15 Noviembre

Eigenvalues & Eigenvectors for Compact Composition operators on H^2, A^2, weighted Bergman spaces, and many others

Recall that on these spaces C_φ compact implies that the D.W point of φ is in $\mathbb{D}$ because compactness implies φ has no finite angular derivatives, i.e. no fixed point b on $\partial\mathbb{D}$ with $\varphi'(b) < \infty$.

This means that, a, the D.W. point of φ is in $\mathbb{D}$.
Without loss of generality $a = 0$ because $\psi(z) = \dfrac{a - z}{1 - \bar{a}z}$ is an automorphism of the disk so that $\psi(a) = 0$ and $\psi(0) = a$ (and indeed $\psi^{-1} = \psi$).

Thus $$C_\psi^{-1} C_\varphi C_\psi = C_{\psi^{-1}} C_\varphi C_\psi = C_\psi C_\varphi C_\psi = C_{\psi\circ\varphi\circ\psi}$$

So C_φ is similar to $C_{\psi\circ\varphi\circ\psi}$ and this means
C_φ compact $\Leftrightarrow$ $C_{\psi\circ\varphi\circ\psi}$ compact and
λ is an eigenvalue of C_φ $\Leftrightarrow$ λ is an eigenvalue of $C_{\psi\circ\varphi\circ\psi}$
and both have same multiplicity, etc.

Notice That $\psi(\varphi(\psi(0))) = \psi(\varphi(a)) = \psi(a) = 0$
so $\psi\circ\varphi\circ\psi$ is a map of $\mathbb{D}$ into $\mathbb{D}$ with D.W. point 0.

Since C_φ is assumed compact and we know $\sigma(C_\varphi)$ is $\{0\}$ together with (possibly) a sequence of eigenvalues converging to 0, we want to find f in H^2 and λ in $\mathbb{C}$ so that $C_\varphi f = \lambda f$. We know $f\circ\varphi \equiv 0 \Leftrightarrow f \equiv 0$ so $\lambda = 0$ is not an eigenvalue and we know that $1\circ\varphi = 1 = 1\cdot 1$ so $\lambda = 1$ is always an eigenvalue for C_φ.

Also, if C_φ is compact φ is <u>not</u> an automorphism.

The functional equation $f \circ \varphi = \lambda f$ for a given φ is called Schroeder's functional Equation and it was first solved by Koenigs in 1884.

Koenigs proved that for $\varphi : \mathbb{D} \to \mathbb{D}$ $\varphi(0) = 0$ and $\varphi'(0) \neq 0$ then the functional equation

$$f \circ \varphi = \lambda f \quad \text{has a solution} \iff \lambda = \varphi'(0)^n$$

for $n = 0, 1, 2, \ldots$ and for each $\lambda = \varphi'(0)^n$ the solution space is 1-dimensional.

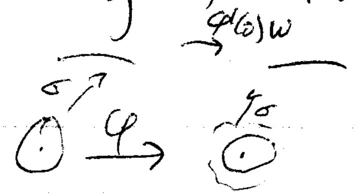
The solution of $f \circ \varphi = \varphi'(0) f$ and $f'(0) = 1$ is often called "Koenigs' function".

We will take a different approach but Koenigs proof is to use a normal families argument to show that

$$\sigma(z) = \lim_{n \to \infty} \frac{\varphi_n(z)}{\varphi'(0)^n} \quad \text{exists as} \ \text{~~an almost uniform~~}$$

a limit uniformly convergent on compact sets, gives $\sigma'(0) = 1$ and $\sigma \circ \varphi = \varphi'(0) \sigma$. This proof does not generalize to $\mathbb{C}^n$, where φ maps ball to ball or polydisk to polydisk.

Of course, since $\varphi'(0) \neq 0$ in Koenigs result, $\varphi : \mathbb{D} \to \mathbb{D}$ and 0 is D.W. point, we know φ is in plane-dilation case. Koenigs function is exactly the map in the model!



Let us look naively at the problem:

$\varphi(0)=0$ so Taylor series for φ is

$$\varphi(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

and we seek $f : \mathbb{D} \to \mathbb{C}$ so that $f \circ \varphi = \lambda f$

Then f has a power series $f(z) = b_0 + b_1 z + b_2 z^2 + \cdots$

Let us equate the series $f\circ\varphi$ and the series λf.

We do not want $f \equiv 0$ so one of $b_j \neq 0$

$$f\circ\varphi(z) = b_0 + b_1(a_1 z + a_2 z^2 + \cdots) + b_2(a_1 z + a_2 z^2 + \cdots)^2 + \cdots$$

$$\lambda f(z) = \lambda b_0 + \lambda b_1 z + \lambda b_2 z^2 + \cdots$$

So $\lambda b_0 = b_0$ from equality of constant terms.

If $b_0 \neq 0$ then $\lambda = 1$

In this case the coefficients of z are $b_1 a_1 = \lambda b_1 = b_1$

so either $a_1 = 1$ or $b_1 = 0$

But Schwartz says (since φ is not an automorphism)

$a_1 = 1$ is not possible! so $b_1 = 0$

Now the coefficients of z^2 are $b_2 a_1^2 = \lambda b_2 = b_2$

So either $a_1^2 = 1$ or $b_2 = 0$

$a_1^2 = 1$ is not possible so $b_2 = 0$ also

etc!

So we see $f \equiv b_0$ is a solution of $f\circ\varphi = 1f$ and these are the only solutions for $\lambda = 1$.

So we assume $b_0 = 0$.

Constant terms of both $f\circ\varphi$ and λf are 0

so look at coefficients of z:

these are $b_1 a_1 = \lambda b_1$ so either $\lambda = a_1 = \varphi'(0)$

or $b_1 = 0$

If $b_1 \neq 0$ then ~~α~~ $\lambda = a_1 = \varphi'(0)$
and the z coefficients are equal.
now the z^2 coefficients are
$$b_1 a_2 + a_1^2 b_2 = \lambda b_2 = \cancel{\varphi'(0)}\ a_1 b_2$$
Notice that if $b_1 \neq 0$, since we know $f \circ \varphi = \varphi'(0) f$ has a solution $\Longrightarrow$ $5f$ or $8f$ or $(3-2i)f$ is also a solution, WLOG $b_1 = 1$

Indeed in $f(z) = b_0 + b_1 z + b_2 z^2 + \cdots$
we may assume WLOG that if $b_0 = b_1 = \cdots = b_{k-1} = 0$ ~~and~~ $b_k \neq 0$ then $b_k = 1$, that is the first coefficient of f that is non-zero is 1.

Now the equation is $a_2 + a_1^2 b_2 = a_1 b_2$
so $a_2 = (a_1 - a_1^2) b_2$
and $b_2 = a_2 / (a_1 - a_1^2)$ Since $a_1 \neq 0$ $a_1 \neq 1$
b_2 is determined.

Similarly coefficients of z^3
are $b_1 a_3 + 2 b_2 a_1 a_2 + b_3 a_1^3 = \lambda b_3 = a_1 b_3$
so $a_3 + 2 b_2 a_1 a_2 = (a_1 - a_1^3) b_3$
and $b_3 = (a_3 + 2 b_2 a_1 a_2)/(a_1 - a_3^3)$ Since $a_1 \neq 0$ and $a_1^2 \neq 1$

b_3 is determined, etc. So there is a 1-dim'l solution for the Taylor coefficients based on $b_1 \neq 0$ and $a_1 = \varphi'(0) \neq 0$

Continuing in this way we see if $b_0 = 0$, $b_1 = 0$, $b_2 \neq 0$ then $\lambda = a_1^2 = \varphi'(0)^2$ and $b_3, b_4, \sim$ are determined by b_2.

this shows if f has power series then $f \circ \varphi = \lambda f \iff \lambda = \varphi'(0)^n$
and solution space is 1-dim'l

Now this does not show f is analytic in $\mathbb{D}$!

Rather it shows if f is analytic & satisfies $f\circ\varphi = \lambda f$ then $\lambda = \varphi'(0)^n$ and all f's are multiples of one solution, say with $b_k = 1$ for 1^{st} non-zero coeff of f.

Now notice if σ solves $\sigma\circ\varphi = \lambda\sigma$ then σ^2 solves $\sigma^2\circ\varphi = \lambda^2\sigma^2$ etc.

Thus, the solutions of $f\circ\varphi = \varphi'(0)^n f$ are just $f = \mu\sigma^n$ for σ the solution of $\sigma\circ\varphi = \varphi'(0)\sigma$ and $\sigma'(0) = 1$, and $\mu \in \mathbb{C}$.

Now $\varphi(0) = 0 \implies C_\varphi \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & 0 & & \\ 0 & a_2 & a_1^2 & 0 & & \\ 0 & a_3 & 2a_1a_2 & a_1^3 & & \\ & \vdots & \vdots & \vdots & & \end{pmatrix}$

with respect to basis $1, z, z^2, \ldots$

and $C_\varphi^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots \\ 0 & 0 & \bar{a}_1^2 & 2\bar{a}_1\bar{a}_2 & \cdots \\ 0 & 0 & 0 & \bar{a}_1^3 & \cdots \end{pmatrix}$

Now C_φ^* clearly has span $[1, z, z^2, \ldots z^n]$ = polynomials of degree $\le n$ as an invariant subspace of H^2 and C_φ^* has eigenvalues $1, \bar{a}_1, \bar{a}_1^2, \ldots \bar{a}_1^n$ on this subspace!

If f is poly of degree $\le n$ and $C_\varphi^* f = \lambda f$ then $f \in H^2$ and $C_\varphi^* f = \lambda f$ so

C_φ bdd op on H^2 and $\varphi(0)=0 \Rightarrow$
$1, \overline{\varphi'(0)}, \overline{\varphi'(0)}^2, \ldots \quad \in \sigma(C_\varphi^*)$.

Moreover, if $0<|\varphi'(0)|<1$, we see each $\overline{\varphi'(0)}^k$ is eigenspace of dimension 1.

Now for C_φ compact and $\lambda \neq 0$
λ eigenvalue of C_φ of eigenspace of dimension k
$\Longleftrightarrow$ $\bar{\lambda}$ eigenvalue of C_φ^* eigenspace of dimension k.

Therefore C_φ compact on $H^2 \Rightarrow \sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\varphi'(0)^n : n=1,2,\ldots\}$
and each eigenspace $1, \varphi'(0)^n$ (if $\varphi'(0) \neq 0$) is dimension 1, and eigenvectors
are $\mu\, \sigma^n$ where $\sigma \circ \varphi = \varphi'(0)\, \sigma$ and $\sigma'(0)=1$
So in particular C_φ compact $\Rightarrow \sigma^n \in H^2$ for all n.

As it happens converse is also true!
$\varphi(0)=0$, $\varphi'(0) \neq 0$, $\sigma^n \in H^2 \ \forall n \Rightarrow C_\varphi$ is compact on H^2.

This generalizes: if $\varphi : B_N \to B_N$ or $\mathbb{D}^N \to \mathbb{D}^N$
ball polydisk in $\mathbb{C}^N$

$\varphi(0)=0$ and $\varphi'(0)$ is $N \times N$ matrix
and C_φ compact on $H^2(B_N)$ $H^2(\mathbb{D}^N)$ etc
then $\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{$ all products of eigenvalues of $\varphi'(0)\}$