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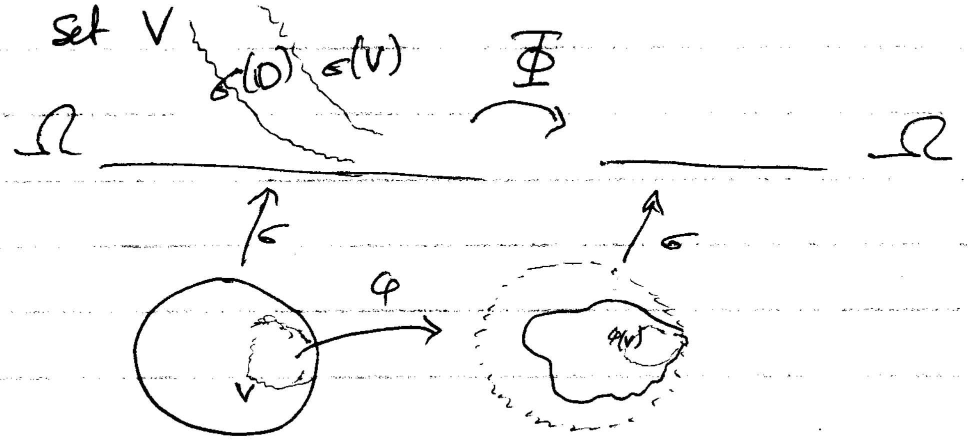
<u>Model for iteration</u> for analytic functions mapping $\mathbb{D}$ into $\mathbb{D}$

Let φ be analytic mapping of $\mathbb{D}$ into itself, non-constant and not an automorphism of $\mathbb{D}$, and let a be Denjoy-Wolff point of φ. If $\varphi'(a) \neq 0$, there is a fundamental set V for φ on $\mathbb{D}$, a domain Ω, either the plane or a half-plane, an automorphism Φ of Ω onto Ω, and a mapping σ of $\mathbb{D}$ into Ω such that φ and σ are univalent on V, $\sigma(V)$ is a fundamental set for Φ on Ω and

$$\Phi \circ \sigma = \sigma \circ \varphi$$

Moreover Φ is unique up to an automorphism of Ω onto Ω and Φ and σ depend only on φ not on the fundamental set V



<u>Def</u> If Ψ maps Δ ~~onto~~ into itself, we say V is a fundamental set for Ψ on Δ if V is an open connected, simply connected subset of Δ such that $\Psi(V) \subset V$ and for every compact set K in Δ, there is a positive integer n so that $\Psi_n(K) \subset V$

The point of the model is that

$$\Phi \circ \sigma = \sigma \circ \varphi \Rightarrow$$

$$\sigma \circ \varphi \circ \varphi = \Phi \circ \sigma \circ \varphi = \Phi \circ \Phi \circ \sigma$$

i.e. $\sigma \circ \varphi_2 = \Phi_2 \circ \sigma$

Similarly $\sigma \circ \varphi_n = \Phi_n \circ \sigma$

So σ is a "change of variables" that changes the (hard to understand) iterates of φ into the (easy to understand) iterates of Φ

The point of fundamental sets — gives uniqueness in the model.

<u>Thm</u> σ is univalent on $\mathbb{D}$ if and only if φ is univalent on $\mathbb{D}$.

<u>Four cases</u>

(plane dilation) $\Omega = \mathbb{C}$, $\sigma(a) = 0$, $\Phi(w) = sw$ where $0 < |s| < 1$

(plane translation) $\Omega = \mathbb{C}$ $\sigma(a) = \infty$ $\Phi(w) = w + 1$

(half-plane dilation) $\Omega = \{w : \operatorname{Re} w > 0\}$ $\sigma(a) = 0$, $\Phi(w) = sw$ where $0 < s < 1$

(half-plane translation) $\Omega = \{w : \operatorname{Im} w > 0\}$ $\sigma(a) = \infty$ $\Phi(w) = w \pm 1$

Proof is a matter of constructing Ω as a virtual domain by looking for "preimages" of points in disk:

we know $\alpha,\ \varphi(\alpha),\ \varphi(\varphi(\alpha)), \dots, \varphi_n(\alpha), \dots \to a$

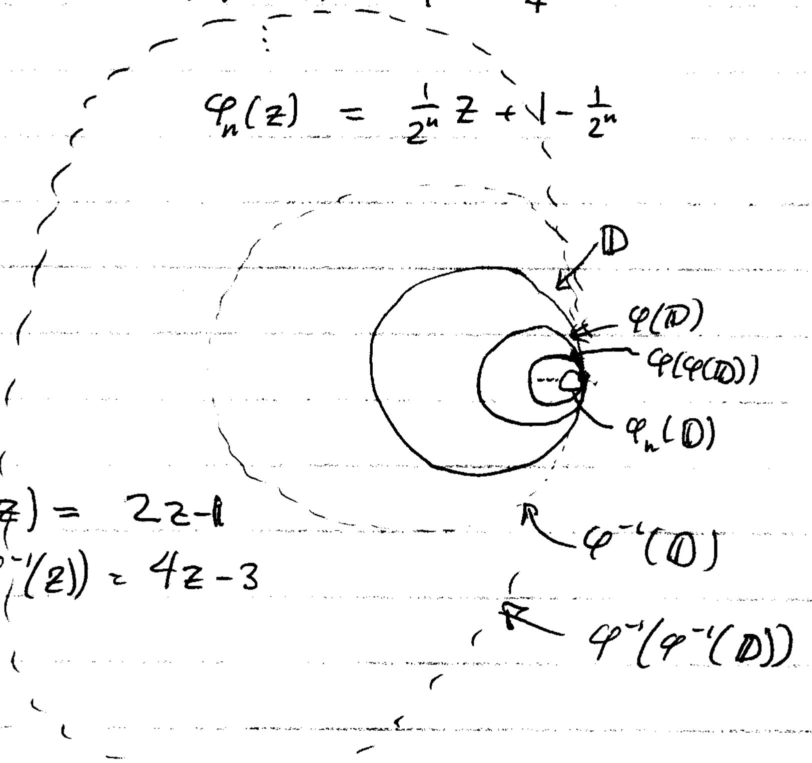
we want to construct "$\varphi^{-1}(\alpha)$", "$\varphi^{-1}(\varphi^{-1}(\alpha))$" ... "$\varphi_{-n}(\alpha)$" ...

Example $\quad \varphi(z) = \frac{1}{2}z + \frac{1}{2}$

$$\varphi(\varphi(z)) = \frac{1}{4}z + \frac{3}{4}$$

$$\vdots$$

$$\varphi_n(z) = \frac{1}{2^n}z + 1 - \frac{1}{2^n} \qquad \varphi_n \to 1 = a$$



$$\varphi^{-1}(z) = 2z - 1$$

$$\varphi^{-1}(\varphi^{-1}(z)) = 4z - 3$$

$$\varphi^{-n}(z) = \varphi_{-n}(z) = 2^n z - 2^n + 1$$

$$\Omega = \bigcup_{n=1}^{\infty} \varphi_{-n}(\mathbb{D}) = \{z : \operatorname{Re} z < 1\}$$

In fact $\varphi(z) = \frac{1}{2}z + \frac{1}{2}$ is in Half-plane dilation case

DW pt is $a = 1$

$w = \sigma(z) = -z + 1 \qquad \Phi(w) = \frac{1}{2}w$

Check $\sigma(\varphi(z)) = -\frac{1}{2}z + \frac{1}{2} = \Phi(\sigma(z))$

For general $\varphi : \mathbb{D} \to \mathbb{D}$ $\quad \varphi'(a) \neq 0$

construct Ω as a Riemann surface of equivalence classes of "inverse images" of points of $V \subset \mathbb{D}$

use complex structure of V to get complex structure on Ω + $\Phi : \Omega$ onto Ω by using φ + virtual structure

→ This works in $\mathbb{C}^n$ for the ball, polydisk, etc in principle.

Last step: for $\mathbb{D}$ show Ω is contractible, connected simply connected non-compact Riemann surface and Φ is an automorphism of Ω onto Ω

Just 2 Riemann surfaces, 2 types automorphisms each!

For ball, polydisk there are potentially uncountably many possible non-equivalent complex n-manifolds.

σ takes D-W pt of φ to D-W point of Φ

$$\varphi'(a) = \Phi'(\sigma(a))$$

Plane dilation: $\quad a \in \mathbb{D},\ 0 < |\varphi'(a)| < 1$

plane translation $\quad a \in \partial\mathbb{D} \quad \varphi'(a) = 1$

half plane dilation $\quad a \in \partial\mathbb{D} \quad 0 < \varphi'(a) < 1$

half plane translation $\quad a \in \partial\mathbb{D} \quad \varphi'(a) = 1$

<u>Def</u> A sequence in $\mathbb{D}$, $\{z_n\}_{n=1}^{\infty}$ is called an <u>interpolating sequence</u> (for H^∞) if for each sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ that is bounded, that is $\exists M>0$ so that $|a_n| \le M$ for all $n=1,2,\dots$, there is φ bounded and analytic on $\mathbb{D}$ ($\varphi \in H^\infty(\mathbb{D})$) such that $\varphi(z_n) = a_n$ for all $n=1,2,\dots$.

Points in an interpolating sequence are "far apart" in a technical sense.

<u>Thm</u> If $\varphi : \mathbb{D} \to \mathbb{D}$ has Denjoy-Wolff point a on ∂D and $\varphi'(a) = 1$
(not useful in practice)
then φ is in the half plane translation case
if and only if
for some α in $\mathbb{D}$, $\{\varphi_n(\alpha)\}$ is an interpolating sequence
if and only if
for all α in $\mathbb{D}$ $\{\varphi_n(\alpha)\}$ is an interpolating sequence.

and

φ is in plane translation case
if and only if
for some α in $\mathbb{D}$, $\{\varphi_n(\alpha)\}$ is <u>not</u> an interpolating sequence
if and only if
for all α in $\mathbb{D}$ $\{\varphi_n(\alpha)\}$ is <u>not</u> an interpolating sequence.

<u>Thm</u> If φ is as above and there is α so that $\varphi_n(\alpha) \to a$ non-tangentially and then φ is in plane translation case.

<u>Cor</u> If $\varphi((-1,1)) \subset (-1,1)$ $\varphi(1)=1=\varphi'(1)$ then φ in plane tra