RATIOS OF NORMS FOR POLYNOMIALS AND CONNECTED n-WIDTH PROBLEMS

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Dedicated to Björn Gustafsson on the occasion of his 60th birthday

ABSTRACT. Let $G \subset \mathbb{C}$ be a bounded simply connected domain with boundary Γ and let $E \subset G$ be a regular compact set with connected complement. In this paper we investigate asymptotics of the extremal constants:

$$\chi_n = \inf_{p \in \mathscr{P}_{k_n}} \sup_{q \in \mathscr{P}_{n-k_n}} \frac{\|pq\|_E}{\|pq\|_\Gamma}, \quad n = 1, 2, \ldots,$$

where $\|\cdot\|_K$ is the supremum norm on a compact set K, $\mathscr{P}_m$ is the set of all algebraic polynomials of degree at most m, and $k_n/n \to \theta \in [0,1]$ as $n \to \infty$. Subsequently, we obtain asymptotic behavior of the Kolmogorov k-widths, $k = k_n$, of the unit ball A_n^∞ of $H^\infty \cap \mathscr{P}_n$ restricted to E in $C(E)$, where H^∞ is the Hardy space of bounded analytic functions on G and $C(E)$ is the space of continuous functions on E.

1. INTRODUCTION

The Kolmogorov k-width of a set A contained in a Banach space X is defined by

$$d_k(A; X) := \inf_{X_k} \sup_{h \in A} \inf_{g \in X_k} \|h - g\|,$$

where X_k runs over all k-dimensional subspaces of X and $\|\cdot\|$ is a norm on X. Let G be a bounded simply connected domain with boundary Γ in the complex plane $\mathbb{C}$, and H^∞ be the Hardy space of bounded analytic functions in G. Denote by $E \subset G$ a regular compact set with connected complement D and A^∞ the unit ball of H^∞ restricted to E. In [12] H. Widom investigated the asymptotic behavior of $d_k(A^\infty; C(E))$, where $C(E)$ is the space of continuous functions on E endowed with the usual supremum norm $\|\cdot\|_E$. It is proved that

$$\lim_{k \to \infty} \left(\frac{1}{k} \log d_k(A^\infty; C(E)) \right) = -\frac{1}{\mathrm{cap}(E, \Gamma)}, \tag{1.1}$$

where $\mathrm{cap}(E, \Gamma)$ is the *condenser (Green) capacity* of E with respect to G (see, for example, [10, Sec. II.5]). Further, in [2] (see also [1, Sec. 7.5]) S. D. Fisher and C. A. Micchelli obtained the following representation for $d_k(A^\infty; C(E))$:

$$d_k(A^\infty; C(E)) = \inf_{z_1, \ldots, z_k} \sup\{\|h\|_E : h \in A^\infty, \ h(z_j) = 0, \ j = 1, \ldots, k\}. \tag{1.2}$$

Clearly, it is enough to consider only the *Blaschke products* instead of all functions from A^∞ in (1.2). Then it is a consequence of [3] that the zero counting measures of any asymptotically extremal

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sequence of Blaschke products swept out to ∂E converge weak-star to the *Green equilibrium distribution* on E relative to G.

In this paper we investigate the n-th root behavior of $d_k(A_n^\infty; C(E))$, $k = k_n$, the Kolmogorov k-widths of the unit ball A_n^∞ of $H^\infty \cap \mathscr{P}_n$ restricted to E in $C(E)$, and show its connection to the following extremal problem:

$$\chi_n = \inf_{p\in\mathscr{P}_{k_n}} \sup_{q\in\mathscr{P}_{n-k_n}} \frac{\|pq\|_E}{\|pq\|_\Gamma}, \tag{1.3}$$

where

$$\lim_{n\to\infty} \frac{k_n}{n} = \theta, \quad \theta \in [0,1]. \tag{1.4}$$

Analogous k-width occur in the study of truncated Hankel operators which the authors will explore in a later paper.

Regarding the minimax problem defined in (1.3), we observe that it connects two well-understood extremal problems of potential theory. It is an simple consequence of the Bernstein-Walsh inequality ([11] and [10, Sec. III.2]) and properties of the Chebyshev polynomials for E that

$$\lim_{n\to\infty} \left(\inf_{p\in\mathscr{P}_n} \frac{\|p\|_E}{\|p\|_\Gamma} \right)^{1/n} = \exp\left\{ -\max_{z\in\Gamma} g(z,\infty) \right\}, \tag{1.5}$$

where $g(\cdot,\infty)$ is the Green function for D with singularity at infinity. It is easy to see that

$$\lim_{n\to\infty} \left(\sup_{q\in\mathscr{P}_n} \frac{\|q\|_E}{\|q\|_\Gamma} \right)^{1/n} = 1 \tag{1.6}$$

and the extremal polynomial is $q \equiv 1$. Furthermore, it is readily verified that polynomials $z^n - R^n$ are asymptotically extremal for (1.6) whenever R is such that $\{|z| < R\} \supset G$. Let us also illustrate extremal problem (1.5). Put E to be the closed unit disk $\overline{\mathbb{D}}$ and Γ to be the circle of radius $R > 2$ centered at 1. In this case $g(z,\infty) = \log|z|$ and therefore the monomials z^n are extremal for (1.5) and the limit is equal to $1/(R+1)$. Moreover, the polynomials $z^n - 1$ are asymptotically extremal for that problem.

This paper is organized as follows. In Section 2 we consider two minimal energy problems, one for the Green potentials and another for the logarithmic potentials, that are vital for our main results. The latter are given in Section 3, which contains results on the behavior of χ_n and the extremal polynomials (Theorems 3.1 and 3.2) as well as connection with n-width (Theorem 3.3). In Section 4 we study some extremal problem of the potential theory which can be considered as an continuous analog of the extremal problem (1.3). In Section 6 we provide a detailed description of the extremal measures defined in Section 2. Sections 5 and 7 of this paper consist of proofs of the stated results. In Section 8 we investigate the asymptotics of k-widths.

2. Equilibrium Measures

Let G, Γ, E, and D be as described. We shall use the standard terminology that a property holds *quasi-everywhere* (q.e.) if it holds everywhere except for a set of zero *logarithmic capacity* (see [10, Sec. I.1] or [9, Sec. 5] for the definition of capacity). In this paper we extensively utilize *logarithmic* and *Green potentials*. The logarithmic potential of a finite positive Borel measure ν with compact support $\mathrm{supp}(\nu)$, is given by

$$U^\nu(z) = -\int \log|z-t| d\nu(t).$$

It is superharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C}\setminus \operatorname{supp}(\nu)$. Unlike the logarithmic case, Green potentials are defined relative to a domain. Let ν be a positive Borel measure compactly supported in D. Then the Green potential of ν relative to D is given by

$$U_D^\nu(z) = \int g(z,t)d\nu(t),$$

where $g(z,t) = g_E(z,t)$ is the Green function for D with singularity at $t \in D$. Since E is a regular compact set, $g(z,t) = 0$ for $z \in \partial D = \partial E$. Here and in what follows we assume that $g(z,t) = 0$ for all $z \in E$. The Green potential of ν is nonnegative and superharmonic in D, harmonic in $D \setminus \operatorname{supp}(\nu)$, and satisfies $U_D^\nu = 0$ on E.

Let K be a compact set. Denote by $\Lambda_\delta(K)$, $\delta > 0$, the set of positive Borel measures λ of mass $\delta = |\lambda| = \int d\lambda$ compactly supported on K.

For each $\theta \in [0,1)$ consider the following weighted Green energy of a measure $\lambda \in \Lambda_{1-\theta}(\Gamma)$:

$$J_\theta(\lambda) := \int\int g(z,t)d\lambda(t)d\lambda(z) - 2\int g(t,\infty)d\lambda(t). \tag{2.1}$$

Then we have the following result.

Theorem 2.1. *For each $\theta \in [0,1)$ there exists a unique measure $\lambda_\theta \in \Lambda_{1-\theta}(\Gamma)$ such that*

$$\min_{\lambda\in\Lambda_{1-\theta}(\Gamma)} J_\theta(\lambda) = J_\theta(\lambda_\theta). \tag{2.2}$$

The extremal measure λ_θ satisfies the following properties:

$$U_D^{\lambda_\theta}(z) - g(z,\infty) = m_\theta, \quad z \in S_\theta := \operatorname{supp}(\lambda_\theta) \subseteq \Gamma, \tag{2.3}$$

and

$$U_D^{\lambda_\theta}(z) - g(z,\infty) \geq m_\theta, \quad z \in \Gamma, \tag{2.4}$$

where

$$m_\theta := \frac{1}{1-\theta}\left(J_\theta(\lambda_\theta) + \int g(t,\infty)d\lambda_\theta(t)\right). \tag{2.5}$$

Remarks. (a) This theorem is a special case of [10, Thm. II.5.10] for the external field $-g(\cdot,\infty)/(1-\theta)$. We exhibit the dependence of λ_θ on θ in Theorem 6.1 (see also [6, Thm. 2.4]). In particular, $\lambda_{\theta_1} - \lambda_{\theta_2}$ is a positive measure for any choice of $\theta_1 < \theta_2$.

(b) In general, (2.3) holds only q.e. on S_θ. However, as pointed out in [6, Thm. 2.2], the regularity of Γ is sufficient for this property to hold at every point of S_θ.

(c) As shown later in Lemma 4.2, $m_0 = 0$, $\lambda_0 = \omega_\Gamma$, and $S_0 = \Gamma$, where ω_K stands for the *logarithmic equilibrium distribution* on a set K.

(d) It follows from Theorem 6.1, the limit of m_θ as θ approaches 1 from the left exists and

$$m_1 := \lim_{\theta\to 1^-} m_\theta = -\max_{z\in\Gamma} g(z,\infty). \tag{2.6}$$

Furthermore, we define λ_1 to be the zero measure.

(e) The measure λ_θ is uniquely determined by conditions (2.3) and (2.4). If $\lambda \in \Lambda_{1-\theta}(\Gamma)$ has a finite Green energy, $U_D^\lambda - g(z,\infty) = c$ on $\operatorname{supp}(\lambda)$ and $U_D^\lambda - g(z,\infty) \geq c$ on Γ, then $\lambda = \lambda_\theta$ and $c = m_\theta$ (see [10, Thm. II. 5.12]).

Let us consider the special case when Γ is a level curve of $g(\cdot,\infty)$.

Example 2.1. *Let $\Gamma = \{z : g(z,\infty) = R\}$ for some constant $R > 0$. Then for every $\theta \in [0,1)$ we have*

$$\lambda_\theta = (1-\theta)\omega_\Gamma \quad \text{and} \quad m_\theta = -\theta R.$$

It is easy to see (cf. Lemma 4.2) that

$$U_D^{(1-\theta)\omega_\Gamma}(z)-g(z,\infty)=-\theta g(z,\infty)=-\theta R,\quad z\in\Gamma.$$

Therefore, by Remark (e), we get

$$\lambda_\theta=(1-\theta)\omega_\Gamma,\quad m_\theta=-\theta R.$$

The second extremal problem that we need and which, in a way, is complimentary to (2.2), is related to the following energy integral:

$$I_\theta(\mu):=\int U^\mu(t)d\mu(t)+2\int U^{\lambda_\theta}(t)d\mu(t),$$

where $\theta\in(0,1]$ and $\mu\in\Lambda_\theta(E)$.

As in the case of Theorem 2.1, the following is known [10, Thm. I.1.3].

Theorem 2.2. *For each $\theta\in(0,1]$ there exists a unique measure $\mu_\theta\in\Lambda_\theta(E)$ such that*

$$\min_{\mu\in\Lambda_\theta(E)} I_\theta(\mu)=I_\theta(\mu_\theta). \tag{2.7}$$

Moreover, the extremal measure μ_θ has the following properties:

$$U^{\mu_\theta+\lambda_\theta}(z)=\widehat{m}_\theta,\quad z\in\operatorname{supp}(\mu_\theta)\subseteq\partial E, \tag{2.8}$$

and

$$U^{\mu_\theta+\lambda_\theta}(z)\geq\widehat{m}_\theta,\quad z\in E, \tag{2.9}$$

where

$$\widehat{m}_\theta:=\frac{1}{\theta}\left(I_\theta(\mu_\theta)-\int U^{\lambda_\theta}(t)d\mu_\theta(t)\right). \tag{2.10}$$

Remarks. (a) For $\theta=1$, (2.7) reduces to the classical (unweighted) minimal energy problem (cf. [9, Sec. 3.3] and [10, Sec. I.1]) when $\theta=1$. In this case, μ_1 is the logarithmic equilibrium distribution ω_E and and $\widehat{m}_1$ is the Robin constant for E, $\widehat{m}_1=-\log\operatorname{cap}(E)$, where $\operatorname{cap}(E)$ is the logarithmic capacity of E.

(b) It is a well-known fact that $\operatorname{supp}(\mu_\theta)\subseteq\partial E$ (see, for example, [10, Thm. IV.1.10(a)]).

(c) As in the case of λ_1, it is convenient for us to define μ_0 to be the zero measure.

(d) The measure μ_θ is uniquely determined by conditions (2.8) and (2.9). If $\mu\in\Lambda_\theta(E)$ has a finite energy and $U^{\mu+\lambda_\theta}=c$ on $\operatorname{supp}(\mu)$ and $U^{\mu+\lambda_\theta}\geq c$ on E, then $\mu=\mu_\theta$ and $c=\widehat{m_\theta}$ (cf. [10, Thm. I. 3.3]).

Further properties of λ_θ and μ_θ and the constants m_θ and $\widehat{m}_\theta$ are given in Section 6, including asymptotics as $\theta\to0$.

3. Main Results

Let χ_n be defined by (1.3) and (1.4). Below we show that $\lim\chi_n^{1/n}$ exists and provide the asymptotic behavior of the zeros of the *extremal polynomials*. The latter are defined as follows. Let $\{p_n,q_n\}_{n\in\mathbb{N}}$, $p_n\in\mathcal{P}_{k_n}$ and $q_n\in\mathcal{P}_{n-k_n}$, be such that

$$\lim_{n\to\infty}\left(\frac{1}{\chi_n}\frac{\|p_nq_n\|_E}{\|p_nq_n\|_\Gamma}\right)^{1/n}=\lim_{n\to\infty}\left(\frac{1}{\chi_n}\sup_{q\in\mathcal{P}_{n-k_n}}\frac{\|p_nq\|_E}{\|p_nq\|_\Gamma}\right)^{1/n}=1. \tag{3.1}$$

We shall call $\{p_n, q_n\}$, satisfying the equalities above, a *sequence of asymptotically extremal pairs* of polynomials. To each such p_n and q_n we associate the zero counting measures, $\nu(p_n)$ and $\nu(q_n)$, respectively, defined by the rule

$$\nu(p_n) := \frac{1}{n} \sum_{p_n(z)=0} \delta_z \quad \text{and} \quad \nu(q_n) := \frac{1}{n} \sum_{q_n(z)=0} \delta_z,$$

where δ_z is the point mass distribution at $z \in \mathbb{C}$ and the sums are taken counting multiplicities of zeros of polynomials p_n and q_n. It also will be convenient for us to sweep out (*balayage*) measures $\nu(p_n)$ and $\nu(q_n)$ onto ∂E and Γ, respectively. Recall that for any finite positive Borel measure ν compactly supported in $\mathbb{C}$ and, with finite energy if $\operatorname{supp}(\nu) \cap \partial D \neq \emptyset$, there exists a unique measure $\widehat{\nu}$, the balayage measure of ν, supported on ∂E, such that $|\nu| = |\widehat{\nu}|$,

$$U^{\widehat{\nu}}(z) = U^{\nu}(z) + \int g(t,\infty) d\nu(t), \quad z \in E, \quad \text{if} \quad \operatorname{supp}(\nu) \subset D,$$

and

$$U^{\widehat{\nu}}(z) = U^{\nu}(z), \quad z \in \overline{D}, \quad \text{if} \quad \operatorname{supp}(\nu) \subseteq E.$$

We remark that for any positive compactly supported in D measure ν,

$$U_D^{\nu}(z) = U^{\nu - \widehat{\nu}}(z) + \int g(t,\infty) d\nu(t), \quad z \in \mathbb{C}. \tag{3.2}$$

Denote by $\widetilde{\nu}$ the balayage of a finite positive Borel measure ν compactly supported in $\mathbb{C} \setminus \overline{G}$ onto Γ. We have $|\widetilde{\nu}| = |\nu|$ and

$$U^{\widetilde{\nu}}(z) = U^{\nu}(z) + \int g_{\overline{G}}(t,\infty)\, d\nu(t), \quad z \in \overline{G}, \tag{3.3}$$

where $g_{\overline{G}}(z,\infty)$ is the Green function of the domain $\overline{\mathbb{C}} \setminus \overline{G}$ with singularity at infinity. Now we define measures $\alpha(p_n)$ and $\beta(q_n)$ as

$$\alpha(p_n) := \nu(p_n)_{|\overline{D}} + \widehat{\nu(p_n)_{|E \setminus \partial E}}, \tag{3.4}$$

and

$$\beta(q_n) := \nu(q_n)_{|\overline{G}} + \widetilde{\nu(q_n)_{|\mathbb{C} \setminus \overline{G}}} + \frac{n - k_n - \deg(q_n)}{n}\, \omega_\Gamma, \tag{3.5}$$

respectively, where a notation $\lambda_{|K}$ means restriction of a measure λ on a set K. Let $\mathcal{M}(E) = \{\nu : \nu \in \Lambda_\theta(E),\ \widehat{\nu} = \mu_\theta\}$.

The following result holds.

Theorem 3.1. *Let $\{k_n\}_{n\in\mathbb{N}}$ satisfy (1.4) for some $\theta \in [0,1]$. Then*

$$\lim_{n\to\infty} \left(\frac{1}{n} \log \chi_n \right) = m_\theta, \tag{3.6}$$

where m_θ was defined in (2.5) and (2.6). If $\{p_n, q_n\}$ is a sequence of asymptotically extremal pairs of polynomials in the sense (3.1), then, for $\theta \in (0,1)$ any weak-star limit point of $\{\nu(p_n)\}$ belongs to $\mathcal{M}(E)$ and

$$\alpha(p_n) \xrightarrow{*} \mu_\theta \quad \textit{as} \quad n \to \infty, \tag{3.7}$$

where $\xrightarrow{*}$ stands for the convergence of measures in the weak-star sense. Moreover,

$$\begin{aligned} \nu(q_n) &\xrightarrow{*} \lambda_\theta \quad \text{as } n \to \infty, \text{ if } \overline{\mathbb{C}} \setminus S_\theta \text{ is connected}, \\ \beta(q_n) &\xrightarrow{*} \lambda_\theta \quad \text{as } n \to \infty, \text{ otherwise}. \end{aligned} \tag{3.8}$$

Remarks. (a) Since $m_0 = 0$ and $m_1 = -\max_{z\in\Gamma} g(z,\infty)$, (1.3) indeed connects extremal problems (1.5) and (1.6).

(b) Observe that $\alpha(p_n) = \nu(p_n)$ when E has empty interior. In this case (3.7) is a statement on the convergence of counting measures themselves, rather than their balayages onto ∂E.

The following theorem is related to the case when $k_n \to \infty$ and $k_n = o(n)$ as $n \to \infty$. To formulate the result, we need to slightly modify the definition of an asymptotically extremal sequence. We say that a sequence $\{p_n\}$ is *asymptotically extremal* if

$$\lim_{n\to\infty} \left(\frac{1}{\chi_n} \sup_{q\in\mathscr{P}_{n-k_n}} \frac{\|p_n q\|_E}{\|p_n q\|_\Gamma} \right)^{1/k_n} = 1. \tag{3.9}$$

Notice that for $\theta > 0$ definitions (3.1) and (3.9) coincide. Let

$$\nu^*(p_n) = \frac{1}{k_n} \sum_{p_n(z)=0} \delta_z \tag{3.10}$$

and

$$\alpha^*(p_n) = \nu^*(p_n)_{|\overline{D}} + \widehat{\nu^*(p_n)_{|E\setminus\partial E}}. \tag{3.11}$$

We remark that $|\nu^*(p_n)| \le 1$ and $|\alpha^*(p_n)| \le 1$. Let $\mathscr{N}(E) = \{\nu : \nu \in \Lambda_1(E), \widehat{\nu} = \omega_{(E,\Gamma)}\}$, where an $\omega_{(E,\Gamma)}$ is the Green equilibrium distribution on E relative to G.

Theorem 3.2. *Let $k_n \to \infty$ and $k_n = o(n)$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \left(\frac{1}{k_n} \log \chi_n \right) = -\frac{1}{\operatorname{cap}(E,\Gamma)}. \tag{3.12}$$

Moreover, if $\{p_n\}$ is an asymptotically extremal sequence in the sense of (3.9), then any weak-star limit point of $\{\nu^*(p_n)\}$ belongs to $\mathscr{N}(E)$ and

$$\alpha^*(p_n) \xrightarrow{*} \omega_{(E,\Gamma)} \quad \text{as} \quad n \to \infty. \tag{3.13}$$

The last theorem provides the asymptotic behavior of the Kolmogorov k-width, $k = k_n$, of A_n^∞ in $C(E)$. To formulate this theorem we need to introduce more notation. Fix $\theta \in (0,1]$ and define

$$G_\theta := \left\{ z \in \mathbb{C} : \; U_D^{\lambda_\theta}(z) - g(z,\infty) > m_\theta \right\}.$$

For $\theta = 0$ we simply set $G_0 := G$. Clearly, the maximum principle for harmonic functions implies that $G_\theta = G$ whenever $S_\theta = \Gamma$ and it follows from (2.3) and (2.4) that $G \subseteq G_\theta$ for all $\theta \in [0,1]$. Let $G \subseteq G' \subseteq G_\theta$, $H^\infty(G')$ be the space of bounded analytic functions on G', and $A_n^\infty(G')$ stand for the restriction to E of the unit ball of $H^\infty(G') \cap \mathscr{P}_n$. The following theorem shows that the n-th root limit of $d_{k_n}(A_n^\infty(G'); C(E))$, $k_n/n \to \theta$, is independent of G'.

Theorem 3.3. *Let $\{k_n\}_{n\in\mathbb{N}}$ satisfy (1.4) for some $\theta \in [0,1]$, G' be a simply connected domain such that $G \subseteq G' \subseteq G_\theta$, and $A_n^\infty = A_n^\infty(G')$. Then*

$$\lim_{n\to\infty} \left(\frac{1}{n} \log d_{k_n}(A_n^\infty; C(E)) \right) = m_\theta. \tag{3.14}$$

In particular, when $\theta = 0$ and $k_n \to \infty$ as $n \to \infty$, we have that

$$\lim_{n\to\infty}\left(\frac{1}{k_n}\log d_{k_n}(A_n^\infty; C(E))\right) = -\frac{1}{\operatorname{cap}(E,\Gamma)}. \tag{3.15}$$

4. An Extremal Problem of the Potential Theory

We now state the main theorem of this section. Let σ be a compactly supported positive Borel measure. Define

$$M(\sigma) := \min_\Gamma U^\sigma - \min_E U^\sigma.$$

Theorem 4.1. *For each $\theta \in [0,1]$ we have*

$$m_\theta = \inf_{\mu\in\Lambda_\theta(E)} \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu+\lambda) = \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} \inf_{\mu\in\Lambda_\theta(E)} M(\mu+\lambda). \tag{4.1}$$

Moreover, if μ^*, $|\mu^*| \le \theta$, and λ^*, $|\lambda^*| \le 1-\theta$, are compactly supported positive Borel measures such that

$$m_\theta = M(\mu^*+\lambda^*) = \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu^*+\lambda) \tag{4.2}$$

then $\operatorname{supp}(\mu^*) \subseteq E$, $\widehat{\mu^*} = \mu_\theta$, *and $\lambda^* = \lambda_\theta$ when S_θ does not separate the plane and* $\operatorname{supp}(\lambda^*) \subset \mathbb{C}\setminus G$, $\widetilde{\lambda}^* = \lambda_\theta - (1-\theta-|\lambda^*|)\omega_\Gamma$, *otherwise.*

The proof of Theorem 4.1 is based on several auxiliary lemmas.

Lemma 4.2. *We have*

$$\lambda_0 = \omega_\Gamma, \quad \widehat{\lambda}_0 = \omega_E, \quad \textit{and} \quad m_0 = 0. \tag{4.3}$$

Proof. Since $U^{\omega_\Gamma}(z) = -\log\operatorname{cap}(\Gamma)$ for $z \in \overline{G}$ and $U^{\omega_E}(z) = -\log\operatorname{cap}(E)$ for $z \in E$, it holds that

$$U^{\omega_E}(z) = U^{\omega_\Gamma}(z) + c, \quad z \in E,$$

where $c = -\log\operatorname{cap}(E) + \log\operatorname{cap}(\Gamma)$. Using now the fact $\operatorname{supp}(\omega_E) = \partial E$ and the uniqueness of the balayage (see, for example, [10, Thm. II.4.4]), we can immediately conclude that

$$\widehat{\omega}_\Gamma = \omega_E \tag{4.4}$$

and

$$\int g(t,\infty)\,d\omega_\Gamma(t) = -\log\operatorname{cap}(E) + \log\operatorname{cap}(\Gamma). \tag{4.5}$$

From this, on account of the formula

$$U^{\omega_E}(z) = -\log\operatorname{cap}(E) - g(z,\infty), \quad z \in \mathbb{C}, \tag{4.6}$$

we obtain that for every $z \in \mathbb{C}$,

$$\begin{aligned} U_D^{\omega_\Gamma}(z) - g(z,\infty) &= U^{\omega_\Gamma - \omega_E}(z) + \int g(t,\infty)\,d\omega_\Gamma(t) - g(z,\infty) \\ &= U^{\omega_\Gamma}(z) + \log\operatorname{cap}(\Gamma). \end{aligned} \tag{4.7}$$

So,

$$U_D^{\omega_\Gamma}(z) - g(z,\infty) = 0, \quad z \in \Gamma.$$

Therefore, relations $\lambda_0 = \omega_\Gamma$ and $m_0 = 0$ follow from the uniqueness of the measure λ_θ satisfying conditions (2.3) and (2.4) (see Remark (e) after Theorem 2.1). □

Lemma 4.3. *For each* $\theta \in [0,1]$ *we have*

$$\mu_\theta = \widehat{\lambda_0 - \lambda_\theta} = \omega_E - \widehat{\lambda_\theta}, \tag{4.8}$$

$$\widehat{m}_\theta = -\log \operatorname{cap}(E) - \int g(t,\infty)\, d\lambda_\theta(t), \tag{4.9}$$

and

$$U_D^{\lambda_\theta}(z) - g(z,\infty) = U^{\lambda_\theta+\mu_\theta}(z) - \widehat{m}_\theta, \quad z \in \mathbb{C}. \tag{4.10}$$

Moreover,

$$\operatorname{supp}(\mu_\theta) = \partial E, \quad \theta \in (0,1]. \tag{4.11}$$

Proof. It is easy to see that for $\theta = 0$ and $\theta = 1$ (4.8), (4.9) and (4.10) are valid. In the case $\theta = 0$, $\mu_0 = 0$ by definition, and Lemma 4.3 implies that $\lambda_0 = \omega_\Gamma$ and $\widehat{\lambda}_0 = \omega_E$. From this, on account of (4.5) and (4.7) we get (4.9) and (4.10). For $\theta = 1$, $\lambda_1 = 0$ by definition. It follows from Theorem 2.2 (see Remark (a) after Theorem 2.2) that $\mu_1 = \omega_E$ and $\widehat{m}_1 = -\log \operatorname{cap}(E)$. We also note that since $\operatorname{supp}(\omega_E) = \partial E$, (4.11) holds for $\theta = 1$.

Let us consider now the case $\theta \in (0,1)$. We start from the next observation. As noted in the Remark (b) after Theorem 3.1, it follows from [6, Thm. 2.4], that $\lambda_0 - \lambda_\theta$ is a positive measure. It is easy to see that $|\lambda_0 - \lambda_\theta| = \theta$. Hence, $\widehat{\lambda_0 - \lambda_\theta}$ is a positive measure, and $|\widehat{\lambda_0 - \lambda_\theta}| = \theta$. Moreover, it is a simple application of the second unicity theorem [10, Thm. II.4.6] to see that

$$\widehat{\lambda_0 - \lambda_\theta} = \widehat{\lambda}_0 - \widehat{\lambda}_\theta.$$

So, $\widehat{\lambda}_0 - \widehat{\lambda}_\theta = \omega_E - \widehat{\lambda}_\theta$ is a positive measure and $|\omega_E - \widehat{\lambda_\theta}| = \theta$. According to the property (3.2) of the Green potential,

$$U_D^{\lambda_\theta}(z) - g(z,\infty) = U^{\lambda_\theta - \widehat{\lambda}_\theta}(z) + \int g(t,\infty)\, d\lambda_\theta(t) - g(z,\infty), \quad z \in \mathbb{C}, \tag{4.12}$$

and, by (4.6),

$$U_D^{\lambda_\theta}(z) - g(z,\infty) = U^{\lambda_\theta + \omega_E - \widehat{\lambda}_\theta}(z) + \log \operatorname{cap}(E) + \int g(t,\infty)\, d\lambda_\theta(t), \quad z \in \mathbb{C}. \tag{4.13}$$

Since $U_D^{\lambda_\theta}(z) - g(z,\infty) = 0$ on E, (4.8) and (4.9) follow from the uniqueness of the measure μ_θ satisfying conditions (2.8) and (2.9). So, we have (4.10). Using now the facts that E is a regular compact set, μ_θ is the balayage of $\lambda_0 - \lambda_\theta$ and properties of the balayage (see, for example, [5]) we can conclude that $\operatorname{supp}(\mu_\theta) = \partial E$. ☐

We can consider equation (4.10) as the basic equation of this section, it allows us to connect the Green potential $U_D^{\lambda_\theta}(z) - g(z,\infty)$ and the logarithmic potentials $U^{\lambda_\theta+\mu_\theta}(z) - \widehat{m}_\theta$ of the extremal problems from Section 2. Since $U_D^{\lambda_\theta}(z) - g(z,\infty) = 0$ on E and $\min_\Gamma(U_D^{\lambda_\theta}(z) - g(z,\infty)) = m_\theta$ on Γ, we get immediately from (4.10) the equality:

$$m_\theta = M(\lambda_\theta + \mu_\theta). \tag{4.14}$$

The function $U^{\lambda_\theta+\mu_\theta}$ satisfies the following property. The logarithmic potential $U^{\lambda_\theta+\mu_\theta}$ of a probability measure $\lambda_\theta + \mu_\theta$ is equal to constants on supports of μ_θ and λ_θ:

$$U^{\lambda_\theta+\mu_\theta} = \widehat{m}_\theta \quad \text{on} \quad E \quad \text{and} \quad U^{\lambda_\theta+\mu_\theta} = \min_\Gamma U^{\lambda_\theta+\mu_\theta} = m_\theta + \widehat{m}_\theta \quad \text{on} \quad S_\theta \subseteq \Gamma. \tag{4.15}$$

Lemma 4.4. *For each $\theta \in [0,1]$, we have*

$$m_\theta = \inf_\mu M(\mu + \lambda_\theta), \tag{4.16}$$

where infimum is taken over all compactly supported positive Borel measures with $|\mu| \le \theta$. Further, the equality in (4.16), for $\theta \in (0,1]$, is possible if and only if $\operatorname{supp}(\mu) \subseteq E$ *and $\widehat{\mu} = \mu_\theta$.*

Proof. Let $\theta = 0$. In this case $\mu_0 = 0$ by definition and by Lemma 4.2 $m_0 = 0$, $\lambda_0 = \omega_\Gamma$. Since $U^{\omega_\Gamma}(z) = -\log \operatorname{cap}(\Gamma)$ for $z \in \overline{G}$, $M(\lambda_0) = M(\omega_\Gamma) = 0$. This yields the equality $m_0 = M(\lambda_0)$.

Let $\theta \in (0,1]$. Consider a logarithmic potential $U^{\mu-\mu_\theta}$. This function is superharmonic and bounded from below in $D = \overline{\mathbb{C}} \setminus E$. Then by the generalized minimum principle for superharmonic functions [10, Thm. I.2.4],

$$\min_E U^{\mu-\mu_\theta} \le U^{\mu-\mu_\theta}(z), \quad z \in D. \tag{4.17}$$

In particular,

$$\min_E U^{\mu-\mu_\theta} \le \min_\Gamma U^{\mu-\mu_\theta}. \tag{4.18}$$

Moreover, there are strict inequalities in (4.17) for $z \in D$ and in (4.18), unless

$$\operatorname{supp}(\mu) \subseteq E, \quad U^{\mu-\mu_\theta}(z) = 0, \quad z \in D.$$

That is if and only if $\widehat{\mu} = \mu_\theta$ by Carleson's unicity theorem (see [10, Thm. II. 4.13]).

With the help of the equality

$$U^{\mu+\lambda_\theta}(z) = U^{\mu-\mu_\theta}(z) + U^{\lambda_\theta+\mu_\theta}(z), \quad z \in \mathbb{C},$$

and (4.15), we can write

$$\min_E U^{\mu+\lambda_\theta} = \min_E U^{\mu-\mu_\theta} + \widehat{m}_\theta$$

and

$$\min_\Gamma U^{\mu+\lambda_\theta} \ge \min_\Gamma U^{\mu-\mu_\theta} + m_\theta + \widehat{m}_\theta.$$

Therefore, by (4.18),

$$M(\mu + \lambda_\theta) = \min_\Gamma U^{\mu+\lambda_\theta} - \min_E U^{\mu+\lambda_\theta} \ge m_\theta,$$

and the equality in (4.16) is possible if and only if $\operatorname{supp}(\mu) \subseteq E$ and $\widehat{\mu} = \mu_\theta$. □

Lemma 4.5. *For each $\theta \in [0,1]$ we have*

$$m_\theta = \sup_\lambda M(\mu_\theta + \lambda), \tag{4.19}$$

where supremum is taken over all compactly supported positive Borel measures with $|\lambda| \le 1 - \theta$. Further, the equality in (4.19), for $\theta \in [0,1)$ is possible if and only if $\lambda = \lambda_\theta$ when S_θ does not separate the plane and $\operatorname{supp}(\lambda) \subset \mathbb{C} \setminus G$, *$\widetilde{\lambda} = \lambda_\theta - (1 - \theta - |\lambda|)\omega_\Gamma$, otherwise.*

Proof. Let $\theta = 1$. In this case $\lambda_1 = 0$ and $m_1 = -\max_\Gamma g(z,\infty)$ by definition and $\mu_1 = \omega_E$. On the basis of (4.6) we can write

$$M(\mu_1) = \min_\Gamma U^{\omega_E} - \min_E U^{\omega_E} = -\max_{z\in\Gamma} g(z,\infty) = m_1.$$

Let us consider the case when $\theta \in [0,1)$. Denote by λ any compactly supported positive Borel measure with mass at most $1 - \theta$. It is enough to show that

$$\min_{S_\theta} U^{\mu_\theta+\lambda} - \min_E U^{\mu_\theta+\lambda} \le m_\theta.$$

Consider a logarithmic potential $U^{\lambda-\lambda_\theta}$. This is superharmonic and bounded below function in $\overline{\mathbb{C}}\setminus S_\theta$. Then by the generalized minimum principle for superharmonic functions,

$$\min_{S_\theta} U^{\lambda-\lambda_\theta} \leq U^{\lambda-\lambda_\theta}(z), \quad z\in\overline{\mathbb{C}}\setminus S_\theta, \tag{4.20}$$

and

$$\min_{S_\theta} U^{\lambda-\lambda_\theta} \leq \min_{E} U^{\lambda-\lambda_\theta}. \tag{4.21}$$

From this, using (4.15), we get

$$\min_{S_\theta} U^{\mu_\theta+\lambda} - \min_{E} U^{\mu_\theta+\lambda} \leq m_\theta$$

and therefore (4.19) holds. Observe also that the equality in (4.19) is possible if and only if we have the equality in (4.21). That is if and only if

$$U^{\lambda_\theta}(z) = U^{\lambda}(z)+c^*, \quad z\in\Omega, \tag{4.22}$$

where c^* is some constant, $\Omega=\mathbb{C}\setminus S_\theta$ if S_θ does not separate the plane and $\Omega=G$ otherwise. In the former case $c^*=0$ and we get $\lambda=\lambda_\theta$ by Carleson's unicity theorem. In the latter situation $\operatorname{supp}(\lambda)\subseteq\mathbb{C}\setminus G$. Using the continuity of potentials in fine topology (see [10, Sec. I.5]) and regularity of Γ, we may continue equality in (4.22) up to $\overline{G}$. Let $\widetilde{\lambda}$ be the balayage of λ onto Γ relative to $\mathbb{C}\setminus G$ (we balayage only the part of λ which is supported outside of $\overline{G}$). Then

$$U^{\widetilde{\lambda}}(z) = U^{\lambda}(z)+c, \quad z\in\overline{G},$$

where $c=\displaystyle\int g_{\overline{G}}(t,\infty)d\lambda(t)$. Thus,

$$U^{\widetilde{\lambda}}(z) = U^{\lambda_\theta}(z)-c^*+c, \quad z\in\overline{G}. \tag{4.23}$$

Using now the maximum principle of harmonic functions in the domain $\overline{\mathbb{C}}\setminus\overline{G}$, we get

$$U^{\widetilde{\lambda}}(z) = U^{\lambda_\theta}(z)+\left(1-\theta-|\widetilde{\lambda}|\right)g_{\overline{G}}(z,\infty)-c^*+c, \quad z\in\overline{\mathbb{C}}\setminus\overline{G} \tag{4.24}$$

(we applied the maximum principle of harmonic functions for the difference of the left and right hand sides). Taking now on an account (4.23), we obtain the equality (4.24) for all $z\in\overline{\mathbb{C}}$. From this with help of the formula $U^{\omega_\Gamma}(z)=-\log\operatorname{cap}(\Gamma)-g_{\overline{G}}(z,\infty)$, and the unicity theorem [10, Thm. II 2.1], we can conclude that $\widetilde{\lambda}+(1-\theta-|\lambda|)\omega_\Gamma=\lambda_\theta$, which finishes the proof of the lemma. □

Proof of Theorem 4.1. It is a straightforward application of Lemmas 4.4 and 4.5 to obtain

$$\inf_{\mu\in\Lambda_\theta(E)}\sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu+\lambda) \geq \inf_{\mu\in\Lambda_\theta(E)} M(\mu+\lambda_\theta) = m_\theta$$

and

$$\inf_{\mu\in\Lambda_\theta(E)}\sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu+\lambda) \leq \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu_\theta+\lambda) = m_\theta.$$

This establishes the first equality in (4.1). Clearly, we have

$$\inf_{\mu\in\Lambda_\theta(E)}\sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu+\lambda) \geq \sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)}\inf_{\mu\in\Lambda_\theta(E)} M(\mu+\lambda).$$

On the other hand, it follows from Lemmas 4.4 and 4.5 that

$$\sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu_\theta+\lambda)=\inf_{\mu\in\Lambda_\theta(E)} M(\mu+\lambda_\theta).$$

Therefore,

$$\inf_{\mu\in\Lambda_\theta(E)}\sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)} M(\mu+\lambda)\le\sup_{\lambda\in\Lambda_{1-\theta}(\Gamma)}\inf_{\mu\in\Lambda_\theta(E)} M(\mu+\lambda),$$

which finishes the proof of (4.1). Let now μ^* and λ^* be as in (4.2). Then by Lemma 4.4, we observe that

$$m_\theta\le M(\mu^*+\lambda_\theta)\le\sup_\lambda M(\mu^*+\lambda)=m_\theta.$$

Thus,

$$m_\theta=M(\mu^*+\lambda_\theta)$$

and $\operatorname{supp}(\mu^*)\subseteq E$ and $\widehat{\mu^*}=\mu_\theta$ again by Lemma 4.4. Furthermore, in this case

$$m_\theta=M(\mu_\theta+\lambda^*)$$

and, by Lemma 4.5, $\lambda^*=\lambda_\theta$ when S_θ does not separate the plane and $\operatorname{supp}(\lambda^*)\subset\mathbb{C}\setminus G$, $\widetilde{\lambda^*}=\lambda_\theta-(1-\theta-|\lambda^*|)\omega_\Gamma$, otherwise. □

5. Proofs of the Theorem 3.1

Before we present the proof of Theorem 3.1, we introduce the analogue of the Tsuji points ([8], [3]) that corresponds to the weighted Green energy problem (2.2). Set

$$\delta_m^G:=\max_{z_1,\dots,z_m\in\Gamma}\left(\prod_{1\le i<j\le m}\exp\left\{-g(z_i,z_j)+\frac{g(z_i,\infty)}{1-\theta}+\frac{g(z_j,\infty)}{1-\theta}\right\}\right)^{2/m(m-1)}.$$

Then

$$\delta_m^G\ge\delta_{m+1}^G,\quad m\in\mathbb{N},\quad\text{and}\quad\lim_{m\to\infty}\log\delta_m^G=-J(\lambda_\theta)/(1-\theta)^2.$$

Moreover, if $\{\zeta_1,\dots,\zeta_m\}$ is any extremal set for δ_m^G, then

$$\lambda_{m,\theta}\xrightarrow{*}\lambda_\theta\quad\text{as}\quad m\to\infty,\quad\lambda_{m,\theta}:=\frac{1-\theta}{m}\sum_{j=1}^m\delta_{\zeta_j}.\tag{5.1}$$

Here and in what follows we keep to the notation

$$\lambda_n=\frac{1}{n}\sum_{j=1}^{n-k_n}\delta_{\xi_j,n-k_n},$$

where $\{\xi_{1,n-k_n},\dots,\xi_{n-n_k,n-k_n}\}$ is an extremal set for $\delta_{n-k_n}^G$. We remark that

$$\lambda_n\xrightarrow{*}\lambda_\theta\quad\text{as}\quad n\to\infty.\tag{5.2}$$

The proof of these facts needs only minor modifications comparing to the case of the logarithmic kernel [10, Thm. III.1.1-3].

We also need a discretization of μ_θ. So, we introduce the Leja points (see [10, Sec. III.1]) that correspond to the weighted minimal energy problem (2.7). Set

$$\delta_m:=\max_{z_1,\dots,z_m\in E}\left(\prod_{1\le i<j\le m}|z_i-z_j|\exp\left\{-\frac{1}{\theta}\left(U^{\lambda_\theta}(z_i)+U^{\lambda_\theta}(z_j)\right)\right\}\right)^{2/m(m-1)}.$$

Then

$$\delta_m \geq \delta_{m+1}, \quad m \in \mathbb{N}, \quad \text{and} \quad \lim_{m\to\infty} \log \delta_m = -I(\mu_\theta)/\theta^2.$$

Moreover, if $\{z_1, \ldots, z_m\}$ is any extremal set for δ_m, then

$$\mu_{m,\theta} \xrightarrow{*} \mu_\theta, \quad \mu_{m,\theta} := \frac{\theta}{m}\sum_{j=1}^{m} \delta_{z_j}. \tag{5.3}$$

Here and in what follows we keep to the notation

$$\mu_n = \frac{1}{n}\sum_{j=1}^{k_n} \delta_{z_{j,k_n}},$$

where $\{z_{1,k_n}, \ldots, z_{k_n,k_n}\}$ is an external set for δ_{k_n}. We have

$$\mu_n \xrightarrow{*} \mu_\theta \quad \text{as} \quad n \to \infty. \tag{5.4}$$

It is easy to see that for any compact set K, $p \in \mathcal{P}_{k_n}$, $p \not\equiv 0$, and $q \in \mathcal{P}_{n-k_n}$, $q \not\equiv 0$, we have

$$\|pq\|_K^{1/n} = \gamma^{1/n} \exp\left\{-\min_K U^{\nu(p)+\nu(q)}\right\},$$

where γ is the leading coefficient of pq, and

$$\nu(p) := \frac{1}{n}\sum_{p(z)=0} \delta_z \quad \text{and} \quad \nu(q) := \frac{1}{n}\sum_{q(z)=0} \delta_z$$

(the sums are taken counting multiplicities of zeros of p and q). Therefore, we get

$$\frac{1}{n}\log\left(\frac{\|pq\|_E}{\|pq\|_\Gamma}\right) = \min_\Gamma U^{\nu(p)+\nu(q)} - \min_E U^{\nu(p)+\nu(q)} = M(\nu(p)+\nu(q)).$$

Proof of Theorem 3.1. Let

$$\sup_{q\in\mathcal{P}_{n-k_n}} M(\mu_n + \nu(q)) = M(\mu_n + \nu(Q_n))$$

for some polynomial $Q_n \in \mathcal{P}_{n-k_n}, Q_n \not\equiv 0$. Denote by

$$\sigma_n = \nu(Q_n)_{|\overline{G}} + \widehat{\nu(Q_n)_{|\mathbb{C}\setminus\overline{G}}}.$$

By properties of the balayage, $\operatorname{supp}(\sigma_n) \subseteq \overline{G}$, $|\sigma_n| = |\nu(Q_n)|$, and

$$M(\mu_n + \nu(Q_n)) = M(\mu_n + \sigma_n).$$

We now choose a convergent subsequence such that

$$\sigma_n \xrightarrow{*} \sigma, \quad n \in \Lambda \subset \mathbb{N}, \tag{5.5}$$

and

$$\limsup_{n\to\infty} M(\mu_n + \sigma_n) = \lim_{n\to\infty, n\in\Lambda} M(\mu_n + \sigma_n).$$

We remark that $\operatorname{supp}(\sigma) \subseteq \overline{G}$ and $|\sigma| \leq 1-\theta$. Since E and Γ are regular sets, conditions (5.4) and (5.5) imply (cf. [4]) that

$$\min_\Gamma V^{\mu_n+\sigma_n} \to \min_\Gamma V^{\mu_\theta+\sigma} \quad \text{as} \quad n \to \infty, \quad n \in \Lambda,$$

$$\min_E V^{\mu_n+\sigma_n} \to \min_E V^{\mu_\theta+\sigma} \quad \text{as} \quad n \to \infty, \quad n \in \Lambda,$$

and then

$$\lim_{n\to\infty, n\in\Lambda} M(\mu_n+\sigma_n)=M(\mu_\theta+\sigma),$$

where, by Lemma 4.5, $M(\mu_\theta+\sigma)\le m_\theta$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\left(\frac{1}{n}\log\chi_n\right) &\le \limsup_{n\to\infty}\sup_{q\in\mathscr{P}_{n-k_n}} M(\mu_n+\nu(q))=\limsup_{n\to\infty} M(\mu_n+\nu(Q_n))\\ &= \lim_{n\to\infty, n\in\Lambda} M(\mu_\theta+\sigma_n)=M(\mu_\theta+\sigma)\le m_\theta. \qquad (5.6)\end{aligned}$$

For any polynomial $p\in\mathscr{P}_{k_n}$, $p\not\equiv 0$, consider the following function

$$u(z)=U^{\nu(p)+\lambda_n}(z)-\min_E U^{\nu(p)+\lambda_n}-U_D^{\lambda_n}(z)+g(z,\infty),\quad z\in\mathbb{C}.$$

This function is superharmonic in D. Using the generalized minimum principle for superharmonic functions, we obtain that $u(z)\ge 0$, $z\in D$. In particular,

$$\min_\Gamma U^{\nu(p)+\lambda_n}-\min_E U^{\nu(p)+\lambda_n}\ge\min_\Gamma\left(U_D^{\lambda_n}(z)-g(z,\infty)\right). \tag{5.7}$$

Since (5.7) is valid for any $p\in\mathscr{P}_{k_n}$, $p\not\equiv 0$, we get

$$\frac{1}{n}\log\chi_n\ge\min_\Gamma\left(U_D^{\lambda_n}(z)-g(z,\infty)\right). \tag{5.8}$$

Further, in view of the properties of weakly convergent sequences,

$$\min_\Gamma\left(U_D^{\lambda_n}(z)-g(z,\infty)\right)\to\min_\Gamma\left(U_D^{\lambda_\theta}(z)-g(z,\infty)\right)=m_\theta\quad\text{as}\quad n\to\infty.$$

Then, by the relation (5.8), we get

$$\liminf_{n\to\infty}\left(\frac{1}{n}\log\chi_n\right)\ge m_\theta. \tag{5.9}$$

So, (3.6) follows from (5.6) and (5.9).

Fix a positive R such that $\overline{G}\subset U$, where $U=\{z:|z|<R\}$. Let $L=\{z:|z|=R\}$.

Let now $\{p_n,q_n\}_{n\in\mathbb{N}}$, $p_n\in\mathscr{P}_{k_n}$ and $q_n\in\mathscr{P}_{n-k_n}$, be a sequence of asymptotically extremal pairs of polynomials. First, we show (3.7). Let $\{\nu(p_n)\}$, $n\in\Lambda_0\subset\mathbb{N}$, be a convergent subsequence. Let

$$\sigma_n=\nu(p_n)_{|\overline{U}}+\tau_n, \tag{5.10}$$

where τ_n is the balayage of $\nu(p_n)_{|\mathbb{C}\setminus\overline{U}}$ on L. By the properties of balayage, for any polynomial $q\in\mathscr{P}_{n-k_n}$, $q\not\equiv 0$,

$$M(\nu(p_n)+\nu(q))=M(\sigma_n+\nu(q)).$$

Hence,

$$\sup_{q\in\mathscr{P}_{n-k_n}} M(\nu(p_n)+\nu(q))\ge M(\sigma_n+\lambda_n). \tag{5.11}$$

We choose a convergent subsequence

$$\sigma_n\xrightarrow{*}\nu,\ \ n\in\Lambda\subset\Lambda_0\subset\mathbb{N}, \tag{5.12}$$

where $|\nu|\le\theta,\operatorname{supp}(\nu)\subseteq\overline{U}$. On the basis of the fact that E and Γ are regular sets, we get

$$M(\sigma_n+\lambda_n)\to M(\nu+\lambda_\theta)\quad\text{as}\quad n\to\infty,\quad n\in\Lambda. \tag{5.13}$$

By (3.1),

$$\limsup_{n\to\infty}\sup_{q\in\mathscr{P}_{n-k_n}} M((p_n)+\nu(q))=m_\theta.$$

From this, on an account of (5.11) and (5.13), we obtain that $M(\nu+\lambda_\theta)\leq m_\theta$. Applying Lemma 4.4, we can write $M(\nu+\lambda_\theta)=m_\theta$, $\mathrm{supp}(\nu)\subseteq E$, and $\widehat{\nu}=\mu_\theta$.

Since $\mathrm{supp}(\nu)\subseteq E$, we obtain from (5.10) and (5.12) that

$$\tau_n\xrightarrow{*}0 \quad \text{as} \quad n\to\infty, \quad n\in\Lambda,$$

$$\nu(p_n)\xrightarrow{*}\nu \quad \text{as} \quad n\to\infty, \quad n\in\Lambda,$$

and, then,

$$\nu(p_n)\xrightarrow{*}\nu \quad \text{as} \quad n\to\infty, \quad n\in\Lambda_0.$$

From this, by properties of the balayage,

$$\alpha(p_n)\xrightarrow{*}\mu_\theta \quad \text{as} \quad n\to\infty, \quad n\in\Lambda_0,$$

and, then,

$$\alpha(p_n)\xrightarrow{*}\mu_\theta \quad \text{as} \quad n\to\infty.$$

The relation (3.7) thereby is obtained.

It only remains to prove (3.8). By properties of the balayage, $M(\nu(p_n)+\nu(q_n))=M(\alpha(p_n)+\nu(q_n))$. Define

$$\nu_n=\nu(q_n)_{|\overline{U}}+\eta_n, \tag{5.14}$$

where η_n is the balayage of $\nu(q_n)_{|\mathbb{C}\setminus\overline{U}}$ on L, when S_θ does not separate the plane and $\nu_n=\beta(q_n)$ otherwise. Then $M(\alpha(p_n)+\nu(p_n))=M(\alpha(p_n)+\nu_n)$ and (3.1) yields that

$$\lim_{n\to\infty} M(\alpha(p_n)+\nu_n)=m_\theta.$$

As above, taking a convergent subsequence, $\nu_n\xrightarrow{*}\nu$, $n\in\Lambda\subset\mathbb{N}$, we get

$$\lim_{n\to\infty,n\in\Lambda} M(\alpha(p_n)+\nu_n)=M(\mu_\theta+\nu)$$

and $M(\mu_\theta+\nu)=m_\theta$.

Let us consider now the case when S_θ does not separate the plane. Since, by Lemma 4.5, $\nu=\lambda_\theta$, $\mathrm{supp}(\lambda_\theta)\subseteq\Gamma$, we obtain that

$$\eta_n\xrightarrow{*}0 \quad \text{as} \quad n\to\infty, \quad n\in\Lambda,$$

$$\nu(q_n)\xrightarrow{*}\nu=\lambda_\theta \quad \text{as} \quad n\to\infty, \quad n\in\Lambda,$$

and then

$$\nu(q_n)\xrightarrow{*}\lambda_\theta \quad \text{as} \quad n\to\infty.$$

In the case when S_θ does separate the plane, we have by Lemma 4.5 that $\mathrm{supp}(\nu)\subset\mathbb{C}\setminus G$, $|\nu|=1$, and $\widetilde{\nu}=\lambda_\theta$. Moreover, by definition of $\beta(q_n)$ (see (3.5)) we can conclude that $\mathrm{supp}(\nu)\subseteq\Gamma$. From this and the fact that $\widetilde{\nu}=\lambda_\theta$ we obtain that $\nu=\lambda_\theta$ and then

$$\beta(q_n)\xrightarrow{*}\lambda_\theta \quad \text{as} \quad n\to\infty. \qquad \square$$

6. Some properties of λ_θ and μ_θ

In the next two theorems we describe some properties of the extremal measures λ_θ and μ_θ, their supports, and the constants m_θ and $\widehat{m}_\theta$. It will be convenient for us to use the notation

$$\omega_{(K,\partial E)} \quad \text{and} \quad \operatorname{cap}(K,\partial E)$$

for the Green equilibrium distribution and the condenser capacity of a compact set $K \subset D$, respectively (cf. [10, Ch. II and VII]).

Theorem 6.1. (a) *The family* $\{S_\theta\}$, $\theta \in [0,1]$, *is a decreasing family of sets such that*

$$S_\theta = \overline{\bigcup_{\theta<\tau<1} S_\tau} \subseteq \bigcap_{0\le\tau<\theta} S_\tau = \left\{ z \in \Gamma :\ U_D^{\lambda_\theta}(z) - g(z,\infty) = m_\theta \right\}$$

and

$$S_1 := \bigcap_{0\le\tau<1} S_\tau = \{ z \in \Gamma :\ g(z,\infty) = -m_1 \};$$

(b) *the family* $\{\lambda_\theta\}$, $\theta \in [0,1)$, *is decreasing and continuous in the weak* sense. Moreover,*

$$\lambda_\theta = \int_\theta^1 \omega_{(S_\tau,\partial E)} d\tau;$$

(c) m_θ *is a continuous and strictly decreasing function of* θ *on* $[0,1]$. *Furthermore,*

$$m_\theta = m_1 + \int_\theta^1 \frac{d\tau}{\operatorname{cap}(S_\tau,\partial E)}$$

and

$$\frac{m_\theta}{\theta} \to -\frac{1}{\operatorname{cap}(\Gamma,\partial E)} \quad as \quad \theta \to 0;$$

(d) $\dfrac{\lambda_0 - \lambda_\theta}{\theta} \xrightarrow{*} \omega_{(\Gamma,\partial E)}$ *as* $\theta \to 0$;

(e) *if* S_1 *has positive capacity then* $\dfrac{\lambda_\theta}{1-\theta} \xrightarrow{*} \omega_{(S_1,\partial E)}$ *as* $\theta \to 1$.

Proof. Statements (a), (b), and the first part of (c) follow from [6, Thm. 2.4]. (We should remark that it is required in [6, Thm. 2.4] for any compact set $K \subset \Gamma$ to have connected complement. However, a direct examination of the proof shows that the theorem still holds when E is contained in G and the later is simply connected.) Further, by Lemma 4.2, $m_0 = 0$ and $S_0 = \Gamma$. This means that

$$\frac{m_\theta}{\theta} = -\frac{1}{\theta}\int_0^\theta \frac{d\tau}{\operatorname{cap}(S_\tau,\partial E)}.$$

Thus, the second part of (c) follows by the continuity of $\operatorname{cap}(S_\theta,\partial E)$ as function of θ at zero from the right [9, Thm. 5.1.3].

It has been proved in [6, Thm. 2.4] that

$$\frac{d\lambda_\theta}{d\theta} = -\omega_{(S_\theta,\partial E)} \tag{6.1}$$

for any point of continuity of $\operatorname{cap}(S_\theta,\partial E)$ as a function of θ. Then (d) follows from continuity of $\operatorname{cap}(S_\theta,\partial E)$ at $\theta = 0$ and the fact $S_0 = \Gamma$. Now, assume that S_1 has positive logarithmic capacity and therefore well-defined Green equilibrium distribution $\omega_{(S_1,\partial E)}$. As above, we can use (6.1). The continuity from the left of $\operatorname{cap}(S_\theta,\partial E)$ at one follows from [9, Thm. 5.1.3] by the definition of S_1. $\square$

The following theorem describes the connection between μ_θ and λ_θ, some properties of μ_θ and $\widehat{m_\theta}$.

Theorem 6.2. (a) *The family* $\{\mu_\theta\}$, $\theta\in(0,1]$, *is increasing, continuous in the weak** *sense, and such that*

$$\mathrm{supp}(\mu_\theta)=\partial E \quad and \quad \mu_\theta=\omega_E-\widehat{\lambda}_\theta.$$

Moreover,

$$\frac{\mu_\theta}{\theta}\xrightarrow{*}\omega_{(E,\Gamma)} \quad as \quad \theta\to 0;$$

(b) $\widehat{m}_\theta$ *is a continuous and strictly increasing function of* θ *on* $[0,1]$. *Furthermore,*

$$\widehat{m}_\theta=-\log\mathrm{cap}(E)-\int g(t,\infty)d\lambda_\theta(t).$$

Proof. Part (b) follows from (4.9) and Theorem 6.1(b). First part of (a) follows from Lemma 4.3, the formula $\mu_\theta=\widehat{\lambda_0-\lambda_\theta}$ (see (4.8)), and Theorem 6.1(b). The continuity of $\{\mu_\theta\}$ follows from continuity of the family $\{\lambda_\theta\}$, formula $\mu_\theta=\widehat{\lambda_0-\lambda_\theta}$, and properties of the balayage (see, for example [5]). We have

$$\frac{\lambda_0-\lambda_\theta}{\theta}\xrightarrow{*}\omega_{(\Gamma,\partial E)} \quad \text{as} \quad \theta\to 0.$$

Thus, by properties of balayage,

$$\frac{\widehat{\lambda_0-\lambda_\theta}}{\theta}\xrightarrow{*}\widehat{\omega_{(\Gamma,\partial E)}} \quad \text{as} \quad \theta\to 0,$$

and then

$$\mu_\theta/\theta\xrightarrow{*}\widehat{\omega_{(\Gamma,\partial E)}} \quad \text{as} \quad \theta\to 0.$$

It remains only to remark that

$$\widehat{\omega_{(\Gamma,\partial E)}}=\omega_{(E,\Gamma)}.$$

□

7. Proof of Theorem 3.2

Proof of Theorem 3.2. Since $k_n=o(n)$ as $n\to\infty$, $[k_n/\theta]\leq n$ for any fixed $\theta\in(0,1)$ and n sufficiently large. Let $l_n=[k_n/\theta]$. Therefore,

$$\inf_{p\in\mathscr{P}_{k_n}}\sup_{q\in\mathscr{P}_{l_n-k_n}}\frac{||pq||_E}{||pq||_\Gamma}\leq\chi_n\leq d_{k_n}(A^\infty;C(E))$$

(compare (1.2) with the definition (1.3) of χ_n). Then by (3.6) and (1.1), we have

$$\frac{m_\theta}{\theta}\leq\liminf_{n\to\infty}\left(\frac{1}{k_n}\log\chi_n\right)\leq\limsup_{n\to\infty}\left(\frac{1}{k_n}\log\chi_n\right)\leq-\frac{1}{\mathrm{cap}(E,\Gamma)}. \tag{7.1}$$

Taking the limit $\theta\to 0$, we obtain (3.12) from Theorem 6.1(c) and the fact that $\mathrm{cap}(E,\Gamma)=\mathrm{cap}(\Gamma,\partial E)$.

Let now p_n be asymptotically extremal polynomials in the sense of (3.9). Fix an arbitrary $\theta\in(0,1)$. Let

$$\sigma_n=\frac{1}{l_n}\sum_{j=1}^{l_n-k_n}\delta_{\xi_j,l_n-k_n},$$

where $\{\xi_{1,l_n-k_n},\dots,\xi_{l_n-k_n,l_n-k_n}\}$ is an extremal set for $\delta^G_{l_n-k_n}$. Observe that

$$\sigma_n \xrightarrow{*} \lambda_\theta \quad \text{as} \quad n\to\infty.$$

Denote by $Q_n(z)=\prod_{j=1}^{l_n-k_n}(z-\xi_{j,l_n-k_n})$ the corresponding polynomial degree l_n-k_n. By (3.9),

$$\limsup_{n\to\infty}\left(\frac{1}{\chi_n}\frac{\|p_nQ_n\|_E}{\|p_nQ_n\|_\Gamma}\right)^{1/k_n}\le 1. \tag{7.2}$$

We have

$$\frac{1}{k_n}\log\frac{\|p_nQ_n\|_E}{\|p_nQ_n\|_\Gamma}=M\left(\nu^*(p_n)+\frac{l_n}{k_n}\sigma_n\right). \tag{7.3}$$

As in the proof of Theorem 3.1, we fix $R>0$ such that $\overline{G}\subset U=\{z:|z|<R\}$. Let $L=\{z:|z|=R\}$. Let

$$\nu_n=\nu^*(p_n)_{|\overline{U}}+\eta_n,$$

where η_n is the balayage of $\nu^*(p_n)_{|\mathbb{C}\setminus\overline{U}}$ onto L. According to the properties of the balayage,

$$M\left(\nu^*(p_n)+\frac{l_n}{k_n}\sigma_n\right)=M\left(\nu_n+\frac{l_n}{k_n}\sigma_n\right).$$

By (3.12) and (7.2),

$$\limsup_{n\to\infty}M\left(\nu_n+\frac{l_n}{k_n}\sigma_n\right)\le-\frac{1}{\operatorname{cap}(E,\Gamma)}. \tag{7.4}$$

We select a convergent subsequence $\nu_n\xrightarrow{*}\nu$, $n\in\Lambda\subset\mathbb{N}$, $|\nu|\le 1$, such that

$$\limsup_{n\to\infty}M\left(\nu_n+\frac{l_n}{k_n}\sigma_n\right)=\lim_{n\to\infty,n\in\Lambda}M\left(\nu_n+\frac{l_n}{k_n}\sigma_n\right).$$

Since E and Γ are regular sets,

$$\lim_{n\to\infty,n\in\Lambda}M\left(\nu_n+\frac{l_n}{k_n}\sigma_n\right)=M\left(\nu+\frac{\lambda_\theta}{\theta}\right).$$

Therefore, by (7.4), we get

$$M\left(\nu+\frac{1}{\theta}\lambda_\theta\right)\le-\frac{1}{\operatorname{cap}(E,\Gamma)}. \tag{7.5}$$

Since $\lambda_0=\omega_\Gamma$ and $U^{\omega_\Gamma}(z)=-\log\operatorname{cap}(\Gamma)$ on $\overline{G}$,

$$M\left(\nu+\frac{1}{\theta}\lambda_\theta\right)=M\left(\nu-\frac{\lambda_0-\lambda_\theta}{\theta}\right)$$

and

$$M\left(\nu-\frac{\lambda_0-\lambda_\theta}{\theta}\right)\le-\frac{1}{\operatorname{cap}(E,\Gamma)}.$$

According to Theorem 6.1(d),

$$\frac{\lambda_0-\lambda_\theta}{\theta}\xrightarrow{*}\omega_{(\Gamma,\partial E)} \quad \text{as} \quad \theta\to 0.$$

Taking now the limit as $\theta \to 0$, we get

$$M(\nu - \omega_{(\Gamma,\partial E)}) \leq -\frac{1}{\operatorname{cap}(E,\Gamma)}. \tag{7.6}$$

Since

$$U^{\omega_{(E,\Gamma)}-\omega_{(\Gamma,\partial E)}} = \frac{1}{\operatorname{cap}(E,\Gamma)} \quad \text{on} \quad E$$

and

$$U^{\omega_{(E,\Gamma)}-\omega_{(\Gamma,\partial E)}} = 0 \quad \text{on} \quad \Gamma,$$

we obtain that

$$M(\nu - \omega_{(\Gamma,\partial E)}) = M(\nu - \omega_{(E,\Gamma)}) - \frac{1}{\operatorname{cap}(E,\Gamma)}.$$

Thus, we derive from this and (7.6) that

$$M(\nu - \omega_{(E,\Gamma)}) \leq 0$$

and

$$\min_{\Gamma} U^{\nu-\omega_{(E,\Gamma)}} \leq \min_{E} U^{\nu-\omega_{(E,\Gamma)}}. \tag{7.7}$$

Applying now the generalized minimum principle for superharmonic functions, we can conclude that $\min_\Gamma U^{\nu-\omega_{(E,\Gamma)}} = \min_E U^{\nu-\omega_{(E,\Gamma)}}$, $U^{\nu-\omega_{(E,\Gamma)}} = 0$ in D and $\operatorname{supp}(\nu) \subseteq E$. By Carleson's unicity theorem, we obtain from this that $\widehat{\nu} = \omega_{(E,\Gamma)}$. Since $\operatorname{supp}(\nu) \subseteq E$, we get

$$\eta_n \xrightarrow{*} 0 \quad \text{as} \quad n \to \infty, \quad n \in \Lambda,$$

and

$$\nu^*(p_n) \xrightarrow{*} \nu \quad \text{as} \quad n \to \infty, \quad n \in \Lambda.$$

From this, by the properties of the balayage, we can write

$$\alpha^*(p_n) \xrightarrow{*} \omega_{(E,\Gamma)} \quad \text{as} \quad n \to \infty, \quad n \in \Lambda,$$

and then

$$\alpha^*(p_n) \xrightarrow{*} \omega_{(E,\Gamma)} \quad \text{as} \quad n \to \infty.$$

□

8. Proof of Theorem 3.3

Proof of Theorem 3.3. We start by showing the lower bounds in (3.14) and (3.15). Since

$$A_n^\infty(G_\theta) \subseteq A_n^\infty(G'), \quad G' \subseteq G_\theta,$$

we may take $G' = G_\theta$.

Let $\{\lambda_n\}$ be a sequence of measures defined as in (5.2). For each $\theta \in [0,1)$ we take $\{q_n\}$ to be the sequence of monic polynomials such that $\nu(q_n) = \lambda_n$, where $\nu(h)$ be the counting measure of the zeros of a polynomial h normalized by $1/n$. For $\theta = 1$ we take $q_n \equiv 1$. Then $\{pq_n : p \in \mathscr{P}_{k_n}\}$ is a linear space of continuous functions on E of dimesion $k_n + 1$. Hence, it follows from [7, pg. 137] that for any linear space of continuous functions on E of dimension k_n, say X_{k_n}, there exists a polynomials $p_{X_{k_n}}$ such that

$$\inf_{g \in X_{k_n}} \|p_{X_{k_n}} q_n - g\|_E \geq \|p_{X_{k_n}} q_n\|_E.$$

In particular, it means that

$$d_{k_n}(A_n^\infty;C(E)) \geq \inf_{X_{k_n}} \frac{\|p_{X_{k_n}} q_n\|_E}{\|p_{X_{k_n}} q_n\|_{\Gamma_\theta}} \geq \inf_{p\in\mathcal{P}_{k_n}} \frac{\|pq_n\|_E}{\|pq_n\|_{\Gamma_\theta}}. \tag{8.1}$$

When $\theta = 1$, we get from the Bernstein-Walsh inequality and (8.1) that

$$d_{k_n}(A_n^\infty;C(E)) \geq \exp\{k_n m_1\}. \tag{8.2}$$

For $\theta \in [0,1)$, the lower estimate in (8.1) yields

$$\begin{aligned} d_{k_n}(A_n^\infty;C(E)) &\geq \inf_{p\in\mathcal{P}_{k_n}} \exp\left\{n\left(\min_{\Gamma_\theta} U^{\nu(p)+\lambda_n} - \min_E U^{\nu(p)+\lambda_n}\right)\right\} \\ &\geq \exp\left\{n \min_{\Gamma_\theta}\left(U_D^{\lambda_n}(z) - g(z,\infty)\right)\right\}, \end{aligned} \tag{8.3}$$

where $\Gamma_\theta := \partial G_\theta$ and we used (5.7) with Γ_θ instead of Γ. As before, by the properties of weakly convergent sequences, it holds that

$$\min_{\Gamma_\theta}\left(U_D^{\lambda_n}(z) - g(z,\infty)\right) \to \min_{\Gamma_\theta}\left(U_D^{\lambda_\theta}(z) - g(z,\infty)\right) = m_\theta \quad \text{as} \quad n\to\infty. \tag{8.4}$$

Thus, we get from (8.2) and (8.3) with (8.4) that

$$\liminf_{n\to\infty}\left(\frac{1}{n}\log d_{k_n}(A_n^\infty;C(E))\right) \geq m_\theta. \tag{8.5}$$

When $\theta = 0$, we have that $G_0 = G$. Further, we get exactly as in the first inequality in (7.1) that

$$\liminf_{n\to\infty}\left(\frac{1}{k_n}\log d_{k_n}(A_n^\infty;C(E))\right) \geq \liminf_{n\to\infty}\left(\frac{l_n}{k_n}\frac{1}{l_n}\log d_{k_n}(A_{l_n}^\infty(G_\tau);C(E))\right) \geq \frac{m_\tau}{\tau}$$

for any $\tau \in (0,1]$, where $l_n := [k_n/\tau]$ and we used (8.5) and the fact that $G \subseteq G_\tau$. Therefore,

$$\liminf_{n\to\infty}\left(\frac{1}{k_n}\log d_{k_n}(A_n^\infty;C(E))\right) \geq -\frac{1}{\operatorname{cap}(E,\Gamma)} \tag{8.6}$$

by Theorem 6.1(c).

Now we shall show the upper bounds in (3.14) and (3.15). Observe that

$$d_{k_n}(A_n^\infty;C(E)) \leq d_{k_n}(A^\infty;C(E)).$$

Thus, (3.15) follows from (8.6) and (1.1). Since for $\theta = 0$ limit (3.14) follows from (3.15), we may assume that $\theta \in (0,1]$. Moreover, since

$$A_n^\infty(G') \subseteq A_n^\infty(G), \quad G \subseteq G',$$

we may take $G' = G$. To proceed with the upper bound we need to construct a special sequence of domains. Fix $\theta \in (0,1]$ and define

$$\Omega_{\theta,\delta} := \left\{z\in\mathbb{C}:\ U_D^{\lambda_\theta}(z) - g(z,\infty) < m_\theta + \delta\right\}, \quad \delta \in (0,-m_\theta).$$

Each such domain $\Omega_{\theta,\delta}$ is unbounded and contains $S_\theta = \operatorname{supp}(\lambda_\theta)$ by (2.3). Also denote

$$G^\delta := \left\{z\in\mathbb{C}:\ g_{\overline{G}}(z,\infty) \leq \delta\right\},$$

where $g_{\overline{G}}(\cdot,\infty)$ is the Green function with pole at infinity for $\overline{\mathbb{C}}\setminus G$. Now, for each fixed $\delta\in(0,-m_\theta)$ take U_δ to be a connected domain (possibly unbounded) with regular boundary and such that

$$S_\theta\subset\overline{\mathbb{C}}\setminus\overline{U}_\delta,\quad E\subset U_\delta,\quad\text{and}\quad L_\delta:=\partial U_\delta\subset\Omega_{\theta,\delta}\cap G^\delta.$$

Then the harmonic measure (cf. [9, Sec. 4.3]) for U_δ, say $\omega_\delta(\cdot,\cdot)$, exists,

$$\|h\|_{L_\delta}\leq\|h\|_{\partial G^\delta}\leq\|h\|_\Gamma\exp\{n\delta\},\quad h\in\mathcal{P}_n, \tag{8.7}$$

by the Bernstein-Walsh inequality, and

$$\max_{L_\delta}\left(U_D^{\lambda_\theta}(z)-g(z,\infty)\right)\leq m_\theta+\delta. \tag{8.8}$$

Therefore, if $S_\theta=\Gamma$, then U_δ is an open subset of G that contains E and whose boundary is regular and close enough to Γ so (8.8) holds. If S_θ is a proper subset of Γ, the U_δ is an unbounded open set that contains E, whose boundary is regular, encompasses S_θ, and is close enough to it so (8.7) and (8.8) hold.

Let $\{\mu_n\}$ be defined as in (5.4) and $\{p_n\}$ be a sequence of monic polynomials such that $\nu(p_n)=\mu_n$. Further, let $\{q_n\}$ be a sequence of polynomials defined as at the beginning of the theorem when $\theta\in(0,1)$ and take q_n to be an arbitrary polynomial of degree $n-k_n$ with zeros on S_1 when $\theta=1$. Define

$$(T_n^\delta h)(z):=p_n(z)q_n(z)\int_{L_\delta}\frac{h(\tau)}{p_n(\tau)q_n(\tau)}d\omega_\delta(\tau,z),\quad z\in E,\quad h\in\mathcal{P}_n,$$

Then T_n^δ is an operator from $\mathcal{P}_n\cap H^\infty(U_\delta)$ to $C(E)$ such that

$$\|T_n^\delta h\|_E\leq\left(\frac{\|p_nq_n\|_E}{\min_{L_\delta}|p_nq_n|}\right)\|h\|_{L_\delta}. \tag{8.9}$$

Recall that

$$\|p_nq_n\|_E^{1/n}\to\exp\left\{-\min_E U^{\mu_\theta+\lambda_\theta}\right\}=\exp\{-\widehat{m}_\theta\}\quad\text{as}\quad n\to\infty$$

by (5.2) and (5.4), (4.15), and since E is regular. Moreover, the counting measures of zeros of p_nq_n, namely $\mu_n+\lambda_n$, are supported on $E\cup\Gamma$ and converge weakly to $\mu+\lambda$ that is supported on $E\cup S_\theta$. Therefore, we always can modify q_n, if needed, in such a manner that no zeros of q_n lie in some neighborhood of L_δ and λ_n still have the same asymptotic behavior. Hence, since the supports of $\mu_n+\lambda_n$ stay away from L_δ, it holds that

$$|p_nq_n|^{1/n}\to\exp\left\{-U^{\mu_\theta+\lambda_\theta}\right\}\quad\text{as}\quad n\to\infty\quad\text{uniformly on}\quad L_\delta.$$

Thus, we get for the operator norm of T_n^δ that

$$\begin{aligned}\limsup_{n\to\infty}\|T_n^\delta\|^{1/n}&\leq\exp\left\{\max_{L_\delta}U^{\mu_\theta+\lambda_\theta}-\widehat{m}_\theta\right\}\\&=\exp\left\{\max_{L_\delta}\left(U_D^{\lambda_\theta}(z)-g(z,\infty)\right)\right\}\leq\exp\{m_\theta+\delta\}\end{aligned} \tag{8.10}$$

by (4.10) and (8.8).

On the other hand, it holds that

$$T_n^\delta h=h,\quad h(z_{j,n})=0,\quad h\in\mathcal{P}_n,$$

where $z_{1,n},\ldots,z_{k_n,n}$ are the zeros of p_n. Indeed, this holds because the ratio h/p_nq_n is analytic in U_δ (including at infinity since $\deg(p_nq_n)=n$) and continuous on L_δ. Let $\phi_1,\ldots,\phi_{k_n}$ be polynomials of degree at most n such that $\phi_j(z_{i,n})=\delta_{ij}$, where δ_{ij} is the usual Kronecker symbol. Then for any $h\in\mathscr{P}_n$ we have

$$(T_n^\delta h)(z)=h(z)-\sum_{j=1}^{k_n}h(z_j)\left(\phi_j(z)-(T_n^\delta\phi_j)(z)\right). \tag{8.11}$$

Clearly, the sum on the right-hand sum of (8.11) belongs to a k_n-dimensional subspace of $C(E)$ spanned by $\phi_j-T_n^\delta\phi_j$, $j=1,\ldots,k_n$. Hence,

$$d_{k_n}(A_n^\infty;C(E))\leq\exp\{n\delta\}\|T_n^\delta\| \tag{8.12}$$

by (8.7). Combining (8.12) with (8.10), we get

$$\limsup_{n\to\infty}\left(\frac{1}{n}\log d_{k_n}(A_n^\infty;C(E))\right)\leq m_\theta$$

since δ was arbitrary. Thus, (3.14) follows from (8.5) and the last limit. □

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