28 Noviembre

$|a| < 0$

Main Theorem

Let $\varphi : \mathbb{D} \to \mathbb{D}$ $|a| < 1$. Let C_φ be operator on H^2

McCluer + C 1994 ① If φ is univalent, not an automorphism

or

Kamowitz 1975 ② φ is analytic in a neighborhood of $\overline{\mathbb{D}}$

Then $\sigma(C_\varphi) = \{\lambda : |\lambda| \le \rho\} \cup \{\varphi'(a)^k : k = 1, 2, \ldots\} \cup \{1\}$
where ρ is the essential spectral radius of C_φ.

Thm If φ is a univalent map of $\mathbb{D} \to \mathbb{D}$
then the essential norm of C_φ, $\|C_\varphi\|_e$ is

$$\|C_\varphi\|_e = \limsup_{|w| \to 1^-} \frac{\|K_{\varphi(w)}\|}{\|K_w\|}$$

and $$\rho = \lim_{k \to \infty} \left(\limsup_{|w| \to 1^-} \left(\frac{\|K_{\varphi_k(w)}\|}{\|K_w\|} \right)^{1/k} \right)$$

Proof idea As $|w| \to 1$ $\langle f, \frac{K_w}{\|K_w\|} \rangle \to 0$ $\forall f$

Thm $\varphi : \mathbb{D} \to \mathbb{D}$ and $b \in \partial\mathbb{D}$
is a point for which $|\varphi(b)| = 1$
then $$\rho \ge \frac{1}{\sqrt{|\varphi'(b)|}}$$

Thm If $\varphi: \mathbb{D} \to \mathbb{D}$ and $|a|<1$

~~φ not an automorphism~~

then $\sigma(C_\varphi) \supset \{1\} \cup \{\varphi'(a)^n\}_{n=1}^{\infty}$.

Proof matrix for C_φ^* after φ adjusted to make $a=0$.

Thm If $\varphi: \mathbb{D} \to \mathbb{D}$ and $|a|<1$

and $|\varphi'(a)^k| > \rho =$ essential spectral radius of C_φ

then ① $\varphi'(a)^k$ is an eigenvalue of C_φ of multiplicity 1 and has eigenvector σ^k

② $\overline{\varphi'(a)}^k$ is an eigenvalue of C_φ^* of multiplicity 1

③ 0 is not an eigenvalue of C_φ for φ non-constant

and ④ if φ is not univalent then 0 is an eigenvalue of C_φ^* of infinite multiplicity.

Proof ① follows from Königs Theorem and the general fact that for $\lambda \in \sigma(A)$ $|\lambda| >$ ess. spec radius of A

$\Rightarrow$ λ is eigenvalue of A

$\bar{\lambda}$ is eigenvalue of A^*

+ both have same finite multiplicity

②

③ φ non-constant $\Rightarrow$ $\varphi(\mathbb{D})$ is open in $\mathbb{D}$

and $f \circ \varphi \equiv 0 \Rightarrow f(\varphi(\mathbb{D})) = 0 \Rightarrow f \equiv 0$

④ Suppose $\alpha, \beta \in \mathbb{D}$ and $\varphi(\alpha) = \varphi(\beta)$

then $C_\varphi^*(K_\alpha - K_\beta) = K_{\varphi(\alpha)} - K_{\varphi(\beta)} = 0$

but because φ is open mapping $\varphi(\alpha) = \varphi(\beta) \Rightarrow \varphi(U) \cap \varphi(V) \neq \emptyset$ is open

for a neighborhoods of α + β respectively so infinitely many pairs

Example $\varphi(z) = \frac{z}{2-z}$ $a=0$ $\varphi'(0) = \frac{1}{2}$

φ univalent

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \le 1/\sqrt{2}\} \cup \{1\}$$

$$\|C_\varphi|_{zH^2}\| = \|C_\varphi|_{zH^2}\|_e = \rho = \frac{1}{\sqrt{2}}$$

so theorem gives result in this case (as expected)

Moreover, we know in this case $\left(C_\varphi|_{zH^2}\right)^* \cong \frac{1}{2} C_{\frac{1}{2}z+\frac{1}{2}}$

and this means every point of $\{\lambda : 0 < |\lambda| < \frac{1}{\sqrt{2}}\}$ is an eigenvalue of C_φ^* of infinite multiplicity.

In fact this is not, I think uncommon!

Conjecture: MacCluer - ~~Cowen~~ ($\sim$1994)

If the spectrum of C_φ has circular symmetry in some part, there is an invariant subspace for C_φ or C_φ^* on which the restriction is similar to a weighted shift.

Not proved ~~by~~ but proved in some special cases, no hints as to an example in which it might be false.

Wahl (1997) $\varphi(z) = \frac{z^2(1-z)}{1-2cz}$ c such that $\varphi : \mathbb{D} \to \mathbb{D}$

then $\varphi(0) = \varphi'(0) = 0$ and $\varphi(1) = 1$, $1 < \varphi'(1) < \infty$

so C_φ not compact. $\rho = 1/\sqrt{\varphi'(1)} = \|C_\varphi\|_e$

Thm (M Neophytou (2011))

Suppose $\varphi(z) = a_j z^j + a_{j+1} z^{j+1} + \cdots \qquad j \geq 2 \quad a_j \neq 0,$
and $\varphi : \mathbb{D} \to \mathbb{D}$.

If there is b with $|b| = 1$ so that b is a fixed point of φ with $\varphi'(b) < \infty$ and [arithmetic hypothesis]
then $\{\lambda : |\lambda| < 1/\sqrt{\varphi'(b)}\} \subset \sigma(C_\varphi)$
and each number in this set is an eigenvalue of infinite multiplicity for C_φ^*.