12 Noviembre

Recall from 31/10:

<u>Thm</u> Suppose φ is an analytic map of $\mathbb{D}$ into $\mathbb{D}$. Then C_φ is bounded on H^2 and

$$\left(\frac{1}{1-|\varphi_n(0)|^2}\right)^{\frac{1}{2}} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}$$

<u>Cor</u> Suppose φ is an analytic map of $\mathbb{D}$ into $\mathbb{D}$ with Donjoy-Wolff point a in $\overline{\mathbb{D}}$.

① If $|a|<1$ then the spectral radius of C_φ on H^2 is 1

② If $|a|=1$ then the spectral radius of C_φ on H^2 is $\varphi'(a)^{-\frac{1}{2}}$.

<u>Proof</u> The spectral radius of C_φ is $\lim_{n\to\infty} \|C_\varphi^n\|^{1/n}$.

Now $C_\varphi^n = C_{\varphi_n}$ and

$$\frac{1}{\sqrt{2}} \frac{1}{(1-|\varphi_n(0)|)^{1/2}} \leq \|C_{\varphi_n}\| \leq \left(\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|}\right)^{1/2} \leq \frac{\sqrt{2}}{(1-|\varphi_n(0)|)^{1/2}}$$

It follows that the spectral radius of C_φ is

$$\lim_{n\to\infty}\left(\left(\frac{1}{1-|\varphi_n(0)|}\right)^{1/n}\right)^{1/2} \quad \text{because } \lim_{n\to\infty}(\sqrt{2})^{1/n}=1$$

$$\text{and } \lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{n}}=1$$

Now if $|a|<1$ then $\lim_{n\to\infty} \frac{1}{1-|\varphi_n(0)|} = \frac{1}{1-|a|}$

and $\sup \frac{1}{1-|\varphi_n(0)|} \leq M$ for some $M \geq 1$

and $\inf \frac{1}{1-|\varphi_n(0)|} \geq 1$

So $1 = \lim_{n\to\infty} 1^{1/n} \leq \lim_{n\to\infty}\left(\frac{1}{1-|\varphi_n(0)|}\right)^{1/n} \leq \lim_{n\to\infty} M^{1/n} = 1$ //

Notice that $C_\varphi 1 = 1\circ\varphi = 1$ so 1 is always an eigenvector with eigenvalue 1. $1\in\sigma(C_\varphi)$ for any $\varphi:\mathbb{D}\to\mathbb{D}$

If $|a|=1$ and $\varphi'(a)<1$ then $\varphi_n(0)\to a$ non-tangentially

and $$\lim_{n\to\infty}\left(\frac{1}{1-|\varphi_n(0)|}\right)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{1-0}{1-|\varphi(0)|}\right)\left(\frac{1-|\varphi(0)|}{1-|\varphi_2(0)|}\right)\cdots\left(\frac{1-|\varphi_{n-1}(0)|}{1-|\varphi_n(0)|}\right)\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{1-|\varphi_{n-1}(0)|}{1-|\varphi_n(0)|} \quad \text{if this limit exists}$$

(If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}=L$ also but sometimes $\lim_{n\to\infty} a_n$ does not exist but $\lim_{n\to\infty}\frac{a_1+\cdots+a_n}{n}$ does!)

because $\varphi_n(0)\to a$ non-tangentially

$$\lim_{n\to\infty}\frac{1-|\varphi_n(0)|}{1-|\varphi_{n-1}(0)|} = \lim_{n\to\infty}\left|\frac{a-\varphi_n(0)}{a-\varphi_{n-1}(0)}\right| = \lim_{n\to\infty}\left|\frac{\varphi(a)-\varphi(\varphi_{n-1}(0))}{a-\varphi_{n-1}(0)}\right|$$

$$=\varphi'(a) \quad \text{by Julia-Carathodory}$$

so $$\lim_{n\to\infty}\left(\left(\frac{1}{1-|\varphi_n(0)|}\right)^{1/n}\right)^{1/2} = \frac{1}{\sqrt{\varphi'(a)}}$$

If $|a|=1$ and $\varphi'(a)=1$ and z_j is a sequence in $\mathbb{D}$ converging to a such that $\varphi(z_j)$ converges to a and $S=\lim\left(\frac{1-|\varphi(z_j)|}{1-|z_j|}\right)$ exists then J-C implies $S\geq\varphi'(a)=1$

so $$\limsup_{n\to\infty}\frac{1-|\varphi_{n-1}(0)|}{1-|\varphi_n(0)|}\leq 1$$

On the other hand for each n, $|\varphi_n(0)|<1$

so $$\lim_{n\to\infty}\left(\frac{1}{1-|\varphi_n(0)|}\right)^{1/n}\geq 1$$

So in fact $\lim_{n\to\infty}\left(\frac{1}{1-|\varphi_n(0)|}\right)^{1/n} = 1 = \varphi'(a)^{-1}$. //

<u>Compactness</u> Recall that an operator A on a Hilbert space $\mathcal{H}$ is called <u>compact</u> if $\overline{A(B(0,1))}$ is a compact set in $\mathcal{H}$.

A is compact $\iff$ ($v_n \to 0$ weakly in $\mathcal{H}$ $\Rightarrow$ $\|Av_n\| \to 0$)

Carleson measure conditions: on most spaces

C_φ is bdd $\iff$ pull back under φ of $S(\zeta,h)$ is $O(h)$

and C_φ is compact $\iff$ pull back under φ of $S(\zeta,h)$ is $o(h)$.

But Carleson conditions are hard to use in specific situations in practice

In most reasonable spaces, if $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$ then C_φ is compact

On the other hand, in H^2 if $|\varphi(\partial D) \cap \partial D| > 0$

then C_φ is not compact because $z^n \to 0$ weakly

but $\|C_\varphi z^n\|^2 \geq |\varphi(\partial D) \cap \partial D| > 0$.

In H^2, A^2, weighted Bergman spaces; $K_\alpha / \|K_\alpha\| \to 0$ weakly.

C_φ is compact $\iff$ C_φ^* is compact, so

C_φ compact $\Rightarrow$ $\left\| C_\varphi^*\left(\frac{K_\alpha}{\|K_\alpha\|}\right)\right\| \to 0$ as $|\alpha| \to 1$

i.e. $\frac{\|K_{\varphi(\alpha)}\|}{\|K_\alpha\|} \to 0$ as $|\alpha| \to 1$

e.g. in H^2 this says C_φ compact $\Rightarrow$ $\lim\limits_{|\alpha|\to 1^-} \frac{\|K_{\varphi(\alpha)}\|^2}{\|K_\alpha\|^2} =$

Schwartz 1969 Thesis: $\iff$ $\frac{1-|\alpha|^2}{1-|\varphi(\alpha)|^2} \to 0$ as $|\alpha| \to 1$ $\Rightarrow$ no finite angular derivatives.

In fact for A^2, weighted Bergman spaces

C_φ compact $\Leftrightarrow$ no finite angular derivatives

But for H^2 more subtle!

$\varphi(z) = \frac{1}{2}z + \frac{1}{2}$

$\varphi^{-1}(\partial D) = \{1\}$

but C_φ is not compact on H^2, A^2 because $\varphi'(1) = \frac{1}{2}$

Eg. If $\varphi(\mathbb{D}) \subset$ polygon inscribed in ∂D then C_φ is compact on H^2 (~~Caughran Schwartz '75~~ Shapiro Taylor '73)

Joel Shapiro ('87): $\varphi: \mathbb{D} \to \mathbb{D}$ ~~C_φ is composition~~

$$\|C_\varphi\|_e^2 = \limsup_{|w| \to 1^-} \frac{N_\varphi(w)}{-\log|w|} \qquad \text{so compact} \Leftrightarrow \lim_{|w| \to 1^-} \frac{N_\varphi(w)}{-\log|w|} = 0$$

where $N_\varphi(w) = \sum_{\varphi(z_j) = w} \log \frac{1}{|z_j|}$ and $N_\varphi(w) = 0$ if $w \notin \varphi(\mathbb{D})$

Thm If C_φ is compact on H^2

then D.W. pt a of φ satisfies $|a| < 1$

and $\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\varphi'(a)^k\}_{k=1}^\infty$

where 1 and $\{\varphi'(a)^k\}_{k=1}^\infty$ are eigenvalues for C_φ corresponding to eigenspaces of dimension 1 for $|\varphi'(a)| > 0$