A NOTE ON THE SHARPNESS OF THE REMEZ-TYPE INEQUALITY FOR HOMOGENEOUS POLYNOMIALS ON THE SPHERE

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Dedicated to Ed Saff on the occasion of his 60th birthday

ABSTRACT. Remez-type inequalities provide upper bounds for the uniform norms of polynomials p on given compact sets K, provided that $|p(x)| \leq 1$ for every $x \in K \setminus E$, where E is a subset of K of small measure. In this note we obtain an asymptotically sharp Remez-type inequality for homogeneous polynomials on the unit sphere in $\mathbb{R}^d$.

1. INTRODUCTION

For any $d, n \in \mathbb{N}$ define the space of homogeneous polynomials as

$$H_n^d := \left\{ \sum_{|\mathbf{k}|_1 = n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},\ a_{\mathbf{k}} \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d \right\},$$

where $|\cdot|_1$ stands for the ℓ_1-norm of $\mathbf{k} \in \mathbb{Z}_+^d$.

Denote by

$$R_{n,d}(\delta) := \sup \left\{ \frac{\|h\|_{S^{d-1}}}{\|h\|_{S^{d-1} \setminus E}} :\ h \in H_n^d,\ E \subset S^{d-1},\ s_{d-1}(E) \leq \delta^{d-1} \right\},$$

where $S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$ is the unit sphere in $\mathbb{R}^d$ (with respect to the usual ℓ_2-norm, $|\cdot|$), $\|f\|_K := \max_{\mathbf{x} \in K} |f(\mathbf{x})|$ for any continuous function f on an arbitrary compact set K, and $s_{d-1}(\cdot)$ stands for the Lebesgue surface measure in $\mathbb{R}^d$.

The classical inequality of Remez [4] (see also [2]) was generalized in numerous ways during the past decades. In particular, in the recent paper by A. Kroó, E. B. Saff, and the author [3] a result for homogeneous polynomials on star-like domains was obtained. Roughly speaking, a simply connected compact set K in $\mathbb{R}^d$ is a *star-like α-smooth* ($0 < \alpha \leq 2$) *domain* if its boundary is given by an even mapping of S^{d-1} which is Lipschitz continuous of order α. Then, by the result mentioned above, for any $0 < \delta < 1/2$ and any $h \in H_n^d$ such that

$$s_{d-1}(\{\mathbf{x} \in \partial K : |h(\mathbf{x})| > 1\}) \leq \delta^{d-1}$$

we have

$$\frac{1}{n} \log \|h\|_K \leq c(K) \varphi_\alpha(\delta),$$

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where

$$\varphi_\alpha(\delta) := \begin{cases} \delta^\alpha, & 0 < \alpha < 1 \\ \delta \log\frac{1}{\delta}, & \alpha = 1 \\ \delta, & 1 < \alpha \leq 2. \end{cases}$$

For instance, in the case of the unit sphere, it follows that

$$\frac{1}{n}\log R_{n,d}(\delta) \leq c(S^{d-1})\delta.$$

The goal of this note is to obtain asymptotically sharp expression for the constant $c(S^{d-1})$ in the previous inequality.

Theorem 1. *Let $\{\delta_n\}_{n=1}^\infty$ be a sequence of positive numbers tending to zero such that*

$$\lim_{n\to\infty} n\delta_n = \infty$$

and $\Gamma(\cdot)$ stand for the Gamma function. Then for any integer $d \geq 2$ we have

$$\lim_{n\to\infty}\frac{\log R_{n,d}(\delta_n)}{n\delta_n} = \varkappa_d, \tag{1}$$

where

$$\varkappa_d := \frac{1}{\sqrt{\pi}}\left(\frac{d-1}{4}\Gamma\left(\frac{d-1}{2}\right)\right)^{1/(d-1)}. \tag{2}$$

In particular, in the case of the unit circle we obtain

Corollary 2. *Let $\{\delta_n\}_{n=1}^\infty$ be as above. Then*

$$\lim_{n\to\infty}\frac{\log R_{n,2}(\delta_n)}{n\delta_n} = \frac{1}{4}.$$

2. PROOFS

The proof of Theorem 1 explores a connection between the restriction of H_n^2 to the unit sphere in $\mathbb{R}^2$, $H_n^2(S^1)$, and $P_{2n}(\mathbb{T})$, the space of complex polynomials of degree at most $2n$ restricted to the unit circle. Namely, for any $h(x,y) \in H_n^2(S^1)$, there exists $q(z) \in P_{2n}(\mathbb{T})$ such that

$$|h(x,y)| = |q(z)|, \quad \text{for any} \quad z = x + iy \in \mathbb{T}.$$

It will allow us to use the known Remez inequality for polynomials in $P_{2n}(\mathbb{T})$. The following result that we shall apply later is due to V. Andrievskii and can be found in [1].

Theorem 3. *Let $n \in \mathbb{N}$, $\delta \geq 0$, and $q \in P_n(\mathbb{T})$ be such that*

$$s_1\{z \in \mathbb{T} : |q(z)| \geq 1\} \leq \delta.$$

Then

$$\|q\|_{\mathbb{T}} \leq \left(\frac{1+\sin(\delta/4)}{\cos(\delta/4)}\right)^n.$$

This estimate is sharp in the asymptotic sense. Namely, let $\{q_n\}$ be a sequence of normalized Fekete polynomials for the set

$$\mathscr{C}_\delta := \left\{z = e^{i\phi} \in \mathbb{T} : \phi \in [-\pi, -\delta/2] \cap [\delta/2, \pi]\right\},$$

where normalization means that $\|q_n\|_{\mathscr{C}_\delta} = 1$. *Then*

$$\lim_{n\to\infty} |q_n(1)|^{1/n} = \frac{1+\sin(\delta/4)}{\cos(\delta/4)}.$$

Next we shall need an auxiliary lemma which will reduce the problem to the two-dimensional case.

Let $S_+^{d-1} := \{\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d : |\mathbf{x}| = 1, \ \ x_d \geq 0\}$ denote the upper half-sphere. Any two-dimensional plane containing the line $\{x_1 = \cdots = x_{d-1} = 0\}$ can be described as follows:

$$L_{\overline{\phi}} = \{\gamma \cdot \mathbf{u} + \beta \cdot \mathbf{e}_d : \gamma, \beta \in \mathbb{R}\},$$

where $\overline{\phi} \in T^{d-2} := [0,\pi]\times[-\pi/2,\pi/2]^{d-3}$, $\mathbf{e}_d := (0,\dots,0,1) \in \mathbb{R}^d$, and $\mathbf{u} = (u_1,\dots,u_{d-1}) \in S^{d-2}$ which can be represented in the spherical coordinates of $\mathbb{R}^{d-1}$ as $(1,\overline{\phi})$ or $(-1,\overline{\phi})$.

Lemma 4. *Let* $\epsilon > 0$ *and* $d \in \mathbb{N}$ *be fixed. Further, let* $E \subset S_+^{d-1}$ *be such that* $\mathbf{e}_d \in E$ *and* $s_{d-1}(E) = \epsilon^{d-1}$. *Then*

$$\inf\left\{s_1\left(L_{\overline{\phi}} \cap E\right) : \overline{\phi} \in T^{d-2}\right\} \leq 2^{d/(d-1)} \varkappa_d \epsilon + o(\epsilon), \quad \text{as} \quad \epsilon \to 0, \tag{3}$$

where $\varkappa_d$ *is defined by* (2).

Proof. Define a projection $P_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ by the rule

$$P_d(x_1,\dots,x_{d-1},x_d) := (x_1,\dots,x_{d-1}).$$

For any $r > 0$ denote by

$$A_r := P_d^{-1}(B_r^{d-1}) \cap S_+^{d-1}$$

a spherical cap around point $\mathbf{e}_d$ on the unit sphere which is the preimage of the ball B_r^{d-1} under the projection P_d, where $B_r^{d-1} := \{\mathbf{x} \in \mathbb{R}^{d-1} : |\mathbf{x}| \leq r\}$. Let $r(\epsilon)$ be chosen in such a way that $s_{d-1}(A_{r(\epsilon)}) = \epsilon^{d-1}$. Denote by

$$E_{\overline{\phi}} = \left\{\rho \in [-1,1] : (\rho,\overline{\phi}) \in P_d(E)\right\},$$

where $(\rho,\overline{\phi}) \in \mathbb{R} \times T^{d-2}$ are spherical coordinates in $\mathbb{R}^{d-1}$.

First we are going to show that

$$\inf\left\{s_1\left(L_{\overline{\phi}} \cap E\right) : \overline{\phi} \in T^{d-2}\right\} \leq 2\arcsin(r(\epsilon)). \tag{4}$$

Suppose (4) is false, i.e., for any $\overline{\phi} \in T^{d-2}$ we have that

$$s_1\left(L_{\overline{\phi}} \cap E\right) > 2\arcsin(r(\epsilon)).$$

The last claim can be restated as

$$\int_{E_{\overline{\phi}}} \frac{d\rho}{\sqrt{1-\rho^2}} > \int_{-r(\epsilon)}^{r(\epsilon)} \frac{d\rho}{\sqrt{1-\rho^2}}, \quad \text{for all } \overline{\phi} \in T^{d-2},$$

which can be written in the following form

$$\int_{E_{\overline{\phi}} \setminus [-r(\epsilon),r(\epsilon)]} \frac{d\rho}{\sqrt{1-\rho^2}} > \int_{[-r(\epsilon),r(\epsilon)] \setminus E_{\overline{\phi}}} \frac{d\rho}{\sqrt{1-\rho^2}}, \quad \text{for all } \overline{\phi} \in T^{d-2}. \tag{5}$$

Since
$$\rho_1:=\min_{E_{\overline{\phi}}\setminus[-r(\epsilon),r(\epsilon)]}|\rho|^{d-2}\geq\max_{[-r(\epsilon),r(\epsilon)]\setminus E_{\overline{\phi}}}|\rho|^{d-2}=:\rho_2,$$
inequality (5) implies that
$$\begin{aligned}\int_{E_{\overline{\phi}}\setminus[-r(\epsilon),r(\epsilon)]}\frac{|\rho|^{d-2}}{\sqrt{1-\rho^2}}d\rho &\geq \int_{E_{\overline{\phi}}\setminus[-r(\epsilon),r(\epsilon)]}\frac{\rho_1^{d-2}}{\sqrt{1-\rho^2}}d\rho\\ &> \int_{[-r(\epsilon),r(\epsilon)]\setminus E_{\overline{\phi}}}\frac{\rho_2^{d-2}}{\sqrt{1-\rho^2}}d\rho\geq\int_{[-r(\epsilon),r(\epsilon)]\setminus E_{\overline{\phi}}}\frac{|\rho|^{d-2}}{\sqrt{1-\rho^2}}d\rho\end{aligned}$$
and consequently
$$\int_{E_{\overline{\phi}}}\frac{|\rho|^{d-2}}{\sqrt{1-\rho^2}}d\rho>\int_{-r(\epsilon)}^{r(\epsilon)}\frac{|\rho|^{d-2}}{\sqrt{1-\rho^2}}d\rho$$
for all $\overline{\phi}\in T^{d-2}$. Then
$$\begin{aligned}\epsilon^{d-1} &= s_{d-1}(E)=\int_{P_d(E)}\left(1-\sum_{k=1}^{d-1}x_k^2\right)^{-1/2}d\mathbf{x}=\int_{T^{d-2}}J(\overline{\phi})\int_{E_{\overline{\phi}}}\frac{|\rho|^{d-2}}{\sqrt{1-\rho^2}}d\rho\, d\overline{\phi}\\ &> \int_{T^{d-2}}J(\overline{\phi})\int_{-r(\epsilon)}^{r(\epsilon)}\frac{|\rho|^{d-2}}{\sqrt{1-\rho^2}}d\rho\, d\overline{\phi}=\int_{B_{r(\epsilon)}^{d-1}}\left(1-\sum_{k=1}^{d-1}x_k^2\right)^{-1/2}d\mathbf{x}\\ &= s_{d-1}\left(A_{r(\epsilon)}\right)=\epsilon^{d-1},\end{aligned}$$
where $|\rho|^{d-2}J(\overline{\phi})$ is the Jacobian of the spherical transformation in $\mathbb{R}^{d-1}$. Thus, we have obtained a contradiction.

Now, to prove (3) we need to get an upper estimate for $r(\epsilon)$. Since
$$\mu_{d-1}\left(B_{r(\epsilon)}^{d-1}\right)\leq s_{d-1}\left(A_{r(\epsilon)}\right)\leq(1+r^2(\epsilon)/2)\mu_{d-1}\left(B_{r(\epsilon)}^{d-1}\right),$$
we have
$$\epsilon^{d-1}+o(\epsilon^{d-1})=\mu_{d-1}\left(B_{r(\epsilon)}^{d-1}\right)=\mu_{d-1}\left(B_1^{d-1}\right)r^{d-1}(\epsilon)=\frac{1}{2}\left(\frac{r(\epsilon)}{\varkappa_d}\right)^{d-1},$$
where $\mu_{d-1}(\cdot)$ stands for the usual Lebesgue measure in $\mathbb{R}^{d-1}$. From the above we obtain that
$$r(\epsilon)=2^{1/(d-1)}\varkappa_d\epsilon+o(\epsilon),$$
which completes the proof. □

Proof of Theorem 1. We start by showing the upper estimate for the limit in (1). Let $h\in H_n^d$ and $E\subset S^{d-1}$ with $s_{d-1}(E)\leq\delta_n^{d-1}$. Without loss of generality we may assume that $||h||_{S^{d-1}\setminus E}=1$ and h attains maximum of its modulus at $\mathbf{e}_d\in E$. Then the auxiliary lemma ensures that there exists a one-dimensional sphere S^1 which goes through the $\mathbf{e}_d$ with the property
$$s_1\left(E\cap S^1\right)\leq 4\varkappa_d\delta_n+o(\delta_n),$$
where $o(\delta_n)$ is understood in the following sense
$$\lim_{n\to\infty}o(\delta_n)\cdot\delta_n^{-1}=0.$$

Since h restricted to S^1 is a homogeneous polynomial of two variables, problem can be reduced to the two-dimensional case.

The unit sphere in $\mathbb{R}^2$ can be viewed as the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$, which allows us to establish a relationship between homogeneous polynomials on S^1 and polynomials with complex coefficients on $\mathbb{T}$.

$$h(x,y)=\sum_{j=0}^{n} h_j x^j y^{n-j}=\sum_{j=0}^{n} h_j \left(\frac{z^2+1}{2z}\right)^j \left(\frac{z^2-1}{2iz}\right)^{n-j}=\frac{q_h(z^2)}{z^n},$$

where $z=x+iy$ and $q_h\in P_n(\mathbb{T})$. Moreover

$$|h(x,y)|=|q_h(z^2)|,\quad z=x+iy\in\mathbb{T}.$$

Which, in particular, means

$$|h(\cos\phi,\sin\phi)|=|h(\cos(\pi+\phi),\sin(\pi+\phi))|=\left|q_h\left(e^{2i\phi}\right)\right|$$

for any $\phi\in[0,\pi]$. Since

$$\begin{aligned} s_1\{z=x+iy\in\mathbb{T}:\ |h(x,y)|>1\} &= 2\mu_1\{\phi\in[0,\pi]:|h(\cos\phi,\sin\phi)|>1\}\\ &\le 4\varkappa_d\delta_n+o(\delta_n), \end{aligned}$$

we obtain

$$\begin{aligned} s_1\{z\in\mathbb{T}:\ |q_h(z)|>1\} &= \mu_1\left\{\phi\in[0,2\pi]:\ \left|q_h\left(e^{i\phi}\right)\right|>1\right\}\\ &= 2\mu_1\left\{\phi\in[0,\pi]:\ \left|q_h\left(e^{2i\phi}\right)\right|>1\right\}\le 4\varkappa_d\delta_n+o(\delta_n). \end{aligned}$$

Thus we can apply Theorem 3, which yields

$$||h||_{S^{d-1}}=||h||_{S^1}=||q_h||_{\mathbb{T}}\le\left(\frac{1+\sin(\varkappa_d\delta_n+o(\delta_n))}{\cos(\varkappa_d\delta_n+o(\delta_n))}\right)^n.$$

The last inequality implies

$$\frac{1}{n}\log R_{n,d}(\delta_n)\le\log\left(1+\sin(\varkappa_d\delta_n+o(\delta_n))\right)-\log\cos(\varkappa_d\delta_n+o(\delta_n))=\varkappa_d\delta_n+o(\delta_n),$$

which gives us the desired upper bound for the limit in (1).

Now we turn our attention to the lower estimate. For $0<\epsilon<1$ consider the n-th Chebyshev polynomials for the interval $[-1+\epsilon,1-\epsilon]$, i.e.,

$$T_n^\epsilon(x):=T_n\left(\frac{x}{1-\epsilon}\right),$$

where $T_n(x)=\{(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n\}/2$ is the classical n-th Chebyshev polynomial. It satisfies

(i) $|T_n^\epsilon(x)|\le 1$ for $x\in[-1+\epsilon,1-\epsilon]$;

(ii) $\max_{x\in[-1,1]}|T_n^\epsilon(x)|=|T_n^\epsilon(1)|=\left|T_n\left(\frac{1}{1-\epsilon}\right)\right|$.

Due to the symmetry of $[-1+\epsilon,1-\epsilon]$ we can write $T_n^\epsilon(x)$ in the next form:

$$T_n^\epsilon(x)=\begin{cases} k_n\prod_{j=1}^m(x^2-t_j^2), & n=2m;\\ k_n x\prod_{j=1}^m(x^2-t_j^2), & n=2m+1. \end{cases}$$

This leads to the following homogeneous polynomials of degree n:

$$h_n^\epsilon(\mathbf{x}) = \begin{cases} k_n \prod_{j=1}^m \left((1-t_j^2)x_d^2 - t_j^2(x_1^2+\cdots+x_{d-1}^2)\right), & n=2m; \\ k_n x_d \prod_{j=1}^m \left((1-t_j^2)x_d^2 - t_j^2(x_1^2+\cdots+x_{d-1}^2)\right), & n=2m+1; \end{cases}$$

which enjoys the property

$$h_n^\epsilon(\mathbf{x})\Big|_{S^{d-1}} = T_n^\epsilon(x_d),$$

and consequently

$$\|h_n^\epsilon\|_{S^{d-1}} = |T_n^\epsilon(1)|.$$

Then the exceptional set E_ϵ (i.e. $E_\epsilon := \{\mathbf{x}\in S^{d-1} : |h_n^\epsilon(\mathbf{x})| \geq 1\}$) can be described as

$$E_\epsilon = \left\{\mathbf{x}\in S^{d-1} : |x_d| \geq 1-\epsilon\right\}.$$

Thus, $E_\epsilon = P_d^{-1}\left(B_{r(\epsilon)}^{d-1}\right)$, where P_d is the orthogonal projection from Lemma 4 and $r(\epsilon) = \sqrt{\epsilon(2+\epsilon)}$. We choose ϵ in such a way that $s_{d-1}(E_\epsilon) = \delta_n^{d-1}$. As was shown before

$$\sqrt{2\epsilon(\delta_n)+\epsilon^2(\delta_n)} = \varkappa_d\delta_n + o(\delta_n),$$

where $\varkappa_d$ is defined by (2). So, we get

$$\begin{aligned}
\frac{1}{n}\log R_{n,d}(\delta_n) &\geq \frac{1}{n}\log\left\|h_n^{\epsilon(\delta_n)}\right\| = \frac{1}{n}\log\left|T_n\left(\frac{1}{1-\epsilon(\delta_n)}\right)\right| \\
&\geq \log\left(\frac{1}{1-\epsilon(\delta_n)} + \sqrt{\frac{2\epsilon(\delta_n)+\epsilon^2(\delta_n)}{(1-\epsilon(\delta_n))^2}}\right) + \frac{1}{n}\log\frac{1}{2} \\
&= \frac{1}{n}\log\frac{1}{2} + \sqrt{2\epsilon(\delta_n)+\epsilon^2(\delta_n)} + o\left(\sqrt{\epsilon(\delta_n)}\right) \\
&= \frac{1}{n}\log\frac{1}{2} + \varkappa_d\delta_n + o(\delta_n).
\end{aligned}$$

We complete the proof by dividing the both sides of the inequality above by δ_n and taking the limit when $n\to\infty$.

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