19 Noviembre

on H^2

Examples ① $\varphi(z) = \zeta z$ $\quad \sigma(C_\varphi) = \text{closure}\{\zeta^k\}$

$|\zeta|=1$ $a=0$ $\varphi'(0)=\zeta$

elliptic automorphism

more generally

$\varphi(z) = \frac{z+r}{1+rz}$ $\quad 0<r<1$

$\varphi(1)=1$ $\quad \varphi'(1) = \frac{1-r}{1+r}$

$f(z) = \exp\left((s+it)\log\frac{1+z}{1-z}\right) = \left(\frac{1+z}{1-z}\right)^{s+it}$

$-\frac12 < s < \frac12$

$f\circ\varphi = \left(\frac{1+r}{1-r}\right)^{s+it} f$

② $\varphi(z) = \frac{3z+1}{z+3}$ $\quad \sigma(C_\varphi) = \{\lambda : \frac{1}{\sqrt2} \le |\lambda| \le \sqrt2\}$

$a=1$ $\varphi'(1)=\frac12$

hyperbolic auto

③ $\varphi(z) = \frac{(1+i)z-1}{z+i-1}$ $\quad \sigma(C_\varphi) = \{\lambda : |\lambda|=1\}$

$a=1$ $\varphi'(1)=1$

parabolic auto

④ $\varphi(z) = \zeta z$ $\quad 0<|\zeta|<1$ $\quad C_\varphi$ compact

$a=0$ $\varphi'(0)=\zeta$ $\quad \sigma(C_\varphi) = \{0\}\cup\{\zeta^n : n=0,1,\dots\}$

$(1-\varphi(z)) = (1-sz-1+s)$

$= s(1-z)$

so $C_\varphi(1-z)^\beta = s^\beta(1-z)^\beta$

⑤ $\varphi(z) = sz+1-s$ $\quad 0<s<1$ $\quad$ not compact

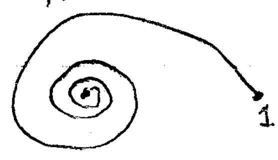
$a=1$ $\varphi'(1)=s<1$ $\quad \sigma(C_\varphi) = \{\lambda : \sqrt{s} \le |\lambda| \le \frac{1}{\sqrt{s}}\}$

⑥ $\varphi(z) = \frac{(2-t)z+t}{-tz+2+t}$ $\quad \sigma(C_\varphi) = \{e^{\beta t} : \beta \le 0\}\cup\{0\}$

Re $t>0$ $\quad a=1$

$\varphi'(1)=1$

$\varphi''(1)=t$



⑦ $\varphi(z) = \frac{rz}{1-(1-r)z}$ $\quad \sigma(C_\varphi) = \{1\}\cup\{\lambda : |\lambda|\le\sqrt{r}\}$

$0<r<1$ $\quad a=0$

$\varphi'(0)=r$

⑧ $\varphi(z) = -\frac12 z + \frac12$

$a=\frac13$ $\varphi'(a) = -\frac12$

C_φ is not compact because $\varphi(-1)=1$ and $|\varphi'(-1)| = |-\frac12| = \frac12 \ne 0$

but C_φ^2 is compact $\quad \varphi\circ\varphi = \frac14 z + \frac14$

so $\sigma(C_\varphi^2) = \{0\}\cup\{\frac14^k : k=0,1,2,3,\dots\}$

By Spectral mapping Thm $(\sigma(C_\varphi))^2 = \sigma(C_\varphi^2)$ and it isn't too hard to see $\sigma(C_\varphi) = \{0\}\cup\{(-\frac12)^k\}$

<u>Thm</u>

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{where } ad-bc \neq 0 \text{ and } \varphi(\mathbb{D}) \subset \mathbb{D}$$

Let $\sigma(z) = \dfrac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}$ let $g(z) = \dfrac{1}{-\bar{b}z+\bar{d}}$, $h(z) = cz+d$.

Then σ maps $\mathbb{D}$ into $\mathbb{D}$ and $g, h \in H^\infty$ and $C_\varphi^* = T_g C_\sigma T_h^*$

<u>Proof</u> Check on K_α $\alpha \in \mathbb{D}$

$$C_\varphi^*\left(\frac{1}{1-\bar{\alpha}z}\right) = C_\varphi^* K_\alpha = K_{\varphi(\alpha)} = \frac{1}{1 - \overline{\frac{a\alpha+b}{c\alpha+d}}\, z}$$

$$T_g C_\sigma T_h^* K_\alpha = \overline{h(\alpha)} \cdot \frac{1}{-\bar{b}z+\bar{d}} \frac{1}{1-\bar{\alpha}\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}}$$

$$= \frac{\bar{c}\bar{\alpha}+\bar{d}}{-\bar{b}z+\bar{d} - \bar{\alpha}\bar{a}z + \bar{\alpha}\bar{c}}$$

$$= \frac{\bar{c}\bar{\alpha}+\bar{d}}{\bar{c}\bar{\alpha}+\bar{d} - (\bar{\alpha}\bar{a}+\bar{b})z} = \frac{1}{1 - \frac{\bar{\alpha}\bar{a}+\bar{b}}{\bar{c}\bar{\alpha}+\bar{d}} z}$$

<u>Cor</u> Let $\varphi(z) = sz + 1 - s = \dfrac{sz+1-s}{0z+1}$ $\quad 0 < s < 1$

Then $C_\varphi^* = T_g C_\sigma T_h^*$ where $\sigma(z) = \dfrac{sz - 0}{-(1-s)z+1}$

$$g(z) = \frac{1}{-(1-s)z+1} = \frac{1}{1-(1-s)z} \qquad h(z) = 1$$

$$= \frac{sz}{1-(1-s)z} = \frac{z}{\frac{1}{s} - (\frac{1}{s}-1)z}$$

$$C_\sigma = 1 \oplus \left(\frac{1}{s} C_\varphi^*\right)$$

$$H^2 = [1] \oplus zH^2$$

And