26 Noviembre

Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ $|a| = 1$ and $\varphi'(a) < 1$; that is suppose φ is in half-plane dilation case of model.

<u>Thm</u> Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ $|a|=1$ $\varphi'(a) < 1$.
Then for θ real, C_φ on $H^2(\mathbb{D})$ is similar to the operator $e^{i\theta} C_\varphi$. In particular, if $\lambda \in \sigma(C_\varphi)$ then $e^{i\theta}\lambda$ is in $\sigma(C_\varphi)$.

<u>Proof</u> Let θ be real number and let $s = \varphi'(a)$, so $0 < s < 1$. Let log be the branch of the logarithm defined on the right half plane with $\log(1) = 0$.
Let $\sigma : \mathbb{D} \to \{ z : \operatorname{Re} z > 0 \} = \mathbb{H}$ and $\sigma(\varphi(z)) = s\sigma(z)$ for all z in $\mathbb{D}$.

Let $F(\omega) = \exp(i\theta \log(\omega)/\log(s))$ for $\omega \in \mathbb{H}$.

Then $F(s\omega) = \exp(i\theta \log(s\omega)/\log(s)) =$
$$\exp\left(i\theta\left(\frac{\log(\omega) + \log(s)}{\log(s)}\right)\right) = \exp\left(i\theta \frac{\log \omega}{\log s} + i\theta\right) = e^{i\theta} F(\omega)$$

$$|F(\omega)| = \exp\left[\operatorname{Re}(i\theta \log(\omega)/\log(s))\right]$$

and $\log(\omega) = x + iy$ where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

so $\operatorname{Re}(i\theta \log(\omega) \log(s))) = -y\theta/\log(s)$

So $-\frac{\pi}{2}\theta/\log(s) < \operatorname{Re}(i\theta \log(\omega)/\log(s)) < \frac{\pi}{2}\theta/\log(s)$

for $\theta < 0$ and

$$+\frac{\pi}{2}\theta/\log(s) < \operatorname{Re}(i\theta \log(\omega)/\log(s)) < -\frac{\pi}{2}\theta/\log(s)$$

for $\theta \geq 0$

$$e^{\frac{\pi\theta}{2\log s}} < |F(\omega)| < e^{\frac{-\pi\theta}{2\log s}} \qquad \theta \geq 0$$

$$e^{-\frac{\pi\theta}{2\log s}} < |F(\omega)| < e^{\frac{\pi\theta}{2\log s}} \qquad \theta < 0$$

Let $f = F \circ \sigma$ so $f \circ \varphi = F(\sigma \circ \varphi) = F(s\sigma) = e^{i\theta} F \circ \sigma = e^{i\theta} f$ and f and $1/f \in H^\infty$

Thus T_f is bdd operator on H^2 and $T_{\frac{1}{f}} = T_f^{-1}$ also bdd.
where $T_f h = fh$

and $T_{1/f}\, C_\varphi\, T_f(h) = T_{1/f}\, C_\varphi(fh) = \frac{1}{f}\, f\circ\varphi\, h\circ\varphi =$

$$\frac{e^{i\theta} f}{f} h\circ\varphi = e^{i\theta} C_\varphi h$$ Thus C_φ is

similar to $e^{i\theta} C_\varphi$ and $\sigma(C_\varphi) = \sigma(e^{i\theta} C_\varphi) = e^{i\theta}\sigma(C_\varphi)$. //

~~Thm~~ Def For K, M integers or $\pm\infty$,
then $\{z_k\}_{k=K}^{M}$ in $\mathbb{D}$ is called an iteration sequence for φ if $\varphi(z_k) = z_{k+1}$ for $K \le k < M$

Thm Suppose φ is analytic map of $\mathbb{D}$ into $\mathbb{D}$ not an automorphism of $\mathbb{D}$ and suppose C_φ is associated comp op on H^2.
If $b \in \partial\mathbb{D}$, b a fixed pt of φ, and $\varphi'(b) > 1$,
φ analytic in a neighborhood of b
then for $\rho < \varphi'(b)^{-1/2}$ the circle of radius ρ intersects the spectrum of C_φ.

Proof There are uncountably many iteration sequences $\{z_j\}_{j=-\infty}^{0}$ so that $z_j \to b$ as $j \to -\infty$
and non-tangentially.
let $k_j = \dfrac{(1-|z_j|^2)^{1/2}}{1-\bar{z}_j z} = \dfrac{1}{\|K_{z_j}\|} K_{z_j}$ and these are approximately an O.N. sequence in H^2.

Let $\rho \le \rho_1 < \varphi'(b)^{-1/2}$ $\quad |\lambda| = \rho$

$$h_\lambda = \sum_{-\infty}^{-1} \lambda^{-j-1} \left(\frac{1-|z_{-1}|^2}{1-|z_j|^2}\right)^{1/2} k_j \in H^2 \qquad \lim_{j\to\infty} \frac{1-|z_{j-1}|^2}{1-|z_j|^2} = \varphi'(b)$$

$$(C_\varphi^* - \lambda)h_\lambda = \sum_{-\infty}^{-1} \lambda^{j-1}\left(\frac{1-|z_j|^2}{1-|z_{j+1}|^2}\right)k_{j+1} - \sum_{-\infty}^{-1} \lambda^{-j}\left(\frac{1-|z_0|^2}{1-|z_j|^2}\right)^{1/2}k_j$$

$$= k_0$$

if $\sigma(C_\varphi)$ does not intersect $|\lambda| = \rho$

then $(C_\varphi^* - \rho e^{i\theta})^{-1}$ exists for each real θ

and we can define Q to be $\int_0^{2\pi} (C_\varphi^* - \rho e^{i\theta})^{-1} \frac{d\theta}{2\pi}$

$Q k_0 = \left(\frac{1-|z_0|^2}{1-|z_{-1}|^2}\right)^{1/2} k_{-1}$ or $Q K_0 = K_{z_{-1}}$

and $C_\varphi^* Q K_{z_0} = K_{z_0}$ $\quad Q$ is rational function of C_φ^*

so $Q C_\varphi^* K_{z_0} = K_{z_0}$ also and $Q C_\varphi^* = C_\varphi^* Q = I$

but φ is not auto so C_φ^* not invertible! //

Poggi-Corradini $\Rightarrow$ OK for b fixed for φ $\varphi'(b) < \infty$.

Lemma Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ $|a|=1$, $\varphi'(a) < 1$.
If λ satisfies $\varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2}$

then λ is an eigenvalue of infinite multiplicity for C_φ on H^2

Proof If $s^{1/2} < \lambda < s^{-1/2}$ let $x = \log \lambda / \log s$

so that $\sigma^x \circ \varphi = (s\sigma)^x = \lambda \sigma^x$

so σ^x is eigenfunction for C_φ if $\sigma^x \in H^2$

$\Psi = \frac{1-\sigma(z)}{1+\sigma(z)}$ maps $\mathbb{D}$ into $\mathbb{D}$ and $F(z) = \left(\frac{1-z}{1+z}\right)^x \in H^2$

so $\sigma^x = F \circ \Psi \in H^2$ so λ is eigenvalue when $s^{1/2} < \lambda < s^{-1/2}$

and $C_\psi \simeq e^{i\theta} C_\varphi$ completes proof. //

Thm If φ is analytic in a neighborhood of $\overline{\mathbb{D}}$ and $|a|=1$ $\varphi'(a)<1$, φ not inner, then
$$\sigma(C_\varphi) = \{\lambda : |\lambda| \le \varphi'(a)^{-1/2}\}$$

Outline of Proof hypotheses $\Rightarrow$ $\exists\, n$ finite so that
$\varphi_n^{-1}(\partial D) = \{$fixed points of $\varphi_n\}$ a finite set
$= \{a, b_1, b_2 \cdots b_K\}$
Previous lemma can be expanded to show
$\min \varphi'(b_j)^{-1/2} < |\lambda| < \varphi'(a)^{-1/2}$ are eigenvalues of C_φ of infinite mult. and $|\lambda| < \min \varphi'(b_j)^{-1/2}$ ~~are~~ in $\sigma(C_\varphi)$
so $\sigma(C_\varphi) =$ disk. //