See: A L Shields Weighted shift operators & analytic function thy in Topics in Op Thy 1974 AMS Surveys # 13 p49-128

Weighted shift operators.

W is a weighted shift operator on ℓ^2 if there is a sequence of complex numbers (w_n) so that

$$We_k = w_k e_{k+1} \quad \text{for} \quad k = 0, 1, 2, \dots$$

So

$$W \sim \begin{pmatrix} 0 & 0 & 0 & 0 & \\ w_0 & 0 & 0 & 0 & \\ 0 & w_1 & 0 & 0 & \\ 0 & 0 & w_2 & 0 & - \\ & & & \ddots & \end{pmatrix}$$

Clearly W bounded $\Rightarrow$ (w_n) is bdd and $\|W\| \geqslant \sup |w_n|$

WLOG $w_n \geqslant 0$ for all n because there is a diagonal unitary operator that shows

$$W \cong \begin{pmatrix} 0 & & \\ |w_0| & & \\ & |w_1| & \ddots \end{pmatrix}$$

Moreover we assume $|w_n| > 0$ otherwise There are finite dimensional invariant subspaces for W on which $W|_M^k = 0$.

For simplicity we will assume $\lim_{n \to \infty} w_n = L$ where $0 < L < \infty$

It is easy to see that $Wv = \lambda v \Rightarrow \lambda = 0$ or $v = 0$

On the other hand, $W^* v = \lambda v$ has a chance!

$$W^* v = \begin{pmatrix} 0 & w_0 & 0 & 0 & 0 \\ 0 & 0 & w_1 & 0 & 0 \\ 0 & 0 & 0 & w_2 & 0 \\ & & \vdots & & \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ w_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} w_0 v_1 \\ w_1 v_2 \\ \vdots \end{pmatrix} \overset{?}{=} \lambda \begin{pmatrix} v_0 \\ v_1 \\ \vdots \end{pmatrix}$$

we get $\quad w_k v_{k+1} = \lambda v_k$

($v \neq 0 \Rightarrow \exists\, v_k \neq 0$ and since eigenspaces are subspaces WLOG $\exists\, k_0$ so that $v_0 = v_1 \cdots = v_{k_0-1} = 0$ and $v_{k_0} = 1$)

Suppose $v_0 = 1$

then $\quad w_0 v_1 = \lambda v_0 = \lambda$

and $\quad v_1 = \frac{\lambda}{w_0}$

then $\quad w_1 v_2 = \lambda v_1 = \frac{\lambda^2}{w_0}$

so $\quad v_2 = \frac{\lambda^2}{w_1 w_0}$

we see by induction that for $n \geq 2$ $\quad v_n = \frac{\lambda^n}{w_{n-1} w_{n-2} \cdots w_1 w_0}$

So the question is When is this v in ℓ^2?

$$\Longleftrightarrow \quad \sum \left| \frac{\lambda^n}{w_{n-1} \cdots w_0} \right|^2 < \infty$$

$$\text{i.e.} \Longleftrightarrow \quad \sum \frac{(|\lambda|^2)^n}{(w_{n-1} \cdots w_0)^2} < \infty$$

$$\Leftarrow \quad \lim_{n \to \infty} \left(\frac{|\lambda|^{2n}}{(w_{n-1} \cdots w_0)^2} \right)^{1/n} < 1$$

$$\Longleftrightarrow \quad \frac{|\lambda|^2}{L^2} < 1 \quad \Longleftrightarrow \quad |\lambda| < L$$

E.g. "The" unilateral shift* has eigenvectors $(1, \alpha, \alpha^2, \ldots) \in \ell^2$ for $|\alpha| < 1$ and no others.

$$\longleftrightarrow \text{ on } H^2 \quad \sum_{n=0}^{\infty} (\bar{\alpha} z)^n = \frac{1}{1 - \bar{\alpha} z} = k_\alpha$$

See H.S. Shapiro & A.L. Shields "On interpolation problems for analytic function" Amer. J. Math 83 (1961) 513–532.
(See 7Nov for Def: $(z_n) \in \mathbb{D}$ is interpolating sequence if
for all $(a_n) \in \ell^\infty$ $\exists\, \varphi \in H^\infty$ so that
$\varphi(z_n) = a_n$)

Suppose $\varphi : \mathbb{D} \to \mathbb{D}$, $|a| = 1$, $0 < \varphi'(a) < 1$
(Half-plane dilation case)
then $(\varphi_n(\alpha))$ is always an interpolating sequence

Shapiro-Shields says (β_n) sequence in disk
is interpolating $\iff$ $\left(\dfrac{K_{\beta_n}}{\|K_{\beta_n}\|} \right)$ is a basic sequence in H^2

in other words, there is a bounded operator V
which has a bounded <u>inverse</u> so that
$$V : \ell^2 \to \overline{\operatorname{span}\{K_{\beta_n}\}}^{\text{closed}} \subset H^2$$
so that $V(e_n) = \dfrac{K_{\beta_n}}{\|K_{\beta_n}\|}$ for each n

Now if $\alpha \in \mathbb{D}$ then $\{\varphi_n(\alpha)\}$ is interpolating
and $\exists\, V : \ell^2 \to \operatorname{span}\{K_{\varphi_n(\alpha)}\} = (BH^2)^\perp$
where B is Blaschke product with $B(\varphi_n(\alpha)) = 0$
and no other zeros

Now $C_\varphi^* K_{\varphi_n(\alpha)} = K_{\varphi_{n+1}(\alpha)}$

So $C_\varphi^* \left(\dfrac{1}{\|K_{\varphi_n(\alpha)}\|} K_{\varphi_n(\alpha)} \right) = \dfrac{1}{\|K_{\varphi_n(\alpha)}\|} K_{\varphi_{n+1}(\alpha)}$

$$= \frac{\|K_{\varphi_{n+1}(\alpha)}\|}{\|K_{\varphi_n(\alpha)}\|} \left(\frac{K_{\varphi_{n+1}(\alpha)}}{\|K_{\varphi_{n+1}(\alpha)}\|} \right)$$

So $\quad V^{-1} C_\varphi^* \Big|_{(BH^2)^\perp} V e_n = V^{-1} C_\varphi^* \dfrac{K_{\alpha_n}}{\|K_{\alpha_n}\|} = V^{-1} \dfrac{\|K_{\varphi_{n+1}(a)}\|}{\|K_{\varphi_n(a)}\|} \dfrac{K_{\varphi_{n+1}}}{\|K\|}$

$$= \frac{\|K_{\varphi_{n+1}(a)}\|}{\|K_{\varphi_n(a)}\|} e_{n+1}$$

$$\lim_{n\to\infty} \frac{\|K_{\varphi_{n+1}(a)}\|}{\|K_{\varphi_n(a)}\|} = \lim_{n\to\infty} \sqrt{\frac{\frac{1}{1-|\varphi_{n+1}(a)|^2}}{\frac{1}{1-|\varphi_n(a)|^2}}} = \sqrt{\lim \frac{1-|\varphi_n(a)|^2}{1-|\varphi_{n+1}(a)|^2}} = \frac{1}{\sqrt{\varphi'(a)}}$$

So $\quad C_\varphi^* \Big|_{(BH^2)^\perp}$ is a weighted Shift

and $\quad \left(C_\varphi^* \Big|_{(BH^2)^\perp} \right)^*$ has eigenvalues

and we can compute spectrum of ~~this~~ the operator $C_\varphi^* \Big|_{(BH^2)^\perp}$