pseudo-hyperbolic metric $\left|\frac{\zeta_1-\zeta_2}{1-\bar{\zeta}_1\zeta_2}\right| = d(\zeta_1,\zeta_2)$ is preserved by automorphisms of disk & is more useful than Euclidean distance, often

Schwarz-Pick

$$\left|\frac{\varphi(w)-\varphi(z)}{1-\overline{\varphi(w)}\varphi(z)}\right| \le \left|\frac{w-z}{1-\bar{w}z}\right|$$

if equality holds for $z \neq w$ then φ is automorphism

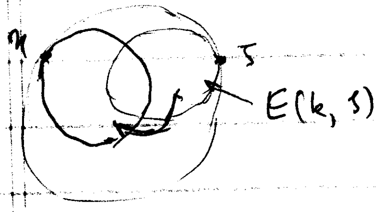
(i.e. maps of disk to itself are "contractions")

Julia's Lemma

ζ is on unit circle and $d(\zeta) = \liminf\limits_{\substack{z\to\zeta \\ z\in D}} \frac{1-|\varphi(z)|}{1-|z|}$ is finite

Suppose $\{a_n\}$ is a sequence along which this lower limit is achieved and for which $\varphi(a_n)\to\eta$. Then $|\eta|=1$ and for every $z\in D$

$$\frac{|\eta-\varphi(z)|^2}{1-|\varphi(z)|^2} \le d(\zeta)\frac{|\zeta-z|^2}{1-|z|^2}$$



$$E(k,\zeta) = \{z\in D : |\zeta-z|^2 \le k(1-|z|^2)\}$$
$$\downarrow$$
$$E(kd(\zeta),\eta)$$

Def We say $b\in\partial D$ is a fixed point of $\varphi: D\to D$ if $\lim\limits_{r\to1^-}\varphi(rb) = b$

non-tangential approach $\leftrightarrow$ radial approach

Def We say φ has a finite angular derivative at $\zeta\in\partial D$ if there is η on the unit circle so that $\frac{\varphi(z)-\eta}{z-\zeta}$ has finite n.t. limit as $z\to\zeta$

$\varphi'(\zeta) \swarrow$

$\varphi: \mathbb{D} \to \mathbb{D}$
$\varphi_0(z) \equiv z,\ \varphi_1(z) = \varphi(z),\ \varphi_2 = \varphi \circ \varphi,\ \varphi_3 = \varphi \circ \varphi \circ \varphi \ \cdots\ \varphi_{n+1} = \varphi \circ \varphi_n \cdots$

Denjoy - Wolff Thm

If φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of $\mathbb{D}$ into itself then there is a point a in $\overline{\mathbb{D}}$ so that φ_n converges to the constant map, a, uniformly on compact subsets of the disk. Moreover, a is the unique fixed point of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

Def: a is called the Denjoy - Wolff point of φ.

Eg. ① $\varphi(z) = \dfrac{z}{2-z}$ $\quad \varphi: \mathbb{D} \to \mathbb{D}$ $\quad \varphi(0) = 0 \quad \varphi(1) = 1$
$\varphi'(0) = \frac{1}{2} \quad \varphi'(1) = 2$

so $a = 0$ is D.W. point

② $\varphi(z) = \frac{2}{3}z + \frac{1}{3}$ $\quad \varphi: \mathbb{D} \to \mathbb{D}$ $\quad \varphi(1) = 1$
$\varphi'(1) = 2/3$

so $a = 1$

③ $\varphi(z) = z^2 \left(\dfrac{z - 1/2}{1 - 1/2 z}\right)^2$ $\quad \varphi(0) = 0 \quad \varphi'(0) = 0$
$\varphi(1) = 1 \quad \varphi'(1) = 8$

$a = 0$

$\varphi\left(\dfrac{1 \pm \sqrt{63}\, i}{8}\right) = \dfrac{1 \pm \sqrt{63}\, i}{8} \quad \varphi'(\cdot) = \frac{10}{3}$

④ $\varphi(z) = \dfrac{1 + z + 2\sqrt{1 - z^2}}{3 - z + 2\sqrt{1 - z^2}}$ $\quad \varphi(1) = 1 \quad \varphi'(1) = 1$

Fixed point set$_n$ for $\varphi: \mathbb{D} \to \mathbb{D}$ not z can be as small as a single point or fairly large : $\{z \mid \varphi(z) = z\} = \{z \mid \varphi(z) - z = 0\}$

Iteration of functions mapping $\mathbb{D}$ into $\mathbb{D}$

aburrido

This is an aspect of dynamical systems, but this is the boring (uninteresting) case: For these functions the open disk is contained in a basin of attraction, whereas complex dynamics concentrates on the Julia set

We are interested in the detailes of the iteration and need quantitative information.