The topological Atiyah–Segal map

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ABSTRACT. Associated to each finite dimensional linear representation of a group G, there is a vector bundle over the classifying space BG. This construction was studied extensively for compact groups by Atiyah and Segal. We introduce a homotopy theoretical framework for studying the Atiyah–Segal construction in the context of infinite discrete groups, taking into account the topology of representation spaces.

We explain how this framework relates to the Novikov conjecture, and we consider applications to spaces of flat connections on the over the 3–dimensional Heisenberg manifold and families of flat bundles over classifying spaces of groups satisfying Kazhdan's property (T).

CONTENTS

1. Introduction

In the 1950s and 1960s, Atiyah and Segal studied a construction that associates a vector bundle to each (complex) representation of a group G. This construction yields a map

$$R[G] \longrightarrow K^*(BG),$$

from the complex representation ring of G to the complex K–theory of its classifying space BG. The classical Atiyah–Segal Completion Theorem [4] states that when G

is a compact Lie group, this map becomes an isomorphism after completing $R[G]$ at its augmentation ideal.

We introduce an analogous map for infinite discrete groups, where the topology of the representation spaces $\mathrm{Hom}(G, U(n))$ plays a key role. Loosely speaking, this map assigns to a spherical family of representations

$$\rho\colon S^m \to \mathrm{Hom}(G, U(n))$$

the K–theory class of the associated vector bundle

$$E_\rho \to S^m \times BG$$

with holonomy ρ. See Section 5 for the definition of E_ρ, and a more precise statement along these lines (Theorem 5.3). The construction $\rho \mapsto E_\rho$ was considered previously in [20], where it was used to study the (strong) Novikov conjecture, and in [5], where it was used to obtain cohomological lower bounds on the homotopy groups of spaces of flat, unitary connections.

The appropriate context for this construction is that of *deformation K–theory*. The reduced, unitary deformation K–theory of a group G can be thought of as the homotopy group completion $\Omega B\mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the topological monoid

$$\mathrm{Rep}(G) = \coprod_n \mathrm{Hom}(G, U(n)),$$

with block sum of matrices as the monoid operation. (A small adjustment is needed to get the correct homotopy group in dimension zero; see Section 3.) General linear deformation K–theory is obtained by replacing $U(n)$ by $GL(n) = GL_n(\mathbb{C})$. The (reduced, unitary) topological Atiyah–Segal map is a homomorphism

$$(1) \qquad\qquad \widetilde{\alpha}_*\colon \widetilde{K}_*^{\mathrm{def}}(G) := \pi_*\Omega B\mathrm{Rep}(G) \longrightarrow \widetilde{K}^{-*}(BG),$$

induced by a map of topological monoids arising from the natural map

$$(2) \qquad\qquad B\colon \mathrm{Hom}(G, U(n)) \longrightarrow \mathrm{Map}_*(BG, BU(n)).$$

The details of this construction appear in Section 5.

Deformation K–theory has proven difficult to compute, and we do not expect any uniform description of its relationship to topological K–theory. This stands in contrast to similar functors such as algebraic K–theory of the group ring, or the K–theory of group C^*–algebras, where there are general conjectures describing this functors in terms of homological data. We view this as a positive feature of the theory: it is subtle enough to capture delicate information about the group in question, so that when computation can be achieved, concrete consequences follow.

We consider several applications in this article.

- In Section 6, we reinterpret a result from [20] to show that rational surjectivity of $\widetilde{\alpha}_*$ in high dimensions implies the strong Novikov conjecture (Theorem 6.2). Thus surjectivity of $\widetilde{\alpha_*}$ should be viewed as a very strong Novikov-type property. Earlier work of the author implies that surjectivity holds for surface groups (Theorem 6.4).
- In Section 7, we use Tyler Lawson's calculations of deformation K–theory for the 3–dimensional Heisenberg group $\pi_1 N^3$ to produce huge families of homotopy classes in the space of flat, unitary connections on bundles over the corresponding 3–dimensional Heisenberg manifold (Theorem 7.6 and Corollary 7.10). This shows a marked difference between gauge theory in

2– and 3–dimensions: over surfaces, homotopy in the space of flat connections is tightly controlled by Yang–Mills theory and complex geometric considerations, but in 3–dimensions the flood gates open.

We note that for these results, the spectrum-level description of the topological Atiyah–Segal map is not really needed; one could instead take Theorem 5.3 as a definition.

- In Section 8, we show that the topological Atiyah–Segal map is actually a map of E_∞ rings, and consider some applications. First, a result of S. P. Wang, combined with computational methods due to Lawson, allows us to calculate the deformation K–theory of groups satisfying Kazhdan's property (T), and to describe the topological Atiyah–Segal map for such groups (Propositions 8.4 and 8.6). We then deduce that for property (T) groups admitting finite classifying spaces, the vector bundles E_ρ associated to spherical families of unitary representations always represent torsion classes in $\widetilde{K}^0(S^m \times BG)$ (Theorem 8.1). Finally, we show that the topological Atiyah–Segal map for the Heisenberg group fails to be surjective (Proposition 8.8).

Acknowledgements

2. Permutative categories arising from group actions

In this section we introduce a framework for producing permutative categories (internal to the category **CGTop** of compactly generated topological spaces) from certain sequences of group actions. All topological notions (e.g. topological groups and group actions) should be interpreted in the category **CGTop**.

The framework introduced here will be used in subsequent sections to give compatible descriptions of deformation K–theory and topological K–theory, facilitating the construction of the topological Atiyah–Segal map as a morphism of spectra. The proofs of the claims made in this section are all routine (and, in fact, relatively short) and will mostly be left to the reader. In Section 2.3, we briefly explain how this theory can be enhanced to produce *bipermutative categories* and hence ring spectra.

We will use the following terminology regarding (topological) group actions: if X is a G–space, Y is an H–space, and $\phi\colon G \to H$ is a (continuous) homomorphism, then a map $f\colon X \to Y$ is called ϕ–equivariant, or equivariant with respect to ϕ, if

$$f(g \cdot x) = \phi(g) \cdot f(x)$$

for all $g \in G$, $x \in X$.

2.1. Action sequences. The canonical example to keep in mind when reading the following definition is the unitary (or general linear) groups acting on themselves by conjugation, with the usual matrix block sum operations (see Example 2.3).

DEFINITION 2.1. A permutative (left) action sequence is a sextuple

$$\mathcal{A} = (I, \{G_i\}_{i \in I}, \{X_i\}_{i \in I}, *, \oplus, \{C_{i,j}\}_{i,j \in I}),$$

where:

- I is a commutative monoid, with identity element 0 and operation $+$;
- Each G_i is a topological group with identity element $e_i \in G_i$, and each X_i is a left G_i–space;
- $* = *_0 \in X_0$ is a non-degenerate basepoint;
- $\oplus = (\oplus^{\mathrm{alg}}, \oplus^{\mathrm{top}})$;
- $\oplus^{\mathrm{alg}}$ is an associative collection of homomorphisms

$$\oplus_{i,j}^{\mathrm{alg}} \colon G_i \times G_j \to G_{i+j}, \ \ i, j \in I;$$

- $\oplus^{\mathrm{top}}$ is an associative collection of $\oplus_{i,j}^{\mathrm{alg}}$–equivariant maps

$$\oplus_{i,j}^{\mathrm{top}} \colon X_i \times X_j \to X_{i+j}, \ \ i, j \in I,$$

  where equivariance refers to the product action of $G_i \times G_j$ on $X_i \times X_j$ (and the action of G_{i+j} on X_{i+j});
- For each $i, j \in I$, we have $C_{i,j} \in G_{i+j}$.

The elements $C_{i,j}$ are subject to the following further axioms for all $i, j, k \in I$. (From here on, we will usually simplify notation by writing $\oplus$ in place of $\oplus_{i,j}^{\mathrm{alg}}$ or $\oplus_{i,j}^{\mathrm{top}}$.)

- $C_{i,j} \cdot (x_i \oplus x_j) = x_j \oplus x_i$ for each $x_i \in X_i$, $x_j \in X_j$
- $C_{i,0} = C_{0,i} = e_i$;
- $C_{i,j} C_{j,i} = e_{i+j}$;
- If $g_i \in G_i$ and $g_j \in G_j$, then $C_{i,j}(g_i \oplus g_j) = (g_j \oplus g_i)C_{i,j}$;
- $(C_{i,k} \oplus e_j)(e_i \oplus C_{j,k}) = C_{i+j,k}$.

Note that it is not necessary to assume $*_0$ is fixed by the action of G_0.

We refer to the operations $\oplus$ as the *monoidal*, or *additive*, structure of $\mathcal{A}$, and we refer to the elements $C_{i,j}$ as the *commutativity operators*.

It will be convenient to use the notation $X = \coprod_i X_i$, $G = \coprod_i G_i$, as well as to set $C = \{C_{i,j}\}$. Then we can write an action sequence in the simplified notation $\mathcal{A} = (I, G, X, *, \oplus, C)$.

A morphism of permutative action sequences

$$\mathcal{A} = (I, G, X, *, \oplus, C) \longrightarrow \mathcal{B} = (J, H, Y, *, \oplus, D)$$

consists of a homomorphism of monoids $f \colon I \to J$ together with group homomorphisms

$$\phi_i \colon G_i \to H_{f(i)}$$

satisfying $\phi_{i+j}(C_{i,j}) = D_{f(i), f(j)}$, and ϕ_i–equivariant maps

$$\zeta_i \colon X_i \to Y_{f(i)}$$

for each $i \in I$, such that $\zeta_0(*) = *$. This defines the category **PAct** of permutative (left) action sequences.

We will always work with left actions, and we drop the adjective *left* from here on. In what follows, we will assume familiarity with the notion of permutative categories (as defined in [**15**]).

CONSTRUCTION 2.2. There is a functor

$$\mathcal{T}\colon \mathbf{PAct} \longrightarrow \mathbf{PCat}$$

from the category of permutative action sequences to the category $\mathbf{PCat}$ of permutative categories internal to $\mathbf{CGTop}$, defined as follows.

Given an action sequence $\mathcal{A} = (I, G, X, *, \oplus, C)$, the object space of $\mathcal{T}(\mathcal{A})$ is simply $X = \coprod_{i \in I} X_i$, while the morphism space is $\coprod_{i \in I} G_i \times X_i$. The domain of (g, x) is x, the codomain is $g \cdot x$, and composition is given by

$$(h, g \cdot x) \circ (g, x) = (hg, x).$$

The operations $\oplus$ give rise to a continuous functor

$$\oplus\colon \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A}) \longrightarrow \mathcal{T}(A),$$

which is (strictly) associative and has the object $*_0 \in X_0$ as (strict) unit. The commutativity isomorphisms are given by $(C_{i,j}, x_i \oplus x_j)$, and our axioms on the $C_{i,j}$ are exactly what is needed to make the coherence diagrams in $\mathcal{T}(\mathcal{A})$ commute.

We call $\mathcal{T}(\mathcal{A})$ as the *translation category* of $\mathcal{A}$. (Note that $\mathcal{T}(\mathcal{A})$ is a groupoid.)

EXAMPLE 2.3. The tautological (additive) unitary permutative action sequence is given by setting $I = \mathbb{N}$, with ordinary addition as the monoid operation, and setting $X_n = G_n = U(n)$ for $n \in \mathbb{N}$. We define $U(0) = \{0\}$, the trivial group. Here we view $U(n)$ as a left $U(n)$–space via *conjugation*, and we use the usual matrix block sum operation to define both $\oplus^{\mathrm{alg}}$ and $\oplus^{\mathrm{top}}$, with $0 \in U(0)$ acting as the unit element. The commutativity operators are the (unitary) permutation matrices

(3) $$I_{m,n} = \left[\begin{array}{cc} 0_{nm} & I_n \\ I_m & 0_{mn} \end{array} \right],$$

where 0_{pq} denotes the $p \times q$ zero matrix. The tautological additive general linear action sequence is defined similarly, by replacing $U(n)$ by $GL(n)$.

DEFINITION 2.4. An (additive) *unitary permutative action sequence* is one in which the underlying monoid is $\mathbb{N}$, with its usual addition, and we have $G_n = U(n)$ and $C_{m,n} = I_{m,n}$ for all $m, n \in \mathbb{N}$. Note that such sequences are completely determined by their topological data, that is, the $U(n)$–spaces X_n (and the basepoint $x_0 \in X_0$) together with the maps $\oplus^{\mathrm{top}}$.

REMARK 2.5. The notion of a permutative action sequence can be generalized by allowing the elements $C_{i,j}$ to depend on $x_i \in X_i$ and $x_j \in X_j$ rather than just on $i, j \in I$, and a small modification again gives a functor from this larger category of sequences to $\mathbf{PCat}$. Furthermore, there is no need to assume the G_i are groups; monoids would suffice.

2.2. The nerve of a permutative action sequence. Consider a permutative action sequence $\mathcal{A} = (I, G, X, *, \oplus, C)$. The continuous functor

$$\oplus\colon \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A}) \longrightarrow \mathcal{T}(\mathcal{A})$$

makes $|N.\mathcal{T}(\mathcal{A})|$ into a topological monoid (note that geometric realization commutes with products of simplicial spaces). May's infinite loop space machine [15] gives a functor $\mathbf{K}$ from $\mathbf{PCat}$ to the category of connective Ω–spectra. One key feature of this functor is that for each permutative category $\mathcal{C}$, the infinite loop space underlying the spectrum $\mathbf{K}(\mathcal{C})$ is naturally weakly equivalent to $\Omega B|N.\mathcal{C}|$. Our next goal is to give an explicit description of the monoid $|N.\mathcal{T}(\mathcal{A})|$.

For a space X with a left action of a topological group G, the homotopy orbit space (also known as the Borel construction) is the quotient space

$$X_{hG} = (EG \times X)/G,$$

where EG is the geometric realization of the category $\overline{G}$, internal to **CGTop**, with object space G and morphism space $G \times G$ (here (g, h) is the unique morphism from h to g, and $(g, h) \circ (h, k) = (g, k)$). Note that $\overline{G}$ admits a right action of G (by functors), defined via right-multiplication in G. This induces a right action of G on EG, and now G acts on $EG \times X$ via $g \cdot (e, x) = (e \cdot g^{-1}, g \cdot x)$. Let BG denote the geometric realization of the category $\underline{G}$ (again internal to **CGTop**) with a single object $*$ and morphism space G, and with composition $g \circ h = gh$. When G is a Lie group, the natural map $EG \to BG$ (induced by the functor sending a morphism (g, h) in $\overline{G}$ to the morphism gh^{-1} in $\underline{G}$) is a universal principal G–bundle [**27**], and the natural map $X_{hG} \to BG$ is a fiber bundle with fiber X.

Given a permutative action sequence $\mathcal{A} = (I, G, X, *, \oplus, C)$, we can form

$$\mathcal{M}(\mathcal{A}) := \coprod_{i \in I} (X_i)_{hG_i}.$$

The maps $\oplus_{i,j}^{\mathrm{alg}}$ induce continuous functors $\overline{G}_i \times \overline{G}_j \longrightarrow \overline{G}_{i+j}$ and hence continuous maps

$$(4) \hspace{4cm} EG_i \times EG_j \longrightarrow EG_{i+j}.$$

Since the maps $\oplus_{i,j}^{\mathrm{alg}}$ are homomorphisms, the maps (4) are equivariant (with respect to $\oplus \colon G_i \times G_j \to G_{i+j}$). Together with the equivariant maps $\oplus \colon X_i \times X_j \to X_{i+j}$, these maps induce a map

$$\mathcal{M}(\mathcal{A}) \times \mathcal{M}(\mathcal{A}) \longrightarrow \mathcal{M}(\mathcal{A}).$$

It is an exercise to check that this map makes $\mathcal{M}(\mathcal{A})$ into a topological monoid with $[*, *_0] \in (X_0)_{hG_0} \subset \mathcal{M}(\mathcal{A})$ as unit element, where $* \in EG$ corresponds to the object in $\overline{G}$ represented by the identity of G.

PROPOSITION 2.6. *There is a natural homeomorphism of topological monoids*

$$\mathcal{M}(\mathcal{A}) \longrightarrow |N.\mathcal{T}(\mathcal{A})|.$$

PROOF. (Sketch) A special case of this statement is proven in [**21**, Proposition 2.4] (and the argument given there is due to Tyler Lawson). That argument immediately generalizes to produce the desired map and to show that it is a continuous bijection. The argument proceeds by viewing each side as the geometric realization of a simplicial space, and providing a map of simplicial spaces that is a homeomorphism on each level. In [**21**], an appeal to compactness was made to deduce continuity of the inverse maps on each level, but it is in fact a simple matter to write down explicit formulas for these inverse maps, from which it is clear that the inverses are continuous.   □

In [**23**], the notion of strongly anchored elements in a (homotopy commutative) monoid M was introduced, in order to give a concrete description of the homotopy groups $\pi_* \Omega BM$ of the homotopy group completion. We say that $x \in M$ is strongly anchored if there exists a homotopy between the maps

$$(m, n) \mapsto mn \quad \text{and} \quad (m, n) \mapsto nm$$

that is constant on (x, x). We write $[X, Y]$ for the set of unbased homotopy classes of maps from X to Y; if $Y = M$ is a topological monoid, this set acquires a monoid structure. (We will also write $[f] \in [X, Y]$ for the class of $f \colon X \to Y$.)

THEOREM 2.7 ([**23**, Theorem 1.1, Proposition 2.8]). *Let M be a topological monoid such that each $x \in M$ is strongly anchored. Then for each $k \geq 0$, there is a natural isomorphism*

$$(5) \qquad \frac{\mathrm{Gr}[S^k, M]}{\mathrm{Gr}(\pi_0 M)} \xrightarrow{\cong} \pi_k(\Omega BM),$$

where Gr *denotes the Grothendieck group and* $\pi_0 M \leqslant [S^k, M]$ *is the submonoid of nullhomotopic maps* $S^k \to M$.

In fact, for $k > 0$, the natural map

$$\pi \colon [S^k, M] \to \frac{\mathrm{Gr}[S^k, M]}{\mathrm{Gr}(\pi_0 M)}$$

is surjective, and $\pi([\phi]) = 0$ if and only if there exists a constant map $c \colon S^k \to M$ such that $\phi \bullet c$ is nullhomotopic (where $\bullet$ denotes pointwise multiplication in M).

We set $\widetilde{\Pi}_k M := \dfrac{\mathrm{Gr}[S^k, M]}{\mathrm{Gr}(\pi_0 M)}$. The above result tells us, in particular, that $\widetilde{\Pi}_k M$ is the quotient, in the category of abelian monoids, of $[S^k, M]$ by $\pi_0 M$. In particular, each element of $[S^k, M]$ is invertible modulo constant maps.

If $\mathcal{A}$ is a unitary or general linear permutative action sequence, then the argument in [**21**, Proof of Corollary 4.4] shows that every element in the monoid $\mathcal{M}(\mathcal{A})$ is (strongly) anchored.

COROLLARY 2.8. *If $\mathcal{A}$ is a unitary or general linear permutative action sequence, then the natural map*

$$\widetilde{\Pi}_k \mathcal{M}(\mathcal{A}) \longrightarrow \Omega B\mathcal{M}(\mathcal{A})$$

is an isomorphism for each $k \geqslant 0$.

REMARK 2.9. The commutativity operators $C_{i,j}$ induce a natural transformation between the functors $\oplus \colon \mathcal{T}(\mathcal{A}) \times \mathcal{T}(\mathcal{A}) \to \mathcal{T}(A)$ and $\oplus \circ \tau$, where τ is the twist functor on the product category. It then follows from basic categorical homotopy theory ([**27**, Proposition 2.1]) that $|N.(\mathcal{T}(\mathcal{A}))| \cong \mathcal{M}(\mathcal{A})$ is homotopy commutative. However, this homotopy does not anchor elements. The argument in [**21**, Proof of Corollary 4.4] involves constructing different homotopies, specific to each element we wish to anchor. The main point in the proof is that for each $x \in X_n$, the matrix $I_{n,n}$ lies in the identity component of the stabilizer of x^2. The fact that this requires no extra assumptions on the stabilizer appears to be a rather special feature of the unitary and general linear groups.

2.3. Bipermutative action sequences. We now extend the notion of permutative action sequence to a notion of *bipermutative action sequence*, in such a way that the translation category inherits the structure of a bipermutative category. As discussed in Section 8, this gives a relatively efficient way of verifying that the topological Atiyah–Segal map is a map of ring spectra.

In short, a bipermutative action sequence is a pair of permutative action sequences, sharing the same indexing set I, the same spaces X_i, and the same groups G_i. Additional coherence axioms relating the two permutative structures must hold

(and these axioms imply that the two monoid structures on I give it the structure of a rig, or a "ring without negatives"). We give the details in Definition 2.10 below; the axioms are just direct translations of the axioms for bipermutative categories. Maps of bipermutative action sequences are just maps that respect both permutative structures.

As an example, the Kronecker product of matrices endows the tautological unitary and general linear action sequences with a bipermutative structure. The details are just an elaboration of the discussion in [**19**, VI §5]. We note that some care must be taken when specifying the exact definition of Kronecker product, so that the coherence axioms hold.

DEFINITION 2.10. A bipermutative action sequence is rig R together with a pair of action sequences

$$((R,+),G,X,*_0,\oplus,C) \text{ and } ((R,\cdot),G,X,*_1,\otimes,D)$$

sharing the same groups G_r and the same G_r–spaces X_r for all $r \in R$. Let $0 \in R$ and $1 \in R$ denote the additive and multiplicative identity elements of R, respectively.

These data must satisfy the following additional axioms for all $r,s,t,u \in R$ and all $x,y,z,w \in X$ and all $g,h,k,l \in G$:

- Zero Axioms: $*_0 \otimes x = *_0 = x \otimes *_0$ and $e_0 \otimes g = e_0 = g \otimes e_0$ (recall that $e_0 \in G_0$ is the identity element);
- Right Distributivity Axioms: $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ and $(g \oplus h) \otimes k = (g \otimes k) \oplus (h \otimes k)$;
- Coherence Axioms: $C_{r,s} \otimes e_t = C_{r \cdot t, s \cdot t}$, and

$$(D_{t,r+s} \oplus D_{u,r+s})D_{r+s,t+u} =$$
$$(e_{r \cdot t} \oplus C_{r \cdot u, s \cdot t} \oplus e_{s \cdot u})\left[[(D_{t,r} \oplus D_{u,r})D_{r,t+u}] \oplus [(D_{t,s} \oplus D_{u,s})D_{s,t+u}]\right].$$

We will sometimes denote these sequences in the simplified form

$$(R,G,X,*_0,*_1,\oplus,C,\otimes,D).$$

A morphism of bipermutative action sequences

$$(R,G,X,*_0,*_1,\oplus,C,\otimes,D) \longrightarrow (S,H,Y,*_0,*_1,\oplus,C',\otimes,D')$$

consists of a function $f\colon R \to S$ that is a monoid homomorphism for both $+$ and $\cdot$, together with homomorphisms $\phi_r\colon G_r \to H_{f(r)}$ and ϕ_r–equivariant maps $X_r \to Y_{f(r)}$ (preserving both basepoints) for all $r \in R$. The homomorphisms ϕ_r must satisfy $\phi_{s+t}(C_{s,t}) = C'_{f(s+t)}$ and $\phi_{s \cdot t}(D_{s,t}) = D'_{f(s \cdot t)}$.

Bipermutative action sequence now form a category **BPAct**, and the translation category construction provides a functor from **BPAct** to the category of bipermutative categories (internal to **CGTop**).

Multiplicative infinite loop space theory, as developed in [**19, 17, 18**], provides a functor taking a bipermutative category $\mathcal{C}$ to a (connective) E_∞ ring spectrum $\mathbb{K}_\infty(\mathcal{C})$. Just as in the permutative case, the underlying infinite loop space $\Omega^\infty \mathbb{K}_\infty(\mathcal{C})$ is naturally weakly equivalent to the group completion $\Omega B\mathcal{C}$, where $B\mathcal{C}$ denotes the geometric realization of the bar construction applied to the nerve of $\mathcal{C}$, using its *additive* monoidal structure (the key step in the proof of this statement is [**18**, Theorem 9.3]).

3. Deformation K–theory

Here explain how to view the unitary and general linear deformation K–theory spectra associated to finitely generated discrete groups in terms of action sequences.

DEFINITION 3.1. Given a discrete group G, let $\mathcal{A}(G)$ denote the unitary permutative action sequence associated to the spaces $X_n = \mathrm{Hom}(G, U(n))$, which we topologize as subsets of the mapping spaces $\mathrm{Map}(G, U(n))$. Note that $\mathrm{Hom}(G, U(0))$ consists of a single point, which will serve as $*$. We let the unitary groups act on these spaces by conjugation, and the block sum operations are induced by block sum of matrices. Note that $\mathcal{A}(G)$ is contravariantly functorial in G.

The deformation K–theory spectrum $K^{\mathrm{def}}(G)$ is the connective Ω–spectrum associated to the permutative translation category $\mathcal{T}(\mathcal{A}(G))$, and we define

$$K_*^{\mathrm{def}}(G) = \pi_* K^{\mathrm{def}}(G) \cong \pi_* \Omega^\infty K^{\mathrm{def}}(G).$$

By Proposition 2.6, we have a natural homeomorphism

$$\Omega^\infty K^{\mathrm{def}}(G) \cong \Omega B \left(\coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))_{hU(n)} \right),$$

where the coproduct on the right has the topological monoid structure induced from the additive structure of $\mathcal{A}(G)$.

DEFINITION 3.2. Given a discrete group G, define

$$\mathrm{Rep}(G) = \mathrm{Rep}(G, U) := \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))$$

and

$$\mathrm{Rep}(G)_{hU} = \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))_{hU(n)}.$$

Block sum of matrices makes $\mathrm{Rep}(G)$ into a topological monoid, with the unique element in $\mathrm{Hom}(G, U(0))$ as the identity. Replacing $U(n)$ by $GL(n)$, we obtain the monoids $\mathrm{Rep}(G, GL)$ and $\mathrm{Rep}(G, GL)_{hU}$.

The main results of [23] give a description of $K_*^{\mathrm{def}}(G)$ in terms of spherical families of representations.

THEOREM 3.3 (Theorem 7.3 in [23]). *Let G be a discrete group. For each $m > 0$, there is are natural isomorphisms*

$$K_m^{\mathrm{def}}(G) \cong \widetilde{\Pi}_m(\mathrm{Rep}(G)) \oplus K^{-m}(*),$$

$$K_0^{\mathrm{def}}(G) \cong \widetilde{\Pi}_0(\mathrm{Rep}(G)) \cong \mathrm{Gr}(\pi_0 \mathrm{Rep}(G)).$$

Moreover, for $k > 0$ the natural map of monoids

$$[S^k, \mathrm{Rep}(G)] \longrightarrow \widetilde{\Pi}_m(\mathrm{Rep}(G))$$

is surjective, *with kernel*

$$\{[\rho] : \exists \text{ a constant map } c \colon S^k \to \mathrm{Rep}(G) \text{ such that } \rho \oplus c \text{ is nullhomotopic}\}.$$

If G is finitely generated, the analogous result holds in the general linear case.

It will also be helpful to consider a reduced form of deformation K–theory. The unitary and general linear cases of this discussion are identical.

For each group G, the map $\{1\} \to G$ induces a map of spectra

$$(6) \qquad q \colon K^{\mathrm{def}}(G) \to K^{\mathrm{def}}(\{1\}),$$

which admits a splitting induced by the map $G \to \{1\}$. Note that $K^{\mathrm{def}}(\{1\})$ is simply the connective K–theory spectrum **ku** (see [**19**, VIII §2], for instance).

DEFINITION 3.4. We define $\widetilde{K}^{\mathrm{def}}(G)$ to be the homotopy fiber of the natural map $K^{\mathrm{def}}(G) \to K^{\mathrm{def}}(\{1\})$, and we set $\widetilde{K}^{\mathrm{def}}_*(G) = \pi_*\widetilde{K}^{\mathrm{def}}(G)$.

COROLLARY 3.5. *There is a natural splitting*

$$K^{\mathrm{def}}_m(G) \cong \widetilde{K}^{\mathrm{def}}_m(G) \oplus K^{-m}(*)$$

for each $m \geqslant 0$, and for $m > 0$ there are natural isomorphisms

$$(7) \qquad \widetilde{K}^{\mathrm{def}}_m(G) \cong \widetilde{\Pi}_m \mathrm{Rep}(G) \cong \pi_m \Omega B\mathrm{Rep}(G).$$

Additionally, $\widetilde{K}^{\mathrm{def}}_0(G)$ is naturally isomorphic to the quotient of

$$\mathrm{Gr}(\pi_0\mathrm{Rep}(G)) \cong \pi_0 \Omega B\mathrm{Rep}(G)$$

by the subgroup generated by the trivial 1-dimensional representation.

PROOF. The first statement is immediate from the fact that (6) splits. For $m > 0$, [**23**, Proof of Theorem 6.5] shows that the sequence

$$(8) \qquad 0 \longrightarrow \widetilde{\Pi}_m \mathrm{Rep}(G) \xrightarrow{i_*} \widetilde{\Pi}_m \left(\mathrm{Rep}(G)_{hU}\right) \xrightarrow{q_*} \widetilde{\Pi}_m \left(\mathrm{Rep}(\{1\})_{hU}\right) \longrightarrow 0.$$

is split exact, which implies (7).

When $m = 0$, the splitting of $q \colon K^{\mathrm{def}}(G) \to K^{\mathrm{def}}(\{1\})$ gives a natural isomorphism between $\widetilde{K}^{\mathrm{def}}_0(G)$ and the cokernel of the map $K^{\mathrm{def}}_0(\{1\}) \to K^{\mathrm{def}}_0(G)$, whose image is generated by the trivial 1-dimensional representation. $\square$

4. Topological K–theory

In this section, $X = (X, x_0)$ will denote a (based) paracompact space having the homotopy type of compact Hausdorff space. (For instance, X might be $B\pi_1(K)$ for some aspherical finite CW complex K.) We will define a permutative category whose homotopy groups agree with the complex topological K–theory of X. Our construction, and the subsequent discussion, is designed to mirror the construction of deformation K–theory in the previous section. This will facilitate our construction and analysis of the topological Atiyah–Segal map in the next section. As in the previous section, there is both a unitary and a general linear version of the constructions given here. We focus on the unitary case; the general linear case is completely analogous (more so than for deformation K–theory).

DEFINITION 4.1. As before, let $BU(n)$ denote the geometric realization of the one-object category $\underline{U(n)}$. Then the (left) conjugation action of $U(n)$ on itself induces an action, by continuous functors, of $U(n)$ on the category $\underline{U(n)}$, and hence an action of $U(n)$ on $BU(n)$. This in turn induces an action of $U(n)$ on the based mapping space $\mathrm{Map}_*(X, BU(n))$. Throughout this section, we will view $\mathrm{Map}_*(X, BU(n))$ as a $U(n)$–space under this action.

The block sum operations $\oplus\colon U(m) \times U(n) \to U(m+n)$ are homomorphisms, and hence induce continuous functors $\underline{U(m)} \times \underline{U(n)} \to \underline{U(m+n)}$, maps

$$BU(m) \times BU(n) \longrightarrow BU(m+n),$$

and equivariant maps

$$\oplus\colon \operatorname{Map}_*(X, BU(m)) \times \operatorname{Map}_*(X, BU(n)) \longrightarrow \operatorname{Map}_*(X, BU(m+n)).$$

Since block sum is equivariant with respect to conjugation (that is, $(CAC^{-1}) \oplus (DBD^{-1}) = (C \oplus D)(A \oplus B)(C \oplus D)^{-1}$), functoriality implies that this data gives a unitary permutative action sequence $\mathcal{A}_K(X)$ with nth space $\operatorname{Map}_*(X, BU(n))$.

Note that when $X = *$, we recover the unitary permutative action sequence whose associated spectrum is $\mathbf{ku}$.

Let $\mathcal{C}_K(X) = \mathcal{T}(\mathcal{A}_K(X))$ be the translation category of $\mathcal{A}_K(X)$, and let $\mathcal{K}(X)$ denote the associated spectrum. Let $\widetilde{\mathcal{K}}(X)$ denote the homotopy fiber of the natural map $\mathcal{K}(X) \to \mathcal{K}(*)$. Finally, set $\mathcal{K}_*(X) = \pi_*\mathcal{K}(X)$ and $\widetilde{\mathcal{K}}_*(X) = \pi_*\widetilde{\mathcal{K}}(X)$.

We have the following consequence of Propositions 2.6 and 2.8.

COROLLARY 4.2. *The geometric realization of the nerve of $\mathcal{C}_K(X)$ is isomorphic, as a topological monoid, to the topological monoid*

$$\mathcal{V}(X)_{hU} := \coprod_n \operatorname{Map}_*(X, BU(n))_{hU(n)},$$

and for each $m \geqslant 0$, there are natural isomorphisms

$$\widetilde{\Pi}_m\left(\mathcal{V}(X)_{hU}\right) \overset{\cong}{\longrightarrow} \pi_m \Omega B\left(\mathcal{V}(X)_{hU}\right) \cong \mathcal{K}_m(X).$$

Our goal in this section is to compare the homotopy groups $\mathcal{K}_*(X)$ with the (complex) topological K–theory of X (for $* \geqslant 0$).

We need to specify a definition of topological K–theory. Note that the Group Completion Theorem gives a natural homotopy equivalence

$$\mathbb{Z} \times BU \longrightarrow \Omega B\left(\coprod_n BU(n)\right),$$

where $BU = \operatorname{colim}_n BU(n)$, the colimit being formed with respect to the maps induced by block sum with the identity $I_1 \in U(1)$.

For based spaces Y and Z, let $\langle Y, Z \rangle$ be the set of based homotopy classes of based maps, and let $\langle f \rangle$ be the based homotopy class of f.

DEFINITION 4.3. *For $m \geqslant 0$, the reduced topological K–theory of X is*

$$\widetilde{K}^{-m}(X) = \widetilde{K}^0(S^m \wedge X) = \langle S^m \wedge X, \mathbb{Z} \times BU \rangle$$

and the unreduced K–theory of X is

$$K^{-m}(X) = \widetilde{K}^{-m}(X) \oplus K^{-m}(*) = \widetilde{K}^{-m}(X) \oplus \pi_m(\mathbf{ku}).$$

Maps $X \to Y$ induce maps on both reduced an unreduced K–theory (in the latter case, all maps act as the identity on $K^{-m}(*)$).

Note that by the Group Completion Theorem, we have a natural zig-zag of homotopy equivalences connecting $\mathbb{Z} \times BU$ and $\Omega B(\coprod_n BU(n))$, so one may replace $\mathbb{Z} \times BU$ by $\Omega B(\coprod_n BU(n))$ in the above definition.

We will need to consider another topological monoid related to $\mathcal{K}(X)$.

DEFINITION 4.4. Define $\mathcal{V}(X) := \coprod_n \mathrm{Map}_*(X, BU(n))$, and equip $\mathcal{V}(X)$ with the monoid structure induced by the block sum operations on $\{BU(n)\}_n$.

We now have the following analogue of the results from Section 3.

COROLLARY 4.5. *Let* (X, x_0) *be a based, path-connected CW complex homotopy equivalent to a finite CW complex. For each* $m \geqslant 0$, *there is a natural splitting*

$$(9) \qquad \mathcal{K}_m(X) \cong \widetilde{\mathcal{K}}_m(X) \oplus \mathcal{K}_m(*) = \widetilde{\mathcal{K}}_m(X) \oplus \pi_m \mathbf{ku}$$

and a natural isomorphism

$$\widetilde{\mathcal{K}}_m(X) \cong \widetilde{K}^{-m}(X).$$

Moreover, for $m > 0$ *there are natural isomorphisms*

$$\widetilde{\mathcal{K}}_m(X) \cong \widetilde{\Pi}_m \mathcal{V}(X) \cong \pi_m \Omega B \mathcal{V}(X),$$

while for $m = 0$, $\widetilde{\mathcal{K}}_0(X) \cong \widetilde{K}^0(X)$ *is naturally isomorphic to the quotient of*

$$\mathrm{Gr}(\pi_0 \mathcal{V}(X)) \cong \pi_0 \Omega B \mathcal{V}(X)$$

by the subgroup generated by the class of nullhomotopic maps $X \to BU(1)$.

PROOF. The splitting (9) is immediate since the inclusion $\{x_0\} \hookrightarrow X$ splits. Next, consider the sequence of groups

$$(10) \quad \widetilde{\Pi}_m(\mathcal{V}(X)) \longrightarrow \widetilde{\Pi}_m(\mathcal{V}(X)_{hU}) = \mathcal{K}_m(X) \longrightarrow \widetilde{\Pi}_m \left(\coprod_n (BU(n)) \right) = \mathcal{K}_m(*)$$

induced by the maps

$$(11) \qquad \mathrm{Map}_*(X, BU(n)) \longrightarrow \mathrm{Map}_*(X, BU(n))_{hU(n)} \longrightarrow BU(n).$$

Using the fact that the sequences (11) are (split) fibrations, one can check directly that (10) is a (split) short exact sequence for each $m > 0$. By naturality, it follows that $\widetilde{\Pi}_m(\mathcal{V}(X)) \cong \widetilde{\mathcal{K}}_m(X)$ for $m > 0$. We will return to the analogous description of $\widetilde{\mathcal{K}}_0(X)$ at the end of the proof.

Next we show that for $m > 0$, there is an isomorphism

$$\widetilde{K}^{-m}(X) = \langle S^m \wedge X, \mathbb{Z} \times BU \rangle \cong \langle S^m, \mathrm{Map}_*(X, BU) \rangle \xrightarrow[\cong]{\Phi} \widetilde{\Pi}_m \mathcal{V}(X).$$

To define the map Φ, note that our hypotheses on X guarantee that every map $f \colon S^m \wedge X \to BU$ factors, up to homotopy, through $BU(n)$ for some n, yielding a well-defined element of $\widetilde{\Pi}_m \mathcal{V}(X)$. To see that Φ is surjective, note that our hypotheses guarantee that every vector bundle over X is a direct summand of a trivial bundle, and hence every class in $\widetilde{\Pi}_m \mathcal{V}(X)$ is represented by a map

$$S^m \to \mathrm{Map}_*(X, BU(n))$$

whose value the basepoint in S^m is nullhomotopic. Since the basepoint of S^m is non-degenerate, we can in fact obtain a representative lying in

$$\mathrm{Map}_*(S^m, \mathrm{Map}_*(X, BU(n))),$$

as desired. To prove injectivity of Φ, say $\Phi(\langle f \rangle) = 0$ for some

$$f \in \mathrm{Map}_*(S^m, \mathrm{Map}_*(X, BU(n))).$$

Note that it since the basepoint of X is non-degenerate, will suffice to show that f is nullhomotopic in the *unbased* sense (see [**9**, Section 4A], for instance). Since $\Phi(\langle f \rangle) = 0$, there exist $h, g \in \mathrm{Map}_*(X, BU(k))$ (for some k) such that

$$[f \oplus c_g] = [c_h]$$

(where c_g and c_h are constant maps out of S^m with values h and g). Now, there exists $g' \in \mathrm{Map}_*(X, BU(l))$ such that $g \oplus g'$ is nullhomotopic, so we have

$$[f] = [f \oplus c_g \oplus c_{g'}] = [c_{h+g'}]$$

in $[S^m, \mathrm{Map}_*(X, BU)]$, and hence f is nullhomotopic as a map into $\mathrm{Map}_*(X, BU)$.

For $m = 0$, connectedness of $U(n)$ yields bijections

$$\pi_0 \mathrm{Map}_*(X, BU(n))_{hU(n)} \cong \pi_0 \mathrm{Map}_*(X, BU(n)) \cong \langle X, BU(n) \rangle \cong [X, BU(n)]$$

so

$$\mathcal{K}_0(X) = \pi_0 \mathcal{V}(X)_{hU} \cong \mathrm{Gr}\left(\pi_0 \mathcal{V}(X)_{hU}\right) \cong \mathrm{Gr}\left(\pi_0 \mathcal{V}(X)\right) = \mathrm{Gr}[X, \coprod_n BU(n)].$$

Now,

$$\mathrm{Gr}[X, \coprod_n BU(n)] \cong \mathrm{colim}([X, \coprod_n BU(n)] \to [X, \coprod_n BU(n)] \to \cdots),$$

where the colimit is taken with respect to addition by the basepoint in $BU(1)$. This colimit is precisely $\mathbb{Z} \times \mathrm{colim}_n[X, BU(n)] \cong \mathbb{Z} \times \langle X, BU \rangle \cong K^0(X)$.

The splitting (9) identifies $\widetilde{\mathcal{K}}_0(X)$ with the cokernel of $\mathcal{K}_0(*) \to \mathcal{K}_0(X)$. By the discussion above, this agrees with cokernel of $\mathrm{Gr}\left(\pi_0 \mathcal{V}(*)\right) \to \mathrm{Gr}\left(\pi_0 \mathcal{V}(X)\right)$. The claimed description of $\widetilde{\mathcal{K}}_0(X)$ now follows from the fact that $\mathcal{V}(*)$ is generated by the constant map from $*$ to the basepoint of $BU(1)$. $\qquad \square$

REMARK 4.6. Corollary 4.5 requires the assumption that X is path-connected. For instance, Corollary 4.2 gives

$$\mathcal{K}_0(S^0) \cong \mathrm{Gr}\left(\coprod_n \mathrm{Map}_*(S^0, BU(n))_{hU(n)}\right),$$

and since $BU(n)$ is simply-connected, each set

$$\pi_0 \mathrm{Map}_*(S^0, BU(n))_{hU(n)} \cong \pi_0 \mathrm{Map}_*(S^0, BU(n))$$

is a singleton. So $\mathcal{K}_0(S^0) = \mathbb{Z}$, whereas $K^0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$.

5. The topological Atiyah–Segal map

Let G be a discrete group whose classifying space BG has the homotopy type of a finite CW complex. We now define reduced and unreduced versions of the topological Atiyah–Segal map, which relates the deformation K–theory of G to the topological K–theory of BG. The unitary and general linear discussions are completely parallel, so we focus on the unitary case.

The classical Atiyah–Segal map associates to each representation $\rho \colon G \to U(n)$ the K–theory class represented by the vector bundle $E_\rho = (EG \times \mathbb{C}^n)/\pi_1 G \to BG$, where $\pi_1 G$ acts via $\gamma \cdot (e, v) = (e \cdot \gamma^{-1}, \rho(\gamma)v)$. We will see that in dimension zero, the classical Atiyah–Segal map factors as

$$R[G] = \mathrm{Gr}(\mathrm{Rep}(G)^\delta) \longrightarrow \mathrm{Gr}(\pi_0 \mathrm{Rep}(G)) \xrightarrow{\alpha_0} K^0(BG),$$

where $\mathrm{Rep}(G)^\delta$ is the discrete monoid underlying the topological monoid $\mathrm{Rep}(G)$, and α_0 is the topological Atiyah–Segal map (in dimension 0).

The simplicial classifying space functor B induces continuous, $U(n)$–equivariant maps

$$(12) \qquad B = B_n \colon \mathrm{Hom}(G, U(n)) \longrightarrow \mathrm{Map}_*(BG, BU(n))$$
$$\rho \longmapsto B\rho$$

which combine to give a map between the associated unitary permutative action sequences. Recall that the spectra associated to these action sequences are $K^{\mathrm{def}}(G)$ and $\mathcal{K}(BG)$, respectively, and the homotopy groups of the latter are the complex K–theory groups of BG.

DEFINITION 5.1. The *topological Atiyah–Segal map*
$$\alpha = \alpha^G \colon K^{\mathrm{def}}(G) \longrightarrow \mathcal{K}(BG)$$

is the map of spectra induced by the above map of permutative action sequences. The reduced topological Atiyah–Segal map
$$\widetilde{\alpha} \colon \widetilde{K}^{\mathrm{def}}(G) \longrightarrow \widetilde{\mathcal{K}}(BG)$$

is the induced map

$$\mathrm{hofib}\left(K^{\mathrm{def}}(G) \to K^{\mathrm{def}}(\{1\})\right) \longrightarrow \mathrm{hofib}\left(\mathcal{K}(BG) \to \mathcal{K}(*)\right).$$

Note that we have $K^{\mathrm{def}}(\{1\}) = \mathcal{K}(*) = \mathbf{ku}$, and $\alpha^{\{1\}}$ is the identity map.

The results in the previous sections combine to give the following descriptions of the induced maps α_* and $\widetilde{\alpha}_*$ on homotopy groups.

COROLLARY 5.2. *For $m \geqslant 0$, the topological Atiyah–Segal map is naturally isomorphic to the maps*

$$\pi_m \Omega B \mathrm{Rep}(G)_{hU} \longrightarrow \pi_m \Omega B \mathcal{V}(BG)_{hU}$$

and

$$\widetilde{\Pi}_m \mathrm{Rep}(G)_{hU} \longrightarrow \widetilde{\Pi}_m \Omega B \mathcal{V}(BG)_{hU}$$

induced by the simplicial classifying space functor B, and there is a natural splitting

$$(13) \qquad\qquad\qquad \alpha_* = \widetilde{\alpha}_* \oplus \mathrm{Id}_{\pi_* \mathbf{ku}}.$$

Moreover, for $m > 0$, $\widetilde{\alpha}_m$ is naturally isomorphic to the maps

$$\pi_m \Omega B \mathrm{Rep}(G) \longrightarrow \pi_m \Omega B \mathcal{V}(BG)$$

and

$$\widetilde{\Pi}_m \mathrm{Rep}(G) \longrightarrow \widetilde{\Pi}_m \mathcal{V}(BG)$$

induced by B.

Our goal in this section is to give an explicit description of the topological Atiyah–Segal map in terms of vector bundles. Consider the diagram

$$(14) \qquad \widetilde{\Pi}_m \mathrm{Rep}(G) \xrightarrow{\ \widetilde{\alpha}_m\ } \widetilde{\Pi}_m \mathcal{V}(BG)$$
$$\Psi \Big\downarrow \cong$$
$$\pi_m \mathrm{Map}_*(BG, BU) \cong \widetilde{K}^{-m}(BG).$$

where Ψ is defined by $\Psi(\langle f \rangle) = [f]$. We will describe the diagonal map $\Psi^{-1} \circ \widetilde{\alpha}_m$.

By Theorem 3.3, each class in $\widetilde{\Pi}_m \mathrm{Rep}(G) \cong K_m^{\mathrm{def}}(G)$ has a representative of the form $[\rho]$, with ρ a family of representations $\rho\colon S^m \to \mathrm{Rep}(G)$. Let E_ρ be the right principal $U(n)$–bundle over $S^m \times BG$ defined by

$$E_\rho = (S^m \times EG \times U(n))\,/G \longrightarrow S^m \times BG$$

$$[z, e, A] \longmapsto (z, q(e)),$$

where $q\colon EG \to BG$ is the bundle projection and $g \in G$ acts via

$$g \cdot (z, x, A) := (z, x \cdot g^{-1}, (\rho(z)(g))A).$$

Basic properties of this construction are reviewed in [**5**, Section 3].

We will use $1 \in S^0 \subset S^m$ as the basepoint of S^m, and for any family

$$\rho\colon S^m \to \mathrm{Hom}(G, U(n)),$$

we let $\widetilde{\rho(1)}\colon S^m \to \mathrm{Hom}(G, U(n))$ denote the constant family at $\rho(1)\colon G \to U(n)$.

For based CW complexes X_1 and X_2, the long exact sequence in K–theory for the pair $(X_1 \times X_2, X_1 \vee X_2)$ yields a (naturally) split short exact sequence

$$0 \longrightarrow \widetilde{K}^0(X_1 \wedge X_2) \xrightarrow{\pi^*} \widetilde{K}^0(X_1 \times X_2) \xrightarrow{i^*} \widetilde{K}^0(X_1 \vee X_2) \longrightarrow 0.$$

If ρ is an S^m–family $(m > 0)$, the bundle $E_\rho \to S^m \times BG$ is trivial when restricted to $S^m \times \{x\}$ (for each point $\{x\} \in BG$): indeed, each point $\widetilde{x} \in q^{-1}(x) \subset EG$ gives rise to a continuous section $z \mapsto [z, \widetilde{x}, I]$ of the restricted bundle. Thus we have

$$E_\rho|_{S^m \vee BG} \cong E_{\widetilde{\rho(1)}}|_{S^m \vee BG},$$

and hence

$$[E_\rho] - [E_{\widetilde{\rho(1)}}] \in \ker(i^*) = \mathrm{Im}(\pi^*).$$

Since π^* is injective, it follows that the class $[E_\rho] - [E_{\widetilde{\rho(1)}}]$ has a well-defined preimage under π^*, which we will denote by

$$(15) \qquad (\pi^*)^{-1}([E_\rho] - [E_{\widetilde{\rho(1)}}]) \in \widetilde{K}^0(S^m \wedge BG) = \widetilde{K}^{-m}(BG).$$

By [**5**, Lemma 3.1(2)], the bundles E_ρ and $E_{\widetilde{\rho(1)}}$ only depend (up to isomorphism) on the unbased homotopy class of ρ. Hence the class (15) depends only the unbased homotopy class of ρ. Note that if ρ is constant, then $E_\rho = E_{\widetilde{\rho(1)}}$, so in this case the class (15) is trivial.

With this understood, we have the following explicit description of $\widetilde{\alpha}_*$ (or more precisely, of the map $\Psi^{-1} \circ \widetilde{\alpha}_*$ in Diagram (14)).

THEOREM 5.3. *Let G be a group whose classifying space BG is homotopy equivalent to a finite CW complex. Then for $m \geqslant 1$, the reduced topological Atiyah–Segal map, viewed as a map*

$$\widetilde{\Pi}_m \mathrm{Rep}(G) \longrightarrow \widetilde{K}^{-m}(BG)$$

via the diagram (14), has the form

$$(16) \qquad [\rho] \mapsto (\pi^*)^{-1}\left([E_\rho] - [E_{\widetilde{\rho(1)}}]\right).$$

When $m = 0$, the map

$$\alpha_0\colon K_0^{\mathrm{def}}(G) \cong \mathrm{Gr}(\pi_0 \mathrm{Rep}(G)) \longrightarrow K^0(BG)$$

is given by $\alpha_0([\rho]) = [E_\rho]$.

Note that the statement for $m = 0$ shows that the classical Atiyah–Segal map factors through α_0, as claimed earlier.

PROOF. We assume $m > 0$; the proof for $m = 0$ is similar but simpler. By definition, $\widetilde{\alpha}_*([\rho]) = [B \circ \rho]$, where B is the map (12). Let $f\colon S^m \wedge BG \to BU(n)$ be a map classifying $(\pi^*)^{-1}\left([E_\rho] - [E_{\widetilde{\rho(1)}}]\right)$. To prove the proposition, we need to show that the adjoint map $f^\vee\colon S^m \to \mathrm{Map}_*(BG, BU(n))$ satisfies

$$\Psi(\langle f^\vee \rangle) = [B \circ \rho]$$

in $\widetilde{\Pi}_m \mathcal{V}(BG)$. (Recall that $\Psi(\langle f^\vee \rangle)$ is simply $[f^\vee]$.)

By choice of f, the composite $f \circ \pi$ classifies $[E_\rho] - [E_{\widetilde{\rho(1)}}]$, and by [5, Lemma 4.1], if c is the constant map $S^m \to \mathrm{Map}_*(BG, BU(p))$ with image $B(\rho(1))$, then $c^\vee \circ \pi$ classifies $[E_{\widetilde{\rho(1)}}]$ (here $c^\vee$ is the adjoint of c), while $(B \circ \rho)^\vee \circ \pi$ classifies $[E_\rho]$. Hence the maps

$$(f \circ \pi) \oplus (c^\vee \circ \pi) = (f \oplus c^\vee) \circ \pi \quad \text{and} \quad (B \circ \rho)^\vee \circ \pi$$

represent the same class in $\widetilde{K}^0(S^m \wedge BG)$. Since π^* is injective, it follows that $f \oplus c^\vee$ and $(B \circ \rho)^\vee$ are homotopic as maps $S^m \wedge BG \to BU(N)$ (for sufficiently large N), and consequently $f^\vee \oplus c$ and $B \circ \rho$ are homotopic as maps $S^m \to \mathrm{Map}_*(BG, BU(N))$. Since c is constant, it follows that $[B \circ \rho] = [f^\vee] = \Psi(\langle f^\vee \rangle)$ in $\widetilde{\Pi}_m \mathcal{V}(BG)$.   □

One may replace $U(n)$ by $GL(n)$ throughout the preceding discussion, yielding a general linear version α^{GL} of the topological Atiyah–Segal map. We note that the unitary topological Atiyah–Segal map factors through this general linear version, which leads to the following natural question.

QUESTION 5.4. Does there exist a group G for which the image of α^{GL}_* is strictly larger than the image of α_*?

6. Relations with previous work

In this section, we reinterpret some of the main results from [5], [20], and [25] in terms of the topological Atiyah–Segal map.

6.1. Bounds on the image of α_*. We now show that $\widetilde{\alpha}_*$ fails to be surjective in dimensions $\mathbb{Q}cd(G) - 2k$ ($k > 0$), where $\mathbb{Q}cd(G)$ is the largest number n for which $H^n(G; \mathbb{Q})$ is non-zero (Theorem 6.1). This low-dimensional failure is closely analogous to the low-dimensional failure of the Quillen–Lichtenbaum conjectures (in the form discussed in [26], for instance), which the relate algebraic K–theory of a field k to its étale K–theory in dimensions greater than the virtual cohomological dimension of k minus 2.

THEOREM 6.1. *The image of $\alpha_*\colon K^{\mathrm{def}}_m(G) \to K^{-m}(BG)$ (and, in fact, of α^{GL}_*) has rank at most $\beta_m(G) + \beta_{m-2}(G) + \cdots$, where $\beta_i(G)$ is the rank of $H_i(BG; \mathbb{Z})$. Hence if $\beta_d(G) \neq 0$, then the maps $\alpha_{d-2}, \alpha_{d-4}, \ldots$ are not surjective.*

PROOF. By Theorem 5.3 and [5, Theorem 3.5], the image of α_m lies in the subgroup of $K^{-m}(BG)$ on which the Chern classes c_{m+i}, $i = 1, 2, \cdots$, vanish rationally. It follows from [5, Lemma 4.2] that the rank of this subgroup is given by the above sum. The last statement follows from the fact that the Chern character is a rational isomorphism, which implies that the rank of $K^{-m}(BG)$ is equal to the sum of all the Betti numbers of BG in dimensions equivalent to m modulo 2.   □

6.2. Relation with the Novikov conjecture. A group G satisfies the *strong Novikov conjecture* if the analytical assembly map (from the K–homology of BG to the K–theory of the maximal C^*–algebra of G) is injective after tensoring with $\mathbb{Q}$. For background on this conjecture, see [**20**] and the references therein.

THEOREM 6.2. *If G is a group such that BG is homotopy equivalent to a finite CW complex, and the unitary topological Atiyah–Segal map α_m is rationally surjective for all sufficiently large m, then G satisfies the strong Novikov conjecture.*

PROOF. Surjectivity of α_m is equivalent to surjectivity of $\widetilde{\alpha}_m$, and surjectivity of $\widetilde{\alpha}_m$ implies that for sufficiently large n, every element in $[S^m, \mathrm{Map}_*(BG, BU(n))]$ has the form $[B \circ \rho]$ for some ρ. It now follows from [**20**, Theorem 3.16] (or rather from the proof of that result) that G lies in the class of *flatly detectable groups*; all such groups satisfy the strong Novikov conjecture by [**20**, Corollary 4.3]. $\square$

6.3. Surface groups. We now translate the results in [**25**] into information about the topological Atiyah–Segal map when $G = \pi_1(\Sigma)$ is the fundamental group of a compact, aspherical surface Σ. Note that we allow Σ to have boundary, in which case $\pi_1(\Sigma)$ is a finitely generated free group. We begin by recalling a result from [**25**], which was proven using Morse theory for the Yang–Mills functional.

THEOREM 6.3 ([**25**], Theorem 3.4). *If Σ is a compact aspherical surface, possibly with boundary, then for each $M \geqslant 0$, there exists N such that for every $n > N$ the natural map*

$$B_*\colon \pi_m\mathrm{Hom}(\pi_1\Sigma, U(n)) \longrightarrow \pi_m\mathrm{Map}_*(B\pi_1\Sigma, BU(n))$$

induces an isomorphism on homotopy groups in dimensions $1 \leqslant m \leqslant M$ (for all choices of compatible basepoints). If Σ is non-orientable, or has non-empty boundary, then this statement holds for $0 \leqslant m \leqslant M$.

THEOREM 6.4. *If Σ is a compact aspherical surface, possibly with boundary, then the topological Atiyah–Segal map*

$$\alpha_m\colon K_m^{\mathrm{def}}(\pi_1\Sigma) \longrightarrow K^{-m}(\Sigma)$$

is an isomorphism for $m \geqslant 1$. If Σ is non-orientable, or has non-empty boundary, then α_0 is an isomorphism as well.

PROOF. Note that it suffices to prove that $\widetilde{\alpha}_m$ is an isomorphism. We work in the case $m > 0$; the same reasoning will apply when $m = 0$ and Σ is non-orientable or has non-empty boundary, using the last part of Theorem 6.3.

We prove injectivity; the proof of surjectivity is similar but simpler. By Theorem 3.3, each element in the kernel of $\widetilde{\alpha}_m$ has the form $[\rho]$ for some

$$\rho\colon S^m \to \mathrm{Hom}(\pi_1\Sigma, U(n))$$

satisfying

(17) $$(B \circ \rho) \oplus d \simeq c$$

for some constant maps $c, d\colon S^m \to \mathcal{V}(B\pi_1\Sigma)$. In fact, by [**22**, Corollaries 4.11 and 4.12] we may assume without loss of generality that $d = BI_p$ for some p, where I_p denotes the constant map $S^m \to \mathrm{Hom}(\pi_1\Sigma, U(p))$ with image the trivial representation. Indeed, for orientable surfaces the homomorphism spaces are all path connected, while in the non-orientable case each homomorphism space has two path components, and adding representations from the non-trivial components

always yields a representation in the trivial component, so (for instance) we may simply replace d by $d \oplus d$ if necessary.

Equation (17) now implies that $B \circ (\rho \oplus I_p)$ is nullhomotopic, and the injectivity portion of Theorem 6.3 implies that $\rho \oplus I_p$ must be nullhomotopic. Hence $[\rho] = 0$ in $\widetilde{\Pi}_m(\mathrm{Rep}(\pi_1 \Sigma))$, as desired. $\square$

7. Families of flat connections over the Heisenberg manifold

In this section we study flat, unitary connections on complex vector bundles over the 3–dimensional Heisenberg manifold by combining the main results of this paper with computations due to Tyler Lawson. We begin with a review of the definition and the basic properties of this manifold.

7.1. Background. The discrete 3–dimensional Heisenberg group H is the group of 3×3 upper triangular integer matrices under ordinary matrix multiplication. This group sits as a (discrete) subgroup of the real Heisenberg group $H_{\mathbb{R}}$, which consists of all real upper triangular matrices. We will identify the real Heisenberg group with $\mathbb{R}^3$ via the function

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto (x, y, z)$$

(note, though, that we are using matrix multiplication to define the operation in $H_{\mathbb{R}}$, not addition in the vector space $\mathbb{R}^3$). The Heisenberg manifold is defined by

$$N^3 = \mathbb{R}^3 / H,$$

where H acts on $\mathbb{R}^3 \cong H_{\mathbb{R}}$ by (left) multiplication.

It is an elementary exercise to check that N^3 is Hausdorff, and that the map $\mathbb{R}^3 \to N^3$ is a covering. In particular, this means N^3 is an aspherical 3-dimensional manifold with fundamental group H (and hence $N^3 \simeq BH$), and N^3 is orientable since the action of H on $\mathbb{R}^3$ is orientation-preserving. Moreover, N^3 is compact; this follows, for instance, from the fact that each closed unit cube in $\mathbb{R}^3$ surjects onto N^3. In fact, N^3 is a circle bundle over $\mathbb{R}^2 / \mathbb{Z}^2$. Indeed, consider the mapping

$$N^3 = \mathbb{R}^3 / H \xrightarrow{q} \mathbb{R}^2 / \mathbb{Z}^2$$

given by sending $[(x, y, z)]$ to $[(x, y)]$. It is elementary to check that this map is a fiber bundle with circle fibers; indeed, for each $[(x, y)] \in \mathbb{R}^2 / \mathbb{Z}^2$, there exists $\epsilon > 0$ such that the mapping

$$[x - \epsilon, x + \epsilon] \times [y - \epsilon, y + \epsilon] \times \mathbb{R}/\mathbb{Z} \longrightarrow N^3$$
$$(x', y', [z]) \longmapsto [(x', y', z)]$$

is a homeomorphism onto

$$q^{-1}\left(\pi([x - \epsilon, x + \epsilon] \times [y - \epsilon, y + \epsilon])\right),$$

where $\pi \colon \mathbb{R}^2 \to \mathbb{R}^2 / \mathbb{Z}^2$ is the quotient map.

The fibration sequence $S^1 \to N^3 \to \mathbb{R}^2 / \mathbb{Z}^2$ gives rise to a short-exact sequence on fundamental groups:

$$1 \longrightarrow \mathbb{Z} \longrightarrow H \xrightarrow{q_*} \mathbb{Z}^2 \longrightarrow 1.$$

Covering space theory gives canonical identifications of $\pi_1(\mathbb{R}^3/H, [(0,0,0)])$ and $\pi_1(\mathbb{R}^2/\mathbb{Z}^2, [(0,0)])$ with H and $\mathbb{Z}^2$ (respectively), and under these identifications the map q_* is simply

$$
\begin{bmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{bmatrix} \mapsto (a, b).
$$

The kernel of q_* is generated by

$$
Z = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
$$

which is the commutator of the elements

$$
X = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \text{ and } Y = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
$$

It is elementary to check that X and Y generate H, and it follows that $\ker(q_*)$ is precisely the commutator subgroup of H, giving

(18) $$H_1(N^3; \mathbb{Z}) \cong \mathbb{Z}^2.$$

Since Z commutes with both X and Y, we see that $\ker(q_*)$ is central, and it follows that H is a nilpotent group. Poincaré Duality, together with (18), yields:

$$
H_i(N^3; \mathbb{Z}) \cong H^i(N^3; \mathbb{Z}) \cong \begin{cases}
\mathbb{Z}^2 & \text{if } i = 1, 2, \\
\mathbb{Z}, & \text{if } i = 0, 3.
\end{cases}
$$

In particular, the cohomology of N^3 is torsion-free.

7.2. Flat bundles over the Heisenberg manifold. To understand flat bundles over N^3, we will use the following fact.

PROPOSITION 7.1. *If X is a finite CW complex with torsion-free integral cohomology, and $E \to X$ is a complex vector bundle whose Chern classes $c_i(E) \in H^{2i}(X; \mathbb{Z})$ vanish for $i \geqslant 1$, then E is stably trivial.*

In particular, if M is a smooth manifold with torsion-free integral cohomology and $E \to M$ is a vector bundle admitting a flat connection, then E is stably trivial.

Note that for finite CW complexes, $H^*(X; \mathbb{Z})$ is torsion-free if and only if $H_*(X; \mathbb{Z})$ is torsion-free.

PROOF. The second statement follows from the first, since by Chern–Weil theory the Chern classes of a flat vector bundle over M map to zero in $H^*(M; \mathbb{Q})$, and when $H^*(M; \mathbb{Z})$ is torsion-free the natural map $H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{Q})$ is injective.

To prove the first statement, consider a vector bundle $E \to X$ with $c_i(E) = 0$ for $i \geqslant 1$. By [**3**, Section 2.5] (see also [**10**, Proposition 6.10]) the complex X–theory of X is torsion-free (and finitely generated), so the natural map $\widetilde{K}^*(X) \to \widetilde{K}^*(X) \otimes \mathbb{Q}$ is injective. Composing with the Chern character gives an injection

$$
\widetilde{K}^*(X) \hookrightarrow \widetilde{K}^*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \widetilde{H}^*(X; \mathbb{Q}).
$$

Since $[E]$ maps to zero under this injection, we have $[E] = 0$ in $\widetilde{K}^*(X)$. $\qquad\square$

COROLLARY 7.2. *Let G be a discrete group whose classifying space BG has torsion-free integral cohomology and has the homotopy type of a CW complex X of dimension at most d. Then if $n \geqslant d/2$, the bundle E_ρ associated to a representation $\rho \colon G \to GL(n)$ is always trivial.*

PROOF. By Proposition 7.1, we know that (the vector bundle associated to) E_ρ is stably trivial, meaning that its classifying map

$$B\rho \colon BG \simeq X \to BGL(n)$$

becomes nullhomotopic after composing with the natural map

$$j \colon BGL(n) \to BGL(n + n', \mathbb{C})$$

for sufficiently large n'. Since j induces an isomorphism on homotopy groups up to dimension $2n$ and a surjection in dimension $2n + 1$, the Whitehead Theorem [**16**, Section 10.3] shows that for all CW complexes X of dimension at most $2n$, the map

$$[X, BGL(n)] \longrightarrow [X, BGL(\infty)]$$

is bijective. In particular, since $j \circ B\rho$ is nullhomotopic, we conclude that $B\rho$ is itself nullhomotopic so long as $d \leqslant 2n$. □

COROLLARY 7.3. *Let $G = \pi_1 K$, where K is an aspherical, 3–dimensional CW complex with torsion-free integral cohomology. Then for each representation $\rho \colon G \to GL_n(\mathbb{C})$, the associated vector bundle E_ρ is trivial.*

PROOF. Since $BG \simeq K$, Corollary 7.2 shows that E_ρ is trivial whenever the degree of ρ is at least 2. Our assumptions imply that the abelianization of K is free abelian, so $\mathrm{Hom}(G, GL_1(\mathbb{C})) \cong (GL_1\mathbb{C})^k$ for some k, and in particular is path-connected. Hence for each $\rho \to G \to GL_1(\mathbb{C})$, we have $E_\rho \cong E_1 \cong BG \times \mathbb{C}$. □

7.3. Homotopy in the space of flat connections over N^3. Let $\mathcal{A}_n^\flat(N^3)$ denote the space of flat connections on the trivial bundle $N^3 \times U(n)$ (or, equivalently, the space of flat unitary connections on $N^3 \times \mathbb{C}^n$). More precisely, $\mathcal{A}_n^\flat(N^3)$ will denote the subspace of flat connections inside the Sobolev completion (with respect to a sufficiently strong Sobolev norm) of the space of all smooth connections on $N^3 \times U(n)$, as in [**5**, Section 5]. In this section we study the homotopy groups of $\mathcal{A}_n^\flat(N^3)$ as $n \to \infty$, using Lawson's calculation of $K_*^{\mathrm{def}}(H)$ [**12**, **13**, **14**].

It was proven in [**5**, Corollary 1.3] that if M^d is a closed, smooth, aspherical d–manifold with $H^3(M; \mathbb{Q}) \neq 0$, then $\mathcal{A}_n^\flat(M)$ has infinitely many path components (so long as $n \geqslant (d+1)/2$). In particular, $\mathcal{A}_n^\flat(N^3)$ has infinitely many path components so long as $n \geqslant 2$. For manifolds M of dimension $d > 3$, [**5**, Corollary 1.3] also gives cohomological lower bounds on the rank of $\pi_m \mathcal{A}_n^\flat(M)$ for $0 < m \leqslant d - 3$, but for 3-manifolds no information about the homotopy groups $\pi_m \mathcal{A}_n^\flat(M)$ ($m \geqslant 1$) is obtained through the methods of that paper.

In this section, we will show that the homotopy groups of $\mathcal{A}_n^\flat(N^3)$ are in fact very large. Moreover, while the classes in $\pi_m \mathcal{A}_n^\flat(M)$ produced by the methods in [**5**] all admit representatives lying inside a single gauge orbit, we produce classes in $\mathcal{A}_n^\flat(N^3)$ that do not admit such representatives (although see Question 7.7).

This result shows a sharp contrast between flat connections over 3–manifolds and over surfaces, where Morse theory for the Yang–Mills functional leads to the conclusion that spaces of flat connections are highly connected (in a range tending

to infinity with the rank of the underlying bundle). For details and precise results, see [**22, 24**] and the references therein.

DEFINITION 7.4. Given a space X together with a choice of representatives $\{x_C\}_{C \in \pi_0(X)}$ for the path components of X, we define

$$\widetilde{\pi}_n(X) = \bigoplus_{C \in \pi_0(X)} \pi_n(X, x_C).$$

Up to isomorphism, this group is independent of the chosen representatives x_C.

Recall that the gauge group $\mathcal{G} = \mathrm{Map}(N^3, U(n))$ acts on the space of all connections on $N^3 \times U(n)$, and this action preserves the subspace $\mathcal{A}_n^\flat(N^3)$ (more precisely, $\mathcal{G}$ is the Sobolev completion of the space of smooth maps with respect to the appropriate Sobolev norm). The based gauge group $\mathcal{G}_0 \leqslant \mathcal{G}$ is the kernel of the restriction map $\mathcal{G} \to U(n)$ induced by evaluation at a fixed basepoint $x \in N^3$. The holonomy map induces a fibration sequence (in fact, a principal $\mathcal{G}_0$–bundle)

$$(19) \qquad \mathcal{G}_0 \xrightarrow{i_A} \mathcal{A}_n^\flat(N^3) \xrightarrow{\mathrm{Hol}} \mathrm{Hom}(H, U(n))$$

for each n. The first map in this sequence is simply the inclusion of the gauge orbit of some flat connection $A \in \mathcal{A}_n^\flat(N^3)$, and sends $g \in \mathcal{G}_0$ to $g \cdot A$. We call maps of the form i_A, and their induced maps on $\widetilde{\pi}_*$, (*based*) *gauge orbit inclusions*.

We need the following result of Lawson.

PROPOSITION 7.5 ([**14**, Section 4.2]). *For each $m \geqslant 0$, the group $K_m^{\mathrm{def}}(H)$ is free abelian of countably infinite rank.*

THEOREM 7.6. *Given $m, R \geqslant 1$, there exists $n_0 \geqslant 1$ such that for all $n \geqslant n_0$, $\widetilde{\pi}_m(\mathcal{A}_n^\flat(N^3))$ contains a subgroup F satisfying:*

(1) The abelianization of F is free of infinite rank;
(2) $\mathrm{Hol}_*(F) \leqslant \widetilde{\pi}_m \mathrm{Hom}(H, U(n))$ has rank R; and
(3) Non-trivial element in F are not in the image of based gauge-orbit inclusions.

When $m = 1$, and when m is even, (3) can be strengthened by replacing the based gauge group by the full gauge group.

PROOF. Since $\widetilde{K}^{-m}(BH) \cong K^{-m}(N^3)$ is finitely generated, Proposition 7.5 implies that the kernel of $\widetilde{\alpha}_m$ is free abelian of countably infinite rank (note here that subgroups of free abelian groups – of any rank – are free [**11**, Appendix 2]).

Now assume $m \geqslant 1$. By Theorem 3.3, the natural map

$$[S^m, \mathrm{Rep}(H)] \longrightarrow \widetilde{\Pi}_m(\mathrm{Rep}(H)) \cong \widetilde{K}_m^{\mathrm{def}}(H)$$

is surjective. This means that we can choose families

$$\rho_i \colon S^m \to \mathrm{Hom}(H, U(n_i)),$$

$i = 1, 2, \ldots$, such that the associated classes in $\widetilde{\Pi}_m(\mathrm{Rep}(H))$ are linearly independent. We have $\widetilde{\alpha}_*[\rho_i] = [B \circ \rho_i]$, and $B \circ \rho_i$ is nullhomotopic by Corollary 7.3.

Now fix an integer $R > 0$. Let $n_0 = \max n_1, \ldots, n_R$. Then for $i = 1, \ldots, R$, and for each $n \geqslant n_0$, the map

$$S^m \xrightarrow{B \circ \rho_i} \mathrm{Map}_*(BH, BU(n_i)) \longrightarrow \mathrm{Map}_*(BH, BU(n_0))$$

is nullhomotopic. Say $n \geqslant n_0$, and let G be the subgroup of $\widetilde{\pi}_m \mathrm{Hom}(H, U(n))$ generated by the elements $\{\langle \rho_i \rangle\}_{i=1}^R$. Then G surjects onto the subgroup of $\widetilde{\Pi}_m(\mathrm{Rep}(H))$ generated by $\{[\rho_i]\}_{i=1}^R$, which is free abelian of rank R, so the abelianization of G must have rank at least R (of course when $m > 1$, the group G is already abelian).

By [**5**, Remark 5.6], the boundary map on homotopy groups associated to the principal bundle (19) can be identified (up to isomorphism) with the map

$$B_* \colon \pi_* \mathrm{Hom}(H, U(n)) \longrightarrow \pi_* \mathrm{Map}_*(BH, BU(n)),$$

so each class $\langle \rho_i \rangle$ maps to zero under the boundary map for the fibration sequence (19) (with appropriately chosen basepoints). Hence there exist maps

$$\alpha_i \colon S^m \to \mathcal{A}_n^\flat(N^3)$$

such that $\mathrm{Hol}_*(\langle \alpha_i \rangle) = \langle \rho_i \rangle$.

The based gauge group $\mathcal{G}_0$ is weakly equivalent to the continuous mapping space

$$\mathrm{Map}_*(N^3, U(n)) \simeq \mathrm{Map}_*(BH, U(n)),$$

and $\mathrm{Map}_*(BH, BU(n))$ is the classifying space of $\mathrm{Map}_*(N^3, U(n))$ (this result is originally due to Gottlieb [**8**]). Hence for $* \geqslant 1$ we have

$$\pi_* \mathrm{Map}_*(BH, BU(n)) \cong \pi_{*-1} \mathcal{G}_0.$$

Let A_i denote a flat connection in the image of α_i, and let $\psi_i = \mathrm{Hol}(A_i)$. The long exact sequence associated the to the fibration sequence (19) ends with

$$\cdots \longrightarrow \pi_1(\mathrm{Hom}(H, U(n)), \psi_i) \xrightarrow{\partial} \pi_0 \mathcal{G}_0 \xrightarrow{(i_{A_i})_*} \pi_0 \mathcal{A}_n^\flat(N^3) \longrightarrow \pi_0(\mathrm{Hom}(H, U(n))).$$

Again, ∂ can be identified (up to isomorphism) with

$$B_* \colon \pi_1(\mathrm{Hom}(H, U(n)), \psi_i) \longrightarrow \pi_1(\mathrm{Map}_*(BH, BU(n)), B\psi_i),$$

and the cokernel of this map has rank 1 by [**5**, Corollary 1.3]. This means there exists an element $g_i \in \mathcal{G}_0$ such that the path component $[g_i]$ has infinite order in

$$\pi_0 \mathcal{G}_0 \cong \pi_1(\mathrm{Map}_*(BH, BU(n)), B\psi_i)$$

and the subgroup of $\pi_0(\mathcal{G}_0)$ generated by $[g_i]$ intersects the image of ∂ trivially. In general, given a (left) principal bundle K–bundle

$$K \longrightarrow P \xrightarrow{q} B,$$

if $k_1, k_2 \in K$ and $p \in P$ are elements such that $[k_1 \cdot p] = [k_1 \cdot p]$ in $\pi_0 P$, then $[k_1^{-1} k_2]$ is in the image of the boundary map $\pi_1(B, q(p)) \xrightarrow{\partial} \pi_0(K)$ (where we identify K with $q^{-1}(p)$ via $p \cdot k \mapsto k$). This implies that for each fixed $i \in \{1, \ldots, R\}$, the path components $[A_i], [g_i \cdot A_i], [g_i^2 \cdot A_i], \cdots$ are distinct in $\pi_0 \mathcal{A}_n^\flat(N^3)$.

Let $F \leqslant \widetilde{\pi}_m \mathcal{A}_n^\flat(N^3)$ be the subgroup generated by the elements $\langle g_i^k \cdot \alpha_i \rangle$, $i = 1, \ldots, R$, $k = 0, 1, \ldots$. We claim that F satisfies the conditions in the theorem. First, note that $\mathrm{Hol}_*(F) = G$, so F satisfies (2).

Next, we show that the abelianization of F is freely generated by the elements $\langle g_i^k \cdot \alpha_i \rangle$. For simplicity, we give the argument when $m > 1$, in which case F is already abelian; the argument for $m = 1$ just requires notational changes. If $\sum_{i,k} \lambda_{i,k} \langle g_i^k \cdot \alpha_i \rangle = 0$, then summing the terms whose images lie in a particular path component C of $\mathcal{A}_n^\flat(N^3)$ will also give zero. By choice of g_i, such a sum contains at most one term of the form $\lambda_{i,k} \langle g_i^k \alpha_i \rangle$ for each i. Thus we have $\sum_i \lambda_{i,k_i} \langle g_i^{k_i} \alpha_i \rangle = 0$

for some collection of natural numbers k_i $(i = 1, \ldots, R)$, and every term from the original sum whose image lies in C appears in this new sum. But now

$$0 = \mathrm{Hol}_*(0) = \mathrm{Hol}_*\left(\sum_i \lambda_{i,k_i}\langle g_i^{k_i}\alpha_i\rangle\right) = \sum_i \lambda_{i,k_i}\langle \rho_i\rangle,$$

and linear independence of the elements $\langle\rho_i\rangle$ implies that $\lambda_{i,k_i} = 0$ for each i. Applying this argument to each path component, we see that all the coefficients $\lambda_{i,k}$ must be zero, as desired.

Finally, we consider gauge-orbit inclusions. Say $f \in F$ is in the image of a based gauge-orbit inclusion. We need to prove that $f = 0$. We can write

$$f = \sum_{i,k} \lambda_{i,k}\langle g_i^k \cdot \alpha_i\rangle$$

(again, for notational convenience we work in the case $m > 1$), and it suffices to consider the case in which all the maps $g_i^k \cdot \alpha_i$ land in the same path component of $\mathcal{A}_n^\flat(N^3)$. Then, as before, there can be at most one term in this sum for each i, so we can write

$$f = \sum_i \lambda_{i,k_i}\langle g_i^k \cdot \alpha_i\rangle.$$

Since f is in the image of a based gauge-orbit inclusion $\mathcal{G}_0 \to \mathcal{A}_n^\flat(N^3)$, and the composite

$$\mathcal{G}_0 \longrightarrow \mathcal{A}_n^\flat(N^3) \longrightarrow \mathrm{Hom}(H, U(n))$$

is constant, we have $\sum_i \lambda_{i,k_i}(\langle\rho_i\rangle) = 0$. But this element maps to $\sum_i \lambda_{i,k_i}[\rho_i]$ in $\widetilde{\Pi}_m(\mathrm{Rep}(H))$, which implies that all of the coefficients λ_{i,k_i} are in fact zero.

Finally, we consider the orbits of the full gauge group. Evaluation at a point gives a split fibration

(20) $$\mathcal{G}_0 \to \mathcal{G} \to U(n),$$

and since $U(n)$ is path-connected, $\widetilde{\pi}_*\mathcal{G} \cong \widetilde{\pi}_*\mathcal{G}_0 \times \pi_*U(n)$ for each $* \in \mathbb{N}$. Now consider the case where $m = 2l$ is even. In constructing the subgroup F, we are free to choose n_0 large with respect to $2l$, and hence we may assume that that $\pi_{2l}U(n) = 0$ whenever $n \geqslant n_0$. Thus when m is even, the image of a full gauge-orbit inclusion is the same (in homotopy) as the image of the corresponding based gauge-orbit inclusion.

To obtain the desired conclusion when $m = 1$, it suffices to show that for every gauge-orbit inclusion, the composite

$$U(n) \xrightarrow{s} \mathcal{G} \longrightarrow \mathcal{A}_n^\flat(N^3) \longrightarrow \mathrm{Hom}(H, U(n))$$

(where s is a splitting of (20)) induces the zero map on π_1. The image of this composite map lies inside a single conjugation orbit inside $\mathrm{Hom}(H, U(n))$, so this composite factors through the projection $U(n) \to U(n)/Z$, where $Z \cong U(1)$ denotes the center of $U(n)$. Since the inclusion $Z \hookrightarrow U(n)$ induces isomorphisms on π_0 and π_1, we see that $\pi_1(U(n)/Z) = 0$. $\square$

Here are two natural questions regarding the above results.

QUESTION 7.7. When m is odd and greater than 1, are the classes in $\widetilde{\pi}_m\mathcal{A}_n^\flat(N^3)$ constructed in Theorem 7.6 in the images of gauge-orbit inclusions $\mathcal{G} \to \mathcal{A}_n^\flat(N^3)$?

QUESTION 7.8. Do the results of this section extended to higher-dimensional Heisenberg manifolds?

7.4. Flat connections and the topological Atiyah–Segal map. The arguments above suggest that when G is a group such that BG has the homotopy type of a closed, smooth (aspherical) manifold M, the homotopy fiber of the topological Atiyah–Segal map should be closely related to spaces of flat connections on bundles over M. Here we give one result along these lines.

First we formulate a general result about topological monoids. Let $p\colon M \to N$ be a homomorphism of homotopy commutative topological monoids. We will write the operations in M and N, and the induced pointwise multiplication operations for maps into M and N, as $\bullet$.

Let $I \subset \pi_0 N$ denote the image of $p_*\colon \pi_0 M \to \pi_0 N$. Assume that I is cofinal in $\pi_0 N$ (that is, for each $c \in \pi_0 N$ there exists $d \in \pi_0 N$ such that $cd \in I$) and that there is a map of monoids $s\colon I \to N$ satisfying $[sc] = c$ for each component $c \in I$, where $[sc] \in \pi_0 N$ denotes the path component of $sc \in N$ (in other words, s is a partially defined section of the tautological map $N \to \pi_0 N$). Then the collection of homotopy fibers $F := \coprod_{c \in I} \mathrm{hofib}_{sc}(p)$ inherits a topological monoid structure from M and N – here it is critical that the basepoints $\{sc : c \in I\} \subset N$ is a submonoid.

PROPOSITION 7.9. *In the setting above, for each $m > 0$, there is a surjection*

$$\widetilde{\Pi}_m F \longrightarrow \ker(\Omega Bp\colon \pi_*(\Omega BM, *) \to \pi_*(\Omega BN, *)).$$

Note that if all elements in M and N are strongly anchored, then the above kernel agrees with $\ker(\Omega Bp_*\colon \pi_*(\Omega BM, *) \to \pi_*(\Omega BN, *))$.

PROOF. The natural maps from the homotopy fibers to M induce maps of monoids $[S^m, F] \to [S^m, M]$ and $\widetilde{\Pi}_m F \to \widetilde{\Pi}_m M$, and it is immediate that the composite map $\widetilde{\Pi}_m F \to \widetilde{\Pi}_m M \to \widetilde{\Pi}_m M$ is zero. It remains only to show that if $\mu\colon S^m \to M$ maps to zero in $\widetilde{\Pi}_m N$, then it lifts (up to homotopy) to F. Without loss of generality, we may assume that $p(\mu(1)) = sc$ for some $c \in I$. Now $p_*[\mu] = 0$ and cofinality of I imply that $[\mu \bullet sc'] = [sd]$ for some $c', d \in I$ (here sc' and sd denote constant maps with these values). Letting $H\colon S^m \times I \to M$ be a nullhomotopy from $\mu \bullet sc'$ to sd, the pair $(\mu \bullet sc, H)$ defines a map $S^m \to \mathrm{hofib}_{sc}(p)$, whose image in $\widetilde{\Pi}_m M$ is $[\mu \bullet sc] \equiv [\mu]$. $\qquad\square$

We note that while $\pi_k \mathrm{hofib}_*(\Omega Bp)$ also surjects onto the above kernel, we do not expect an isomorphism $\widetilde{\Pi}_m F \cong \pi_m \mathrm{hofib}_*(\Omega Bp)$ (except in much more restricted situations): one might try to construct a map $\widetilde{\Pi}_{m+1} N \to \widetilde{\Pi}_m F$ using the boundary maps for the homotopy fiber sequences $\mathrm{hofib}_{sc}(p) \to M \to N$, but for each component of N, there may be many such boundary maps, coming from different path components (and hence choices of basepoint) in $\mathrm{hofib}_{sc}(p)$.

Before sketching the proof of Proposition 7.9, we consider what it tells us about flat connections. Let G be the fundamental group of a closed, aspherical 3-manifold X with torsion-free cohomology, e.g. G could be the Heisenberg group H. Then setting $M = \mathrm{Rep}(G)$, $N = \mathcal{V}(BG)$, and $p = B$, Corollary 7.3 tells us that the image of B_* on components is just the components of nullhomotopic maps in $\mathcal{V}(BG)$, which is cofinal in $\pi_0 \mathcal{V}(BG)$ since every bundle over BG is a direct summand of a trivial bundle. Now we may define s by choosing the constant maps $BG \to BU(n)$ to represent their components. By [5, Theorem 5.5], the homotopy fibers of B are

weakly equivalent to spaces of flat connections over the manifold X, meaning that there are natural zig-zags of maps between these spaces that induce isomorphisms on π_0 and on homotopy groups for all compatible choices of basepoints. Since the unbased homotopy sets are naturally in bijection with the π_1–orbits of the unbased homotopy sets, we obtain a bijection $[S^m, F] \cong [S^m, \mathcal{A}^\flat(X)]$, where

$$\mathcal{A}^\flat(X) := \coprod_n \mathcal{A}_n^\flat(X).$$

Hence $\widetilde{\Pi}_m \mathcal{A}^\flat(X)$ surjects onto the kernel of the reduced topological Atiyah–Segal map. In the case of the Heisenberg manifold N^3, we saw above that this kernel is a non-finitely generated free abelian group. In summary, we have:

COROLLARY 7.10. *The monoid $\widetilde{\Pi}_m \mathcal{A}^\flat(N^3)$ has infinite rank for each $m > 0$.*

8. Multiplicativity of the topological Atiyah–Segal map

We now explain how to enhance the topological Atiyah–Segal map into a map of E_∞ ring spectra, so that the induced map α_* on homotopy becomes a ring homomorphism. This additional structure allows us to deduce further constraints on the image of α_* for the Heisenberg group and for groups satisfying Kazhdan's property (T), leading to the following result regarding families of flat vector bundles.

THEOREM 8.1. *Let G be a discrete group satisfying property* (T), *and assume that BG has the homotopy type of a finite CW complex. Then for every family $\rho\colon S^m \to \mathrm{Hom}(G, U(n))$, the bundle E_ρ represents a torsion class in $\widetilde{K}^0(S^m \times BG)$.*

Note that by Lemma 7.1, when $H^*(BG; \mathbb{Z})$ is torsion-free the conclusion of Theorem 8.1 can be strengthened: E_ρ is in fact stably trivial.

There are many interesting groups to which this result applies. All torsion-free word hyperbolic groups admit finite CW models for BG (such a model can be built using Rips complexes - see [**1**, Corollary 4.12] for instance). There are many such groups with property (T), including cocompact, torsion-free lattices in $\mathrm{Sp}(n, 1)$ [**6**].

8.1. Bipermutative structures. Kronecker product of matrices makes the unitary permutative action sequences giving rise to $K^{\mathrm{def}}(G)$ and $\mathcal{K}(BG)$ into *bipermutative* action sequences, in the sense described in Section 2.3; the details are just a routine extension of the computations in May [**19**, VI §5]. We thus obtain functors $K_\otimes^{\mathrm{def}}$ and $\mathcal{K}_\otimes$ from the category of discrete groups to the category of E_∞ ring spectra, which become naturally equivalent to K^{def} and $\mathcal{K}$ after applying the forgetful functor to spectra.

THEOREM 8.2. *There is a natural transformation $\alpha^\otimes$ between the functors $K_\otimes^{\mathrm{def}}$ and $\mathcal{K}_\otimes$, which becomes equivalent to α after applying the forgetful functor from E_∞ ring spectra to spectra. In particular, α_* is a homomorphism of unital rings, and $\widetilde{\alpha}_*$ is a homomorphism of non-unital rings.*

PROOF. The desired natural transformation is again induced by the simplicial classifying space functor B, which respects the multiplicative structure as well as the additive structures (by functoriality, essentially). The statement regarding $\widetilde{\alpha}_*$ follows from the fact that $\widetilde{K}_*(G)$ and $\widetilde{\mathcal{K}}_*(BG)$ are simply the kernels of the compatible (ring) homomorphisms induced by the inclusion $\{1\} \to G$. $\square$

The defect in this construction is that while the homotopy groups $\mathcal{K}_*(X)$ agree *additively* with the complex topological K–theory of X, the ring structure is not immediately accessible in general, and we postpone discussion of this point to future work. Nevertheless, applications of Theorem 8.2 are provided in Section 8 below, based on the following (rather limited) information regarding the rings $\mathcal{K}_*(X)$. When $X = \{*\}$, May [**19**, VIII §2] showed that the ring $\mathcal{K}_*(*)$ is isomorphic to $\pi_*\mathbf{ku} = \mathbb{Z}[\beta]$, where $\beta \in \pi_2(\mathbf{ku}) \cong \mathbb{Z}$ is a generator. In other words, this is the standard Bott-periodic connective K–theory ring of a point. For each finite CW complex X, the injective ring map $\mathcal{K}_*(*) \to \mathcal{K}_*(X)$ (induced by the projection $X \to *$) now embeds the ring $\pi_*\mathbf{ku}$ in $\mathcal{K}_*(X)$.

8.2. Deformation K–theory and spaces of irreducible representations.
We need to review some of Lawson's results from [**13, 14**], which allow one to compute (unitary) deformation K–theory from homological information about spaces of irreducible representations.

First, consider the space

$$\overline{\mathrm{Rep}}(G) = \coprod_n \mathrm{Hom}(G, U(n))/U(n),$$

where the quotient on the right is taken with respect to the conjugation action. Block sum makes this into a strictly commutative topological monoid. In fact, the sequence of spaces

$$\mathrm{Rep}_n(G) = \mathrm{Hom}(G, U(n))/U(n)$$

form a *bipermutative* action sequence for the trivial groups $G_n = \{1\}$ (using Kronecker product to define the multiplicative structure) and the associated E_∞ ring spectrum $R^{\mathrm{def}}(G)$ satisfies $\Omega^\infty R^{\mathrm{def}}(G) \simeq \overline{\mathrm{Rep}}(G)$ by Proposition 2.6. We define

$$R_*^{\mathrm{def}}(G) = \pi_* R^{\mathrm{def}}(G).$$

The quotient maps $\mathrm{Hom}(G, U(n)) \to \mathrm{Hom}(G, U(n))/U(n)$ respect block sum and Kronecker product, so we obtain an induced map of E_∞ ring spectra

$$K^{\mathrm{def}}(G) \longrightarrow R^{\mathrm{def}}(G).$$

At this point, we need to pass from the category of E_∞ ring spectra to the category of $\mathbf{S}$–algebras, as constructed in [**7**]. The desired functor is discussed in [**7**, II.3], and for us the important point is that it induces an isomorphism on the underlying homotopy rings. We will continue to use the same notation for our E_∞ ring spectra and their associated $\mathbf{S}$–algebras, but it should be noted that smash products will be formed in the derived category of $\mathbf{S}$ modules or $\mathbf{ku}$–modules, as appropriate.

In [**13**], it is shown that when G is finitely generated, there is an equivalence $H\mathbb{Z} \wedge K^{\mathrm{def}}(G) \simeq R^{\mathrm{def}}(G)$. Fix a generator of $\pi_2\mathbf{ku}$ and a map $\beta\colon \mathbf{S}^2 \to \mathbf{ku}$ representing it. We call β the Bott element. The natural map $G \to \{1\}$ induces a map $\mathbf{ku} \to K^{\mathrm{def}}(G)$, and the image of β (which we still denote simply by β) is the Bott element in $K_2^{\mathrm{def}}(G)$. Smashing $\mathbf{S}^2 \xrightarrow{\beta} \mathbf{ku}$ with $\mathbf{ku}$ induces a map

$$\Sigma^2\mathbf{ku} \longrightarrow \mathbf{ku}$$

which we call the Bott map (and also denote by β). Bott periodicity implies that the homotopy cofiber of β is the Eilenberg–MacLane spectrum $H\mathbb{Z}$.

Smashing the homotopy cofiber sequence

$$\Sigma^2\mathbf{ku} \longrightarrow \mathbf{ku} \longrightarrow H\mathbb{Z}$$

with $K^{\mathrm{def}}(G)$ (as **ku**–modules; that is, applying $\wedge_{\mathbf{ku}}$) and taking homotopy groups now gives a long exact sequence of the form

$$(21) \qquad \cdots \xrightarrow{\partial} K^{\mathrm{def}}_*(G) \xrightarrow{\beta} K^{\mathrm{def}}_{*+2}(G) \longrightarrow R^{\mathrm{def}}_{*+2}(G) \xrightarrow{\partial} K^{\mathrm{def}}_{*-3}(G) \xrightarrow{\beta} \cdots ,$$

wherein the map β is multiplication by the Bott element

Lawson also developed a spectral sequence for computing $R^{\mathrm{def}}_*(G)$ from the integral homology of the spaces

$$\mathrm{Irr}^+_n(G) := \mathrm{Rep}_n(G)/\mathrm{Sum}_n(G),$$

where $\mathrm{Sum}_n(G)$ denotes the subspace of reducible representations. Note that $\mathrm{Irr}^+_n(G)$ is the one-point compactification of complement of $\mathrm{Sum}_n(G)$ in $\mathrm{Rep}_n(G)$, and this complement is precisely the subspace of irreducible representations. The spectral sequence is constructed by considering the tower of spectra

$$* = R^{\mathrm{def}}_{\leqslant 0}(G) \longrightarrow R^{\mathrm{def}}_{\leqslant 1}(G) \longrightarrow R^{\mathrm{def}}_{\leqslant 2}(G) \longrightarrow \cdots ,$$

where $R^{\mathrm{def}}_{\leqslant k}$ is the spectrum associated to the subspaces of $\mathrm{Rep}_n(G)$ consisting of representations whose irreducible summands all have dimension at most k; note that these subspaces provide a submonoid of $\overline{\mathrm{Rep}(G)}$, and in fact a permutative subsequence of $(\mathrm{Rep}_n(G))^\infty_{n=0}$. The homotopy colimit of this sequence is $R^{\mathrm{def}}(G)$, and Lawson proves that there are homotopy cofiber sequences of spectra

$$R^{\mathrm{def}}_{k-1}(G) \longrightarrow R^{\mathrm{def}}_k(G) \longrightarrow H\mathbb{Z} \wedge \mathrm{Irr}^+_k(G)$$

for each $k \geqslant 1$.

In general, a sequence of spectra

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

gives rise to an exact couple

and hence to a spectral sequence of the form

$$E^1_{p,q} = \pi_{p+q}(\mathrm{hocofib}\, f_p) \implies \pi_{p+q} \operatorname*{hocolim}_i X_i,$$

with differentials

$$d^r_{p,q} \colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

Since $H\mathbb{Z}$ is the spectrum representing integral homology, in the case at hand we obtain a spectral sequence

$$(22) \qquad\qquad E^1_{p,q} = \widetilde{H}_{p+q}(\mathrm{Irr}^+_p(G); \mathbb{Z}) \implies \pi_{p+q} R^{\mathrm{def}}(G).$$

Note that with this indexing, the spectral sequence can be non-zero in the quadrant where $p, q \geqslant 0$ and in the region where $-p \leqslant q < 0$.

8.3. Groups satisfying Kazhdan's property (T). Property (T) has been widely studied since its introduction by Kazhdan in the late 1960s. Loosely speaking, property (T) is a weak rigidity property for (possibly infinite-dimensional) unitary representations of locally compact groups. See [**6**] for background.

We need a lemma regarding unitary representations of property (T) groups.[1]

LEMMA 8.3. *Let G be a discrete group with property* (T). *Then the space* $\mathrm{Hom}(G, U(n))/U(n)$ *is a finite, discrete space.*

PROOF. By Wang [**28**, Theorem 2.5], if $\rho\colon G \to U(n)$ is irreducible, then the path component of ρ in $\mathrm{Hom}(G, U(n))$ coincides with its conjugation orbit O_ρ. Since $\mathrm{Hom}(G, U(n))$ is compact and triangulable, it has finitely many path components, so we conclude that there are only finitely many irreducible unitary representations in each dimension. Since every unitary representation is a direct sum of irreducibles, finiteness of $\mathrm{Hom}(G, U(n))/U(n)$ follows immediately, and discreteness follows as well since this space is Hausdorff. $\qquad\square$

PROPOSITION 8.4. *Let G be a discrete group satisfying property* (T). *Then* $K^{\mathrm{def}}_{2m+1}(G)$ *is trivial for all $m \geqslant 0$, and the iterated Bott map*

$$\beta^m_*\colon K^{\mathrm{def}}_0(G) \cong \mathrm{Gr}(\pi_0\mathrm{Rep}(G)) \longrightarrow K^{\mathrm{def}}_{2m}(G)$$

is an isomorphism for all $m \geqslant 1$.

PROOF. Lemma 8.3 implies that $\mathrm{Hom}(G, U(n))/U(n)$ is *discrete* for all n, so its cohomology vanishes in positive dimensions. Discrete groups with property (T) are finitely generated [**6**, Theorem 1.3.1], so the spectral sequence (22) yields $\pi_*(R^{\mathrm{def}}(G)) = 0$ for $* > 0$. The long exact sequence (21) shows that $K^{\mathrm{def}}_1(G) \cong R^{\mathrm{def}}_1(G) = 0$ and that $\beta_*\colon K^{\mathrm{def}}_m(G) \to K_{m+2}(G)$ is an isomorphism for $m \geqslant 0$. $\quad\square$

We need a standard fact about flat bundles, coming from Chern–Weil theory.

LEMMA 8.5. *Let G be a discrete group such that BG has the homotopy type of a finite CW complex, and let ϵ^n denote the trivial bundle $BG \times U(n)$. Then for every representation $\rho\colon G \to U(n)$, the class $[E_\rho] - [\epsilon^n]$ is torsion in $\widetilde{K}^0(BG)$.*

For a complete proof (of a much more general statement) see [**5**, Theorem 3.5].

PROPOSITION 8.6. *Let G be a discrete group satisfying property* (T), *and assume that BG has the homotopy type of a finite CW complex. Then the reduced unitary topological Atiyah–Segal map*

$$\widetilde{\alpha}_m\colon \widetilde{K}^{\mathrm{def}}_*(G) \longrightarrow \widetilde{K}^{-m}(G)$$

is zero when m is odd, and its image is torsion when m is even.

PROOF. For m odd, this is immediate from Proposition 8.4, so we consider the even case. By Theorem 5.3, the image of α_0 consists of K–theory classes of the form $[E_\rho]$, where $\rho\colon G \to U(n)$ is a single representation. Lemma 8.5 implies that the image of α_0 becomes torsion after modding out the summand $\pi_0\mathbf{ku} \cong \mathbb{Z}$ corresponding to the trivial bundles. But since $\alpha_0 = \widetilde{\alpha}_0 \oplus \mathrm{Id}_{\pi_0\mathbf{ku}}$ (see (13)), this quotient is isomorphic to the image of $\widetilde{\alpha}_0$.

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[1]I learned this result from Rufus Willett.

Now consider the commutative diagram

(23)
$$
\begin{array}{ccc}
K_{2m}^{\mathrm{def}}(G) & \xrightarrow{\alpha_{2m}} & \mathcal{K}_{2m}(BG) \\
\cdot\beta^m \big\uparrow \cong & & \big\uparrow \cdot\alpha_{2m}(\beta^m) \\
K_0^{\mathrm{def}}(G) & \xrightarrow{\alpha_0} & \mathcal{K}_0(BG),
\end{array}
$$

where the vertical maps are given by multiplication. Each group in the diagram contains a $\mathbb{Z}$ summand arising from the homotopy of **ku**, via the maps induced by $G \to \{1\}$ and $BG \to \{*\}$, and these summands are complementary to the reduced subgroups. All four maps in the diagram are isomorphisms when restricted to these $\mathbb{Z}$ summands, so Proposition 8.6 implies that the image of the lower composite, and hence also the upper composite, has rank 1. Since $\cdot\beta^m \colon K_0^{\mathrm{def}}(G) \to K_{2m}^{\mathrm{def}}(G)$ is an isomorphism, we conclude that the image of α_{2m} has rank 1, and finally that the image of $\widetilde{\alpha}_{2m}$ is torsion, as desired. $\qquad\square$

PROOF OF THEOREM 8.1. By Proposition 8.6 and Theorem 5.3, we know that
$$
\widetilde{\alpha}_m([\rho]) = \pi_*^{-1}\left([E_\rho] - [E_{\widetilde{\rho(1)}}]\right)
$$
is torsion in $\widetilde{K}^0(S^m \wedge BG)$. But the projection $\pi \colon S^m \times BG \to S^m \wedge BG$ induces a (split) injection on reduced K–theory, so for some $k \geqslant 1$ we have $k[E_\rho] = k[E_{\widetilde{\rho(1)}}]$ in $K^0(S^m \times BG)$. Hence it will suffice to show that $l[E_{\widetilde{\rho(1)}}]$ represents a torsion class in reduced K–theory for some $l \geqslant 1$. This holds for the bundle $E_{\rho(1)} \to BG$ by Lemma 8.5, and since $E_{\widetilde{\rho(1)}}$ is a pullback of $E_{\rho(1)}$, the proof is complete. $\qquad\square$

REMARK 8.7. Theorem 8.1 provides a partial answer to [**20**, Question 3.20]. If G satisfies the hypotheses of Theorem 8.1, then no non-trivial class in the rational K–homology of BG can be detected by a spherical family of representations (in the sense of [**20**, Definition 3.4]). However, it remains possible that non-trivial classes can be detected by non-spherical families of representations.

8.4. The topological Atiyah–Segal map for the Heisenberg group. In the case of the 3–dimensional integral Heisenberg group H, Lawson showed in [**12, 13**] that $R_m^{\mathrm{def}}(H) = 0$ for $m \geqslant 3$. In dimension 1, Theorem 6.1 tells us that the image of $\alpha_1 = \alpha_1^H$ has rank at most $\beta_1(N^3) = 2$. Reasoning similar to the proof of Proposition 8.6 yields the following result.

PROPOSITION 8.8. *The image of the unitary topological Atiyah–Segal map*
$$
\alpha_{2m+1} \colon K_{2m+1}^{\mathrm{def}}(H) \longrightarrow K^{-(2m+1)}(BH) \cong K^{-(2m+1)}(N^3)
$$
has rank at most 2 for each $m \geqslant 0$. Hence α_* is not surjective in odd dimensions.

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