}_+=\mathscr{N}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}$;

It can be easily checked using (5.14) that a solution of RHP-$\mathscr{N}$ when $w\equiv 1$ is given by

$$\mathscr{N}_* = \begin{pmatrix} S_{\mathfrak{w}^+}^{-1} & i(\varphi S_{\mathfrak{w}^+})^{-1} \\ -i(\varphi S_{\mathfrak{w}^+})^{-1} & S_{\mathfrak{w}^+}^{-1} \end{pmatrix}. \tag{7.2}$$

Then a solution of RHP-$\mathscr{N}$ for arbitrary w is given by

$$\mathscr{N} = (G_w)^{\sigma_3/2}\mathscr{N}_*(G_w S_w^2)^{-\sigma_3/2}. \tag{7.3}$$

7.2.2. *Auxiliary parametrix near the endpoints.* The following construction was introduced in [33, Thm. 6.3]. Let I_α and K_α be the modified Bessel functions and $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ be the Hankel functions [1, Ch. 9]. Set Ψ to be the following sectionally holomorphic matrix function:

$$\Psi(\zeta)=\Psi(\zeta;\alpha):=\begin{pmatrix} I_\alpha\left(2\zeta^{1/2}\right) & \frac{i}{\pi}K_\alpha\left(2\zeta^{1/2}\right) \\ 2\pi i\zeta^{1/2}I_\alpha'\left(2\zeta^{1/2}\right) & -2\zeta^{1/2}K_\alpha'\left(2\zeta^{1/2}\right)\end{pmatrix}$$

for $|\mathrm{Arg}(\zeta)|<2\pi/3$;

$$\Psi(\zeta):=\begin{pmatrix} \frac{1}{2}H_\alpha^{(1)}\left(2(-\zeta)^{1/2}\right) & \frac{1}{2}H_\alpha^{(2)}\left(2(-\zeta)^{1/2}\right) \\ \pi\zeta^{1/2}\left(H_\alpha^{(1)}\right)'\left(2(-\zeta)^{1/2}\right) & \pi\zeta^{1/2}\left(H_\alpha^{(2)}\right)'\left(2(-\zeta)^{1/2}\right)\end{pmatrix}e^{\frac{1}{2}\alpha\pi i\sigma_3}$$

for $2\pi/3<\mathrm{Arg}(\zeta)<\pi$;

$$\Psi(\zeta):=\begin{pmatrix} \frac{1}{2}H_\alpha^{(2)}\left(2(-\zeta)^{1/2}\right) & -\frac{1}{2}H_\alpha^{(1)}\left(2(-\zeta)^{1/2}\right) \\ -\pi\zeta^{1/2}\left(H_\alpha^{(2)}\right)'\left(2(-\zeta)^{1/2}\right) & \pi\zeta^{1/2}\left(H_\alpha^{(1)}\right)'\left(2(-\zeta)^{1/2}\right)\end{pmatrix}e^{-\frac{1}{2}\alpha\pi i\sigma_3}$$

for $-\pi<\mathrm{Arg}(\zeta)<-2\pi/3$, where $\mathrm{Arg}(\zeta)\in(-\pi,\pi]$ is the principal determination of the argument of ζ. Let further Σ_1, Σ_2, and Σ_3 be the rays $\{\zeta:\mathrm{Arg}(\zeta)=2\pi/3\}$, $\{\zeta:\mathrm{Arg}(\zeta)=\pi\}$, and $\{\zeta:\mathrm{Arg}(\zeta)=-2\pi/3\}$, respectively, oriented from infinity to zero. Using known properties of I_α, K_α, $H_\alpha^{(1)}$, $H_\alpha^{(2)}$, and their derivatives, it can be checked that Ψ is the solution of the following Riemann-Hilbert problem RHP-Ψ:

(a) Ψ is a holomorphic matrix function in $\mathbb{C}\setminus(\Sigma_1\cup\Sigma_2\cup\Sigma_3)$;
(b) Ψ has continuous traces, $\Psi_\pm$, on Σ_j°, $j\in\{1,2,3\}$, and

$$\begin{aligned}\Psi_+ &= \Psi_-\begin{pmatrix}1 & 0\\ e^{\alpha\pi i} & 1\end{pmatrix} && \text{on} \quad \Sigma_1^\circ,\\ \Psi_+ &= \Psi_-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} && \text{on} \quad \Sigma_2^\circ,\\ \Psi_+ &= \Psi_-\begin{pmatrix}1 & 0\\ e^{-\alpha\pi i} & 1\end{pmatrix} && \text{on} \quad \Sigma_3^\circ;\end{aligned}$$

(c) Ψ has the following behavior near ∞:

$$\Psi(\zeta)=\left(2\pi\zeta^{1/2}\right)^{-\sigma_3/2}\frac{1}{\sqrt{2}}\begin{pmatrix}1+O\left(\zeta^{-1/2}\right) & i+O\left(\zeta^{-1/2}\right)\\ i+O\left(\zeta^{-1/2}\right) & 1+O\left(\zeta^{-1/2}\right)\end{pmatrix}e^{2\zeta^{1/2}\sigma_3}$$

uniformly in $\mathbb{C}\setminus(\Sigma_1\cup\Sigma_2\cup\Sigma_3)$;
(d) For $\alpha<0$, Ψ has the following behavior near 0:

$$\Psi=O\begin{pmatrix}|\zeta|^{\alpha/2} & |\zeta|^{\alpha/2}\\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2}\end{pmatrix}\text{ as }\zeta\to 0;$$

For $\alpha=0$, Ψ has the following behavior near 0:

$$\Psi=O\begin{pmatrix}\log|\zeta| & \log|\zeta|\\ \log|\zeta| & \log|\zeta|\end{pmatrix}\text{ as }\zeta\to 0;$$

For $\alpha > 0$, Ψ has the following behavior near 0:

$$\Psi = \begin{cases} O\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} & \text{as } \zeta \to 0 \text{ in } |\mathrm{Arg}(\zeta)| < 2\pi/3, \\ O\begin{pmatrix} |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} & \text{as } \zeta \to 0 \text{ in } 2\pi/3 < |\mathrm{Arg}(\zeta)| < \pi. \end{cases}$$

Further, if we set

$$\widetilde{\Psi} := \sigma_3 \Psi(\cdot;\beta)\sigma_3,$$

then this matrix function satisfies RHP-Ψ with α replaced by β and reversed orientation for Σ_j, $j \in \{1,2,3\}$.

7.2.3. *Parametrix near* 1. As shown in [33], the main term of the asymptotics of the solution of RHP-$\mathscr{B}$ near the endpoints of Δ_n is described by a solution of a special Riemann-Hilbert problem for which Ψ will be instrumental. Let U_δ be as in the construction of Σ_n (see Section 6.3). Below we describe the solution of the following Riemann-Hilbert problem (RHP-$\mathscr{P}$):

(a) $\mathscr{P}$ is a holomorphic matrix function in $U_\delta \setminus \Sigma_n^{md}$;
(b) $\mathscr{P}$ has continuous boundary values, $\mathscr{P}_\pm$, on $U_\delta \cap (\Sigma_n^{md})^\circ$ and

$$\begin{aligned} \mathscr{P}_+ &= \mathscr{P}_- \begin{pmatrix} 1 & 0 \\ r_n c_n \widetilde{c}/w & 1 \end{pmatrix} \quad \text{on} \quad U_\delta \cap (\Delta_{n+}^\circ \cup \Delta_{n-}^\circ), \\ \mathscr{P}_+ &= \mathscr{P}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \quad \text{on} \quad U_\delta \cap \Delta_n^\circ; \end{aligned}$$

(c) $\mathscr{P}\mathscr{N}^{-1} = \mathscr{I} + O(1/n)$ uniformly on ∂U_δ;
(d) For $\alpha < 0$, $\mathscr{P}$ has the following behavior near $z = 1$:

$$\mathscr{P} = O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix}, \quad \text{as} \quad U_\delta \setminus \Sigma_n^{md} \ni z \to 1;$$

For $\alpha = 0$, $\mathscr{P}$ has the following behavior near $z = 1$:

$$\mathscr{P} = O\begin{pmatrix} \log|1-z| & \log|1-z| \\ \log|1-z| & \log|1-z| \end{pmatrix}, \quad \text{as} \quad U_\delta \setminus \Sigma_n^{md} \ni z \to 1;$$

For $\alpha > 0$, $\mathscr{P}$ has the following behavior near $z = 1$:

$$\mathscr{P} = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ outside the lens } \Sigma_n^{md}, \\ O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ inside the lens } \Sigma_n^{md}. \end{cases}$$

To present a solution of RHP-$\mathscr{P}$, we need to introduce more notation. Denote by U_δ^+ and U_δ^- the subsets of U_δ that are mapped by g_n^2 into the upper and lower half planes, respectively. Without loss of generality we may assume that functions θ_n are holomorphic in U_δ and the branch cut of w in U_δ coincides with the preimage of the positive reals under g_n^2. In particular, we have that w is analytic in U_δ^+ and U_δ^- and therefore across $\Delta_{n\pm}^\circ$. Set

$$A_n(z) := \frac{\exp\left\{\frac{1}{2}\left(\theta_n(z) - \mathfrak{w}(z)\ell(z)\right)\right\}}{\sqrt{G_{h_n}} S_{h_n}(z)} (z-1)^{\alpha/2}(z+1)^{\beta/2},$$

where we use the same branch of $(z+1)^{\beta/2}$ as in definition of w and a branch of $(z-1)^{\alpha/2}$ analytic in $U_\delta \setminus \Delta_n$ and positive for positive reals large enough. Then

$$A_n^2 = \begin{cases} e^{\alpha\pi i} w/c_n\widetilde{c}, & \text{in} \quad U_\delta^+, \\ e^{-\alpha\pi i} w/c_n\widetilde{c}, & \text{in} \quad U_\delta^-, \end{cases} \tag{7.4}$$

by the definition of $\widetilde{c}$ and on account of (6.6). Moreover, it readily follows from (7.4) and (6.3) that

$$A_n^+ A_n^- = w \quad \text{on} \quad \Delta_n^\circ. \tag{7.5}$$

Observe that $g_n^{1/2}$ is a holomorphic function on $U_\delta \setminus \Delta_n$ such that

$$(g_n^{1/2})^+ = i(g_n^{1/2})^- \quad \text{on} \quad \Delta_n \tag{7.6}$$

by (3.8). Then the following lemma holds.

Lemma 19. *A solution of* RHP-$\mathscr{P}$ *is given by*

$$\mathscr{P} = \mathscr{E}\Psi\left(\frac{n^2 g_n^2}{4}\right) A_n^{-\sigma_3} r_n^{\sigma_3/2}, \quad \mathscr{E} := \mathscr{N} A_n^{\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (\pi n g_n)^{\sigma_3/2}.$$

Proof. Except for some technical differences, the proof is analogous to the considerations in [33, eqn. (6.27) and after]. First, we must show that $\mathscr{E}$ is holomorphic in U_δ. This is clearly true in $U_\delta \setminus \Delta_n$. It is also clear that $\mathscr{E}$ has continuous boundary values on each side of Δ_n°. Since

$$\begin{aligned} \mathscr{E}_+ &= \mathscr{N}_+ \left(A_n^+\right)^{\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (\pi n)^{\sigma_3/2} \left(\left(g_n^{1/2}\right)^+\right)^{\sigma_3} \\ &= \mathscr{N}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \left(\frac{w}{A_n^-}\right)^{\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (\pi n)^{\sigma_3/2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\left(g_n^{1/2}\right)^-\right)^{\sigma_3} \\ &= \mathscr{N}_- \left(A_n^-\right)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix} (\pi n)^{\sigma_3/2} \left(\left(g_n^{1/2}\right)^-\right)^{\sigma_3} = \mathscr{E}_-, \end{aligned}$$

where we used RHP-$\mathscr{N}$(b), (7.5), and (7.6), $\mathscr{E}$ is holomorphic across Δ_n°. Thus, it remains to show that $\mathscr{E}$ has no singularity at 1. For this observe that

$$\left(g_n^{1/2}(z)\right)^{\sigma_3} = O\begin{pmatrix} |1-z|^{1/4} & 0 \\ 0 & |1-z|^{-1/4} \end{pmatrix}, \quad \text{as} \quad z \to 1,$$

since g_n^2 has a simple zero at 1. Furthermore, by the very definition it holds that

$$\mathscr{N}_* = O\begin{pmatrix} |1-z|^{-1/4} & |1-z|^{-1/4} \\ |1-z|^{-1/4} & |1-z|^{-1/4} \end{pmatrix}, \quad \text{as} \quad z \to 1.$$

Finally, $(A_n/S_w)(z) \to 2^{-(\alpha+\beta)/2}$ as $z \to 1$ by (5.12). Hence, the entries of $\mathscr{E}$ can have at most square-root singularity at 1, which is impossible since $\mathscr{E}$ is analytic in $U_\delta \setminus \{1\}$, and therefore $\mathscr{E}$ is analytic in the whole disk U_δ.

The analyticity of $\mathscr{E}$ implies that the jumps of $\mathscr{P}$ are those of $\Psi\left(n^2 g_n^2/4\right) A_n^{-\sigma_3} r_n^{\sigma_3/2}$. Clearly, the latter has jumps on $\Sigma_n^{md} \cap U_\delta$ by the very definition of g_n^2 and Ψ. Moreover, it is a routine exercise, using RHP-Ψ(b) and (7.4), to verify that these jumps are described exactly by RHP-$\mathscr{P}$(b). It is also clear that RHP-$\mathscr{P}$(a) is satisfied. Further, we get directly from RHP-Ψ(c) that the behavior of $\Psi\left(n^2 g_n^2/4\right)$ on ∂U_δ can be described by

$$\Psi\left(\frac{n^2 g_n^2}{4}\right) = (\pi n g_n)^{-\sigma_3/2} \begin{pmatrix} 1+O(1/n) & i+O(1/n) \\ i+O(1/n) & 1+O(1/n) \end{pmatrix} r_n^{-\sigma_3/2},$$

where the property $O(1/n)$ holds uniformly on ∂U_δ. Hence, using that the diagonal matrices $A_n^{-\sigma_3}$ and $r_n^{\sigma_3/2}$ commute, we get that

$$\begin{aligned}\mathscr{P}\mathscr{N}^{-1} &= \mathscr{E}\left(\pi n g_n\right)^{-\sigma_3/2}\begin{pmatrix} 1+O(1/n) & i+O(1/n) \\ i+O(1/n) & 1+O(1/n)\end{pmatrix}A_n^{-\sigma_3}\mathscr{N}^{-1} \\ &= \mathscr{N}A_n^{\sigma_3}(\mathscr{I}+O(1/n))A_n^{-\sigma_3}\mathscr{N}^{-1} = \mathscr{I}+O(1/n)\end{aligned}$$

since the moduli of all the entries of $\mathscr{N}A_n^{\sigma_3}$ are uniformly bounded above and away from zero on ∂U_δ. Thus, RHP-$\mathscr{P}$(c) holds. Finally, RHP-$\mathscr{P}$(d) follows immediately from RHP-Ψ(d) upon recalling that $|g_n^2(z)| = O(|1-z|)$ and $|A_n(z)| \sim |1-z|^{\alpha/2}$ as $z \to 1$. □

7.2.4. *Parametrix near* -1. In this section we describe the solution of the Riemann-Hilbert problem that plays the same role with respect to -1 as RHP-$\mathscr{P}$ did for 1. Below we describe the solution of the following Riemann-Hilbert problem (RHP-$\widetilde{\mathscr{P}}$):

(a) $\widetilde{\mathscr{P}}$ is a holomorphic matrix function in $\widetilde{U}_\delta \setminus \Sigma_n^{md}$;

(b) $\widetilde{\mathscr{P}}$ has continuous boundary values, $\widetilde{\mathscr{P}}_\pm$, on $\widetilde{U}_\delta \cap (\Sigma_n^{md})^\circ$ and

$$\begin{aligned}\widetilde{\mathscr{P}}_+ &= \widetilde{\mathscr{P}}_-\begin{pmatrix} 1 & 0 \\ r_n c_n \widetilde{c}/w & 1\end{pmatrix} \quad \text{on} \quad \widetilde{U}_\delta \cap (\Delta_{n+}^\circ \cup \Delta_{n-}^\circ), \\ \widetilde{\mathscr{P}}_+ &= \widetilde{\mathscr{P}}_-\begin{pmatrix} 0 & w \\ -1/w & 0\end{pmatrix} \quad \text{on} \quad \widetilde{U}_\delta \cap \Delta_n^\circ;\end{aligned}$$

(c) $\widetilde{\mathscr{P}}\mathscr{N}^{-1} = \mathscr{I} + O(1/n)$ uniformly on $\partial \widetilde{U}_\delta$;

(d) For $\beta < 0$, $\widetilde{\mathscr{P}}$ has the following behavior near $z=-1$:

$$\widetilde{\mathscr{P}} = O\begin{pmatrix} 1 & |1+z|^\beta \\ 1 & |1+z|^\beta\end{pmatrix}, \quad \text{as} \quad \widetilde{U}_\delta \setminus \Sigma_n^{md} \ni z \to -1;$$

For $\beta = 0$, $\widetilde{\mathscr{P}}$ has the following behavior near $z=-1$:

$$\widetilde{\mathscr{P}} = O\begin{pmatrix} \log|1+z| & \log|1+z| \\ \log|1+z| & \log|1+z|\end{pmatrix}, \quad \text{as} \quad \widetilde{U}_\delta \setminus \Sigma_n^{md} \ni z \to -1;$$

For $\beta > 0$, $\widetilde{\mathscr{P}}$ has the following behavior near $z=-1$:

$$\widetilde{\mathscr{P}} = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}, & \text{as } z \to -1 \text{ outside the lens } \Sigma_n^{md}, \\ O\begin{pmatrix} |1+z|^{-\beta} & 1 \\ |1+z|^{-\beta} & 1\end{pmatrix}, & \text{as } z \to -1 \text{ inside the lens } \Sigma_n^{md}.\end{cases}$$

This problem is solved exactly in the same manner as RHP-$\mathscr{P}$. Thus, we set

$$\widetilde{A}_n(z) := \frac{\exp\left\{\frac{1}{2}\left(\theta_n(z) - \mathfrak{w}(z)\ell(z)\right)\right\}}{\sqrt{G_{h_n}}S_{h_n}(z)}(1-z)^{\alpha/2}(-1-z)^{\beta/2},$$

where the branch of $(1-z)^{\alpha/2}$ is the same as the corresponding one in w, and $(-1-z)^{\beta/2}$ is holomorphic in $\widetilde{U}_\delta \setminus \Delta_n$ and positive for negative real large enough. As in (7.4), we have that

$$\widetilde{A}_n^2 = \begin{cases} e^{\beta\pi i} w/c_n\widetilde{c}, & \text{in} \quad \widetilde{U}_\delta^+, \\ e^{-\beta\pi i} w/c_n\widetilde{c}, & \text{in} \quad \widetilde{U}_\delta^-,\end{cases}$$

where $\widetilde{U}_\delta^\pm$ have the same meaning as in the previous section. However, here one needs to be cautious since $\widetilde{g}_n$ reverses the orientation on Σ_2, i.e. Σ_2 is now oriented from zero to infinity, and therefore $\widetilde{U}_\delta^+$ is mapped into $\{z:\ \mathrm{Im}(z)<0\}$ and $\widetilde{U}_\delta^-$ into $\{z:\ \mathrm{Im}(z)>0\}$. Again, it can be checked that

$$\widetilde{A}_n^+\widetilde{A}_n^- = w \quad \text{on} \quad \Delta_n^\circ.$$

The following lemma can be proven exactly as Lemma 19 using that $(-1)^n r_n^{1/2} = e^{n\widetilde{g}_n}$.

Lemma 20. *The solution of* RHP-$\widetilde{\mathscr{P}}$ *is given by*

$$\widetilde{\mathscr{P}} = \widetilde{\mathscr{E}}\widetilde{\Psi}\left(\frac{n^2\widetilde{g}_n^2}{4}\right)\widetilde{A}_n^{-\sigma_3}(-1)^{n\sigma_3} r_n^{\sigma_3/2},\ \widetilde{\mathscr{E}} := \mathscr{N}\widetilde{A}_n^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1\end{pmatrix}(\pi n\widetilde{g}_n)^{\sigma_3/2}.$$

Finally, we are prepared to solve RHP-$\mathscr{A}$.

7.3. **Solution of** RHP-$\mathscr{A}$**.** Denote by Σ_n^{rd} the reduced system of contours obtained from Σ_n^{md} by removing Δ_n and $\Delta_{n\pm}\cap(U_\delta\cup\widetilde{U}_\delta)$ and adding $\partial U_\delta\cup\partial\widetilde{U}_\delta$. For this new system of contours we consider the following Riemann-Hilbert problem (RHP-$\mathscr{R}$):

(a) $\mathscr{R}$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n^{rd}$ and $\mathscr{R}(\infty)=\mathscr{I}$;

(b) the traces of $\mathscr{R}$, $\mathscr{R}_\pm$, are continuous on Σ_n^{rd} except for the branching points of Σ_n^{rd}, where they have definite limits from each sector and along each branch of Σ_n^{rd}. Moreover, $\mathscr{R}_\pm$ satisfy

$$\mathscr{R}_+ = \mathscr{R}_-\begin{cases} \mathscr{P}\mathscr{N}^{-1} & \text{on} \quad \partial U_\delta, \\ \widetilde{\mathscr{P}}\mathscr{N}^{-1} & \text{on} \quad \partial\widetilde{U}_\delta \\ \mathscr{N}\begin{pmatrix} 1 & 0 \\ r_n c_n\widetilde{c}/w & 1\end{pmatrix}\mathscr{N}^{-1} & \text{on} \quad \Sigma_n^{rd}\setminus(\partial U_\delta\cup\partial\widetilde{U}_\delta). \end{cases}$$

Then the following lemma takes place.

Lemma 21. *The solution of* RHP-$\mathscr{R}$ *exists for all n large enough and satisfies*

$$\mathscr{R} = \mathscr{I} + O(1/n), \tag{7.7}$$

where $O(1/n)$ holds uniformly in $\overline{\mathbb{C}}$. Moreover, $\det(\mathscr{R})=1$.

Proof. By RHP-$\mathscr{P}$ (c) and RHP-$\widetilde{\mathscr{P}}$ (c), we have that RHP-$\mathscr{R}$ (b) can be written as

$$\mathscr{R}_+ = \mathscr{R}_-(\mathscr{I}+O(1/n)) \tag{7.8}$$

uniformly on $\partial U_\delta\cup\partial\widetilde{U}_\delta$. Further, since the jump of $\mathscr{R}$ on $\Sigma_n^{rd}\setminus(\partial U_\delta\cup\partial\widetilde{U}_\delta)$ is analytic, it allows us to deform the problem RHP-$\mathscr{R}$ to a fixed contour, say Σ^{rd}, obtained from Σ like Σ_n^{rd} was obtained from Σ_n^{md} (the solutions exist, are simultaneously unique, and can be easily expressed through each other as in (7.1)). Moreover, by the properties of r_n, the jump of $\mathscr{R}$ on $\Sigma^{rd}\setminus(\partial U_\delta\cup\partial\widetilde{U}_\delta)$ is geometrically uniformly close to $\mathscr{I}$. Hence, (7.8) holds uniformly on Σ^{rd}. Thus, by [18, Cor. 7.108], RHP-$\mathscr{R}$ is solvable for all n large enough and $\mathscr{R}_\pm$ converge to zero on Σ^{rd} in L^2-sense as fast as $1/n$. The latter yields (7.7) locally uniformly in $\overline{\mathbb{C}}\setminus\Sigma^{rd}$. To show that (7.7) holds at $z\in\Sigma^{rd}$, deform Σ^{rd} to a new contour that avoids z (by making δ smaller or chosing different arcs to connect U_δ and $\widetilde{U}_\delta$). As the jump in RHP-$\mathscr{R}$ is given by analytic matrix functions, one can state an equivalent problem on this new contour, the solution to which is an analytic continuation of $\mathscr{R}$. However, now we have that (7.7) holds locally around z. Compactness of Σ^{rd} finishes the proof of (7.7).

Finally, as $\mathscr{N}$, $\mathscr{P}$, and $\widetilde{\mathscr{P}}$ have determinants equal to 1 throughout $\mathbb{C}$ [33, Rem. 7.1], $\det \mathscr{R}$ is an analytic function in $\overline{\mathbb{C}} \setminus \Sigma^{rd}$ that is equal to 1 at infinity, has equal boundary values on each side of $\Sigma^{rd} \setminus \{\text{branching points}\}$, and is bounded near the branching points. Thus, $\det(\mathscr{R}) \equiv 1$. □

Finally, we provide the solution of RHP-$\mathscr{A}$.

Lemma 22. *The solution of* RHP-$\mathscr{A}$ *exists for all n large enough and is given by* (7.1) *with*

$$\mathscr{B} := \begin{cases} \mathscr{R}\mathscr{N}, & \text{in} \quad \overline{\mathbb{C}} \setminus (\overline{U}_\delta \cup \overline{\widetilde{U}_\delta} \cup \Sigma_n), \\ \mathscr{R}\mathscr{P}, & \text{in} \quad U_\delta, \\ \mathscr{R}\widetilde{\mathscr{P}}, & \text{in} \quad \widetilde{U}_\delta, \end{cases} \tag{7.9}$$

where $\mathscr{R}$ is the solution of RHP-$\mathscr{R}$. *Moreover,* $\det \mathscr{A} \equiv 1$.

Proof. It can be easily checked from the definition of $\mathscr{P}$, $\widetilde{\mathscr{P}}$, and $\mathscr{N}$, that $\mathscr{B}$, given by (7.9), is the solution of RHP-$\mathscr{B}$. As $\det \mathscr{R} = \det(\mathscr{P}) = \det(\widetilde{\mathscr{P}}) \equiv 1$, it holds that $\det(\mathscr{A}) = \det(\mathscr{B}) \equiv 1$ in $\overline{\mathbb{C}}$, which finishes the proof of the lemma. □

8. $\bar\partial$-Problem

In the previous section we completed the first step in solving RH$\bar\partial$P-$\mathscr{S}$. That is we solved RHP-$\mathscr{A}$, the problem with the same conditions as in RH$\bar\partial$P-$\mathscr{S}$ except for the deviation from analyticity, which was entirely ignored. In this section, we solve a complementary problem, namely, we show that a solution of a certain $\bar\partial$-problem for matrix functions exists. Set

$$\mathscr{W} = \mathscr{A}\mathscr{W}_0\mathscr{A}^{-1}, \tag{8.1}$$

where $\mathscr{W}_0$ was defined in (6.8) and $\mathscr{A}$ is the solution of RHP-$\mathscr{A}$. In what follows, we seek the solution of the following $\bar\partial$-problem ($\bar\partial$P-$\mathscr{D}$):

(a) $\mathscr{D}$ is a continuous matrix function in $\overline{\mathbb{C}}$ and $\mathscr{D}(\infty) = \mathscr{I}$;
(b) $\mathscr{D}$ deviate from an analytic matrix function according to $\bar\partial \mathscr{D} = \mathscr{D}\mathscr{W}$, where the equality is understood in the sense of distributions.

Then the following lemma holds.

Lemma 23. *If* (2.9) *is fulfilled, then* $\bar\partial$P-$\mathscr{D}$ *is solvable for all n large enough. The solution $\mathscr{D}$ is unique and satisfies*

$$\mathscr{D} = \mathscr{I} + o(1), \tag{8.2}$$

where $o(1)$ satisfies (2.13).

Proof. We start by examining the summability and smoothness of the entries of $\mathscr{W}$. As $\mathscr{A}$ is the solution of RHP-$\mathscr{A}$, it is an analytic matrix function in $\Omega_\pm$ and its behavior near ± 1 is given by RH$\bar\partial$P-$\mathscr{S}$(c)–(d). Since $\det \mathscr{A} \equiv 1$ in $\overline{\mathbb{C}}$, the behavior of $\mathscr{A}^{-1}$ near ± 1 is also governed by the matrices in RH$\bar\partial$P-$\mathscr{S}$(c)-(d) with the elements on the main diagonal interchanged. Observe also that $|c_n c|$ is uniformly bounded in $\Omega_\pm$. Then a simple computation combined with Lemma 16 yields that

$$|\mathscr{W}_{lk}| \le \text{const.}|r_n f_{\alpha,\beta} f|, \quad l,k = 1,2, \tag{8.3}$$

where f comes from the decomposition of $\bar\partial c$ in Lemma 16 and $f_{\alpha,\beta}(z) := \log^2|1-z^2|$ if $\upsilon := \max\{|\alpha|,|\beta|\} = 0$ and $f_{\alpha,\beta}(z) := |1-z^2|^\upsilon$ otherwise.

Let s be as in (2.9) and denote by Ω the union $\Omega_+ \cup \Omega_- \cup \Delta^\circ$. When $\theta \in \mathrm{W}_p^{1-1/p}$, $p \in (2,\infty)$, it holds that $f \in \mathrm{L}^p(\Omega)$ by Lemma 16. Then we get from the Hölder inequality that

$$f_{\alpha,\beta} f \in \mathrm{L}^q(\Omega), \quad q \in \begin{cases} \left(2, \frac{2}{1+\upsilon-s}\right), & s-\upsilon \le 1, \\ (2,\infty], & s-\upsilon > 1, \end{cases} \tag{8.4}$$

since $f_{\alpha,\beta} \in \mathrm{L}^q(\Omega)$, $q \in \left(\frac{2p}{p-2}, \frac{2}{\upsilon}\right)$. When $\theta \in \mathrm{C}^{0,\varsigma}$, $\varsigma \in \left(\frac{1}{2}, 1\right]$, (8.4) can be obtained from the inclusion $\mathrm{C}^{0,\varsigma} \subset \mathrm{W}_q^{1-1/q}$ for $q \in \left(2, \frac{1}{1-\varsigma}\right)$. When $\theta \in \mathrm{C}^{m,\varsigma}$, $m \in \mathbb{N}$, $\varsigma \in (0,1]$, we have that[5] $f \in \mathrm{C}_0^{m-1,\varsigma-\epsilon}(\overline{\Omega})$, $\epsilon \in (0,\varsigma)$, by Lemma 16. This, in particular, implies that

$$|(f_{\alpha,\beta} f)(z)| \le \text{const.}|z^2-1|^{s-1-\upsilon-\epsilon},$$

which consequently yields (8.4).

Suppose now that $\bar\partial$P-$\mathscr{D}$ is solvable and $\mathscr{D}$ is a solution. Let Γ be a smooth arc encompassing $\overline{\Omega}$. Convolve $\mathscr{D}$ with a family of mollifiers so that the result is smooth and converges in the Sobolev sense to $\mathscr{D}$. Then by applying the Cauchy-Green formula (5.8) to this convolution and taking the limit, we get that

$$\mathscr{D} = \mathscr{C}_\Gamma(\mathscr{D}) + \mathscr{K}_\Gamma(\mathscr{D}\mathscr{W}) = \mathscr{I} + \mathscr{K}_{\mathscr{W}}\mathscr{D}$$

since $\mathscr{W}$ has compact support $\overline{\Omega}$, i.e., $\mathscr{D}$ is analytic outside of $\overline{\Omega}$, and $\mathscr{D}(\infty) = \mathscr{I}$, where $\mathscr{K}_{\mathscr{W}}(\cdot) = \mathscr{K}(\cdot\mathscr{W})$. Hence, every solution of $\bar\partial$P-$\mathscr{D}$ is a solution of the following integral equation

$$\mathscr{I} = (\mathscr{I} - \mathscr{K}_{\mathscr{W}})\mathscr{D}, \tag{8.5}$$

where $\mathscr{I}$ is the identity operator. As explained in Section 5.1, $\mathscr{I} - \mathscr{K}_{\mathscr{W}}$ is a bounded operator from $\mathrm{L}^\infty(\mathbb{C})$ into itself that maps continuous functions into continuous functions preserving their value at infinity. Conversely, if $\mathscr{D}$ is a solution of (8.5) in $\mathrm{L}^\infty(\mathbb{C}^{2\times 2})$ then $\mathscr{D}$ is, in fact, continuous in $\overline{\mathbb{C}}^{2\times 2}$, analytic outside of $\overline{\Omega}$, $\mathscr{D}(\infty) = \mathscr{I}$, and $\bar\partial\mathscr{D} = \mathscr{D}\mathscr{W}$ by (5.5) in the distributional sense. Thus, $\bar\partial$P-$\mathscr{D}$ is equivalent to uniquely solving (8.5) in $(\mathrm{L}^\infty(\mathbb{C}))^{2\times 2}$ because $\mathscr{D} - \mathscr{K}_{\mathscr{W}}\mathscr{D}$ is holomorphic in $\mathbb{C}$ and is identity at infinity.

We claim that

$$\|\mathscr{K}_{\mathscr{W}}\| \le \frac{C_0}{n^a}, \quad a \in \begin{cases} \left(0, \frac{s-\upsilon}{2}\right), & s-\upsilon \le 1, \\ \left(0, \frac{1}{2}\right), & s-\upsilon > 1, \end{cases} \tag{8.6}$$

where $\|\cdot\|$ is the norm of $\mathscr{K}_{\mathscr{W}}$ as an operator from $\mathrm{L}^\infty(\mathbb{C})$ into itself and the constant C_0 depends on a. Assuming this claim to be true, we get that $(\mathscr{I} - \mathscr{K}_{\mathscr{W}})^{-1}$ exists as a Neumann series and

$$\mathscr{D} = \mathscr{I} + O\left(\frac{\|\mathscr{K}_{\mathscr{W}}\|}{1 - \|\mathscr{K}_{\mathscr{W}}\|}\right),$$

which finishes the proof of the lemma granted the validity of (8.6). Thus, it only remains to prove estimate (8.6). To this end, observe that (8.3), (8.4), and the Hölder inequality imply

$$\|\mathscr{K}_{\mathscr{W}}\| \le C_1 \max_{z\in\overline{\Omega}} \left\| \frac{r_n f_{\alpha,\beta} f}{z - \cdot} \right\|_{1,\Omega} \le C_2 \max_{z\in\overline{\Omega}} \left\| \frac{r_n}{z-\cdot} \right\|_{q,\Omega}, \quad q \in \begin{cases} \left(\frac{2}{s-\upsilon+1}, 2\right), & s-\upsilon \le 1, \\ [1,2), & s-\upsilon > 1, \end{cases} \tag{8.7}$$

[5]Recall that by Lemma 16, f and all its partial derivatives up to and including the order $m-1$ have well-defined vanishing boundary values on Δ and therefore f is indeed $C^{m-1,\varsigma}$ throughout Ω.

where C_1, C_2 are constants depending on q and $\|\cdot\|_{q,\Omega}$ is the usual norm on $\mathrm{L}^q(\Omega)$. Thus, it holds that

$$\|\mathscr{K}_{\mathscr{W}}\| \le C_3 \|r_n\|_{q,\Omega}, \quad q \in \begin{cases} \left(\frac{2}{s-\upsilon}, \infty\right), & s-\upsilon \le 1, \\ (2,\infty), & s-\upsilon > 1, \end{cases} \tag{8.8}$$

by Hölder inequality, where C_3 is a constant depending on q.

In another connection, let ψ be the conformal map of D onto $\mathbb{D}$, $\psi(\infty)=0$, $\psi'(\infty)>0$, and let b_n be a Blaschke product with respect to D that has the same zeros as r_n counting multiplicities, i.e.,

$$b_n(z) = \prod_{r_n(e)=0} \frac{\psi(z)-\psi(e)}{1-\overline{\psi(e)}\psi(z)}, \quad z \in D.$$

Then by the maximum modulus principle for analytic functions and Definition 1-(1), we have that

$$|r_n(z)| \le \max_{t\in\Delta} |r_n^{\pm}(t)||b_n(z)| \le C_4 |b_n(z)|, \quad z \in D, \tag{8.9}$$

where C_4 is independent of n and z. Denote by L_ρ, $\rho \in (0,1)$, the level line of ψ, i.e. $L_\rho := \{z \in D : \ |\psi(z)| = \rho\}$. Due to Definition 1-(2), there exist $1 > \rho_0 > \rho_1 > 0$ such that $\overline{\Omega}$ is contained within the bounded domain with boundary L_{ρ_0}, say Ω_{ρ_0}, and all the zeros of b_n are contained within the unbounded domain with boundary L_{ρ_1}. Then

$$\|b_n\|_{q,\Omega} \le \|b_n\|_{q,\Omega_{\rho_0}} = \left\| (b_n \circ \psi^{-1}) \left((\psi^{-1})' \right)^2 \right\|_{q,\mathbb{A}_{\rho_0,1}}, \tag{8.10}$$

where $\mathbb{A}_{\rho_0,1} := \{z : \ \rho_0 < |z| < 1\}$, ψ^{-1} is the inverse of ψ, and the index q is as in (8.8). As ψ^{-1} is a conformal map of $\mathbb{D}$ onto D, it holds that $|(\psi^{-1})'| \le C_5$ in $\mathbb{A}_{\rho_0,1}$. Set

$$b_n^*(z) := (b_n \circ \psi^{-1})(z) = \prod_{b_n(\psi^{-1}(e^*))=0} \frac{z-e^*}{1-z\overline{e^*}}, \quad z \in \mathbb{D}.$$

Then by (8.8), (8.9), and (8.10) and after, we get that

$$\|\mathscr{K}_{\mathscr{W}}\| \le C_5 \|b_n^*\|_{q,\mathbb{A}_{\rho_0,1}}, \tag{8.11}$$

where the constant C_5 depends on q. Observe now that for $|z|=\rho$, $\rho \in (\rho_0, 1)$, it holds that

$$|b_n^*(z)| \le \prod \frac{\rho+|e^*|}{1+\rho|e^*|} \le \exp\left\{ -(1-\rho) \sum \frac{1-|e^*|}{1+\rho|e^*|} \right\} \le \exp\left\{ -2n \frac{(1-\rho)(1-\rho_1)}{1+\rho} \right\} \tag{8.12}$$

since $|e^*| < \rho_1$ by the definition of ρ_1. Clearly, (8.11) and (8.12) yield that

$$\|\mathscr{K}_{\mathscr{W}}\| \le C_5 \left(\int_0^{2\pi} \int_{\rho_0}^1 \exp\left\{ -2nq \frac{(1-\rho)(1-\rho_1)}{1+\rho} \right\} \rho \, d\rho \, dt \right)^{1/q} \le C_6 n^{-1/q},$$

which is exactly (8.6) with $a = 1/q$. □

9. Solution of RHP-$\mathscr{Y}$ and Proof of Theorem 3

In this last section, we gather the material from Sections 6–8 to prove Theorem 3. It is an immediate consequence of Lemmas 14, 15, and 17, combined with Lemmas 18, 22, and 23 that the following result holds.

Lemma 24. *If (2.9) is fulfilled, then the solution of* RHP-$\mathscr{Y}$ *uniquely exists for all n large enough and can be expressed by reversing the transformations $\mathscr{Y} \to \mathscr{T} \to \mathscr{S}$ using (6.5) and (6.7) with $\mathscr{S} = \mathscr{D}\mathscr{A}$, where $\mathscr{A}$ is the solution of* RHP-$\mathscr{A}$ *and $\mathscr{D}$ is the solution of* $\bar\partial$P-$\mathscr{D}$.

9.1. **Asymptotics away from Δ, formula (2.17).** We claim that (2.17) holds locally uniformly in D. Clearly, for any given closed set in D, it can be easily arranged that this set lies exterior to the lens Σ_n^{rd}, and therefore to the lenses Σ_n and Σ_{ext}. Thus, the asymptotic behavior of $\mathscr{Y}$ on this closed set is given by

$$\mathscr{Y} = \left(2^n \epsilon_n\right)^{-\sigma_3} \mathscr{R}\mathscr{N}\mathscr{D} E_n^{\sigma_3}$$

due to Lemma 24, where ϵ_n and E_n were defined in (6.2), $\mathscr{R}$ is the solution of RHP-$\mathscr{R}$ given by Lemma 22, and $\mathscr{N}$ is the solution of RHP-$\mathscr{N}$ given by (7.3). Moreover, we have that

$$\mathscr{R}\mathscr{N}\mathscr{D} = \left([1+o(1)]\,\mathscr{N}_{lk}\right)_{l,k=1,2}, \tag{9.1}$$

where $o(1)$ satisfies (2.13) locally uniformly in $\overline{\mathbb{C}} \setminus \{\pm 1\}$, including on $(\Delta^{\pm})^{\circ}$, on account of (7.7) and (8.2). Thus, it holds that

$$\begin{cases} \mathscr{Y}_{11} &= [1+o(1)]\mathscr{N}_{11}E_n/(2^n\epsilon_n) \quad = \varphi^n/(2^n S_{w\mathfrak{w}^+} S_{(hh_n/v_n)}) \\ \mathscr{Y}_{12} &= [1+o(1)]\mathscr{N}_{12}/(E_n 2^n \epsilon_n) \quad = 2iG_w G_{(hh_n/v_n)} S_w S_{(hh_n/v_n)}/(\varphi^{n+1} S_{\mathfrak{w}^+}) \end{cases}$$

by (6.2) and (7.3), where $o(1)$ satisfies (2.13) locally uniformly in D. Recall now that the entries of $\mathscr{N}$ are, in fact, continued Szegő functions defined with respect to Δ_n. However, we have already mentioned that they coincide with $S_{\mathfrak{w}^+}$ and S_w outside of a set exterior to $\Delta_n \cup \Delta$. Thus, the equations above indeed hold true. Hence, asymptotic formulae (2.17) follow from (6.1), (2.18), and (5.13).

9.2. **Asymptotics in the Bulk, Formula (2.19).** To derive asymptotic behavior of q_n and R_n on $\Delta \setminus \{\pm 1\}$, we need to consider what happens within the lens Σ_{ext} and outside the disks U_δ and $\widetilde{U}_\delta$. We shall consider the asymptotics of $\mathscr{Y}$ from within Ω_+, the upper part of the lens Σ_{ext}, the behavior of $\mathscr{Y}$ in Ω_- can be deduced in a similar fashion.

Recall that Δ_n either coincides with Δ or intersects it at finite number of points, as both arcs are images of $[-1,1]$ under holomorphic maps. Set

$$\Delta_n^* := \Delta \cap \Omega_{n+} \quad \Delta_n^{**} := \Delta \cap \Omega_{n-},$$

where Ω_{n+} and Ω_{n-} are the upper and lower parts of the lens Σ_n^{md}. Then, it holds that

$$\mathscr{A}_+ = \begin{cases} \mathscr{B}_+, & \text{on} \quad \Delta_n^*, \\ \mathscr{B}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}, & \text{on} \quad \Delta_n^{**}, \end{cases} \tag{9.2}$$

by (7.1), where with a slight abuse of notation we denote by $\mathscr{B}_\pm$ the values of $\mathscr{B}$ in $\Omega_{n\pm}$ and on $\Delta_n^{\pm}$. Then it holds on Δ° by RHP-$\mathscr{N}$(b) and on account of Lemma 24, (9.2), and (7.9) that

$$\mathscr{S}_+ = \begin{cases} \mathscr{R}\mathscr{N}_+\mathscr{D}, & \text{on} \quad \Delta_n^* \\ \mathscr{R}\mathscr{N}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \mathscr{D}, & \text{on} \quad \Delta_n^{**} \end{cases} = \mathscr{R}\widetilde{\mathscr{N}}_+\mathscr{D},$$

where, again, under $\mathscr{N}_\pm$ we understand the values of $\mathscr{N}$ in $\Omega_{n\pm}$ and on $\Delta_n^{\pm}$ and $\widetilde{\mathscr{N}}$ is the analytic continuation of $\mathscr{N}$ that satisfies RHP-$\mathscr{N}$, only with a jump across Δ. Clearly, $\widetilde{\mathscr{N}}$ is defined by (7.2) and (7.3), where $S_{\mathfrak{w}^+}$ and S_w are the Szegő functions of $\mathfrak{w}^+$ and w with respect to Δ and not

the continued functions that actually appear in (7.2) and (7.3). Thus, we deduce from Lemma 24 and (9.1) that

$$\begin{aligned}\mathscr{Y}_+ &= \left(2^n\epsilon_n\right)^{-\sigma_3}\left([1+o(1)]\,\widetilde{\mathscr{N}}^+_{lk}\right)_{l,k=1,2}\begin{pmatrix}1 & 0\\ (r_nc_nc)^+/w & 1\end{pmatrix}(E_n^+)^{\sigma_3}\\ &= \left(2^n\epsilon_n\right)^{-\sigma_3}\left([1+o(1)]\,\widetilde{\mathscr{N}}^+_{lk}\right)_{l,k=1,2}\begin{pmatrix}E_n^+ & 0\\ E_n^-/w & 1/E_n^+\end{pmatrix}\end{aligned}$$

where $o(1)$ satisfies (2.13) locally uniformly on Δ° and we used (6.4) to obtain the second equality. Therefore, it holds that

$$\begin{cases}\mathscr{Y}_{11} &= [1+o(1)]\widetilde{\mathscr{N}}^+_{11}E_n^+/(2^n\epsilon_n)+[1+o(1)]\widetilde{\mathscr{N}}^+_{12}E_n^-/(2^n\epsilon_n w)\\ \mathscr{Y}^+_{12} &= [1+o(1)]\widetilde{\mathscr{N}}^+_{12}/(E_n^+2^n\epsilon_n)\end{cases}$$

with $o(1)$ satisfying (2.13) locally uniformly on Δ°. As in the end of the previous section, we deduce (2.19) from (6.1), the formulae

$$\frac{\widetilde{\mathscr{N}}^\pm_{11}E_n^\pm}{2^n\epsilon_n}=\frac{1}{S_n^\pm}\quad\text{and}\quad\frac{\widetilde{\mathscr{N}}^+_{12}}{E_n^+2^n\epsilon_n}=\frac{S_n^+}{\mathfrak{w}^+},$$

and by noticing that

$$\frac{1}{w}\frac{\widetilde{\mathscr{N}}^+_{12}}{\widetilde{\mathscr{N}}^-_{11}}=\frac{1}{w}\frac{G_wS_w^+S_w^-S^-_{\mathfrak{w}^+}}{\varphi^+S^+_{\mathfrak{w}^+}}=\frac{iS^-_{\mathfrak{w}^+}}{\varphi^+S^+_{\mathfrak{w}^+}}\equiv 1$$

on Δ° by (2.12) and (5.14).

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