Exercises to Accompany
Lectures on Composition Operators on Spaces of Analytic Functions

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Universidad Complutense de Madrid, Otoño 2012

Exercise 1: Prove that $L^1([0,1])$ is *not* a functional Banach space with $X = [0,1]$ and the obvious identification of integrable functions as vectors in L^1:

The continuous functions on $[0,1]$ are a dense subset of $L^1([0,1])$ in the L^1 norm and if f is a continuous function on $[0,1]$ then $f(1/2)$ is defined. Even more, for a continuous function on $[0,1]$, the value of f at $x = 1/2$ cannot be changed and still have f continuous at $x = 1/2$.

Show that, considering the continuous functions on $[0,1]$ as a vector subspace of $L^1([0,1])$, the linear functional on this subset $f \mapsto f(1/2)$ is *not* bounded.

Exercise 2: Prove that the Bergman space is a Hilbert space, that is, that it is a *complete* inner product space. Equivalently, since it is obvious that $A^2(\mathbb{D}) \subset L^2(\mathbb{D})$ and we know L^2, it is enough to show that $A^2(\mathbb{D})$ is a closed subset of $L^2(\mathbb{D})$. That is, show that if f_n is a sequence of functions in $A^2(\mathbb{D})$, and $\lim_{n\to\infty} f_n = f$ in L^2, then actually f is analytic also and is $A^2(\mathbb{D})$.

Exercise 3: Just as for $H^2(\mathbb{D})$, we want another way to think about $A^2(\mathbb{D})$.

(a) Show that the set $\{z^n\}_{n=0}^{\infty}$ is an *orthogonal* basis for $A^2(\mathbb{D})$.

(b) Find the norm of z^n in $A^2(\mathbb{D})$ for each non-negative integer n.

(c) Find a condition $(*)$ on the coefficients a_n so that if f is an analytic function on the disk with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then f is in $A^2(\mathbb{D})$ if and only if $(*)$.

(d) Use the ideas of (a)–(c) to show that for α in the disk, the function K_α in $A^2(\mathbb{D})$ so that $\langle f, K_\alpha\rangle = f(\alpha)$ for every f in $A^2(\mathbb{D})$ is

$$K_\alpha(z) = \frac{1}{(1-\overline{\alpha}z)^2}$$

(e) Conclude that $A^2(\mathbb{D})$ is a functional Hilbert space.

Exercise 4: For the Hardy space H^2, find the kernel function $K_\alpha^{(1)}$ in H^2 so that for each f in H^2 and each α in the disk, $f'(\alpha) = \langle f, K_\alpha^{(1)} \rangle$. Check your answer by using the formula you got for $K_\alpha^{(1)}$ to find $K_0^{(1)}$ the kernel for evaluating the derivative of a function in H^2 at $\alpha = 0$, which should look familiar to you.

Exercise 5: Find all the fixed points of the listed functions and their derivatives there. Then find the Denjoy–Wolff point. ($\sqrt{\ }$ means the branch of the square root that is positive on the positive axis.)

(a) $\varphi(z) = \exp((z+1)/(z-1))$

(b) $\varphi(z) = \left(\dfrac{z+1/3}{1+z/3}\right)^2$

(c) $\varphi(z) = w^{-1}(\psi(w(z))$ where $w(z) = (1+z)/(1-z)$ maps the disk to the right halfplane and $\psi(w) = \sqrt{4w^2+3}$ maps the right half plane to itself.

(d) $\varphi(z) = \dfrac{1+z+2\sqrt{1-z^2}}{3-z+2\sqrt{1-z^2}}$

Exercise 6: For the maps in Exercise 5, find the Case of φ, that is, decide, for each φ, if φ is in the plane dilation, half-plane dilation, plane translation, or half-plane translation case.

Exercise 7: For the maps in Exercise 5, find the spectral radius of C_φ as an operator on H^2.

Exercise 8: Consider the map $\varphi(z) = z/2 + z^2/3$.

(a) Show that φ maps the disk into the disk.

(b) Explain why C_φ is compact on H^2.

(c) Find the spectrum of C_φ.

(d) According to the theory, the eigenvectors are multiples of powers of the Koenigs' function σ, which is also the map in the model for the function φ. In this case, we know $\sigma(\varphi(z)) = \lambda\sigma(z)$ and $0 < |\lambda| < 1$, and this $|\lambda|$ is the largest possible eigenvalue, with $|\lambda| < 1$. Find λ in this case, and find the first 7 Taylor coefficients, i.e. up to a_6, the coefficient of z^6, of σ explicitly. Looking at these Taylor coefficients, can you find σ explicitly, that is, can you guess the rest of the coefficients and write down σ as an elementary function?

Exercise 9: For the maps in Exercise 5, find the spectrum of C_φ as an operator on H^2 if you can, or to the extent that you can.

Exercise 10: For each of the following maps φ, say as much as you can about C_φ and the spectrum of C_φ.

- (a) $\varphi(z) = z^2/3 + z/2 + 1/6$.
- (b) $\varphi(z) = z^2/2 - z/3 + 1/6$.
- (c) $\varphi(z) = z^2/6 + 2z/3 + 1/6$.
- (d) $\varphi(z) = z^2/6 + z/3 + 1/2$.

Exercise 11: In the Theorem for the model for iteration in the case in which the Denjoy-Wolff point, a, is inside the disk, the hypothesis is $\varphi'(a) \neq 0$. Using $\varphi(z) = z^2$, which has Denjoy-Wolff point $a = 0$ and $\varphi'(a) = \varphi'(0) = 0$, explain why the hypothesis is what it is by finding possible analytic functions f and numbers λ so that $f(\varphi(z)) = \lambda f(z)$ in a neighborhood of 0.

Exercise 12: If φ maps the disk into itself and has Denjoy-Wolff point 1 with $\varphi'(1) = .5$, the theory says that the inductively defined sequence $z_{n+1} = \varphi(z_n)$ starting with any point z_0 in the disk is an interpolating sequence. For the function $\varphi(z) = .5z + .5$, find z_n explicitly satisfying $z_{n+1} = \varphi(z_n)$ starting with $z_0 = 0$. Show that, at least, $\{z_n\}$ is a Blaschke sequence, that is, that $\sum(1 - |z_n|) < \infty$, so that there are analytic functions f with $f(z_n) = 0$ for all n but f is not the zero function.

Exercise 13: Let φ be an analytic function mapping the unit disk into itself, with $\varphi(1) = 1$ and $\varphi'(1) = s$ where $0 < s < 1$. According to the theory, φ is in the half-plane dilation case and there is σ analytic, mapping $\mathbb{D}$ into the right half plane $H_+ = \{z : \mathrm{Re}\, z > 0\}$ where $\Phi(w) = sw$ and $\Phi \circ \sigma = \sigma \circ \varphi$. Suppose, in addition, φ is real on the real axis. Using the functions $\widetilde{\varphi}(z) = \overline{\varphi(\overline{z})}$, $\widetilde{\Phi}(z) = \overline{\Phi(\overline{z})}$, and $\widetilde{\sigma}(z) = \overline{\sigma(\overline{z})}$, show that σ is real on the real axis as well.

Exercise 14: Let φ be an analytic function mapping the unit disk into itself. If Φ is an automorphism mapping Ω onto itself, and σ maps the disk into Ω such that $\Phi \circ \sigma = \sigma \circ \varphi$, then it must be the case that σ maps the fixed points of φ to the fixed points of Φ in such a way that the attracting fixed point of φ, the Denjoy-Wolff point of φ, is mapped to the attracting fixed point of Φ and the other fixed points of φ are mapped to the other fixed point of Φ, or at least the iterates of φ leaving a fixed point of φ are mapped to iterates of Φ leaving a fixed point of Φ. Moreover, if φ maps a point of the circle to the circle, then σ must map that point to a point of the boundary of $\sigma(\mathbb{D})$ that Φ maps to a point of the boundary of $\sigma(\mathbb{D})$. Use these ideas to draw a connected, simply connected domain U in the complex plane that U contains 0, so that z in U implies $z/2$ is in U, and so that if σ is the Riemann map of $\mathbb{D}$ onto U that takes 0 to 0 and has $\sigma'(0) > 0$, then using $\Phi(w) = w/2$, the map $\varphi(z) = \sigma^{-1}(\Phi(\sigma(z)))$ has $\varphi(0) = 0$, $\varphi'(0) = 1/2$, and 1 and -1 as fixed points of φ. Can you describe explicitly, a map, φ, of $\mathbb{D}$ into $\mathbb{D}$ with $\varphi(0) = 0$, $\varphi'(0) = 1/2$, $\varphi(1) = 1$, and $\varphi(-1) = -1$?