C_φ bounded on H^2: PLMS$^{(2)}$ 23 (1925) 481–519.

Littlewood's Subordination Thm (1925):

Let $\varphi : D \to D$ analytic and $\varphi(0) = 0$

If G is a subharmonic function in D then for $0<r<1$

$$\int_0^{2\pi} G(\varphi(re^{i\theta}))\,d\theta \le \int_0^{2\pi} G(re^{i\theta})\,d\theta$$

Cor For $0<p<\infty$ μ a finite positive measure on $[0,1]$ and

Suppose Y is space of analytic functions for which

$$\|f\|^p = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\,d\mu(r) < \infty$$

If $\varphi : D \to D$ is analytic & $\varphi(0)=0$ then

C_φ is bdd on Y and $\|C_\varphi\| = 1$

Proof $|f|^p$ is subharmonic

so $\int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p \frac{d\theta}{2\pi} \le \int |f(re^{i\theta})|^p \frac{d\theta}{2\pi}$

So $\|(C_\varphi f)\|^p = \int |f(\varphi(re^{i\theta}))|^2 \frac{d\theta}{2\pi}\,d\mu \le \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}$

Thm

φ an automorphism of D then for $H^2(D)$

$$\left(\frac{1-|\varphi(0)|}{1+|\varphi(0)|}\right)^{1/2} \|f\| \le \|C_\varphi f\| \le \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2} \|f\|$$

i.e. C_φ is bdd & $\|C_\varphi\| = \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}$

Proof For f a polynomial

$\|f\circ\varphi\|^2 = \int |f(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi}$ $\quad \varphi(z) = \lambda \frac{z+u}{1+\bar{u}z}$ $\quad |\lambda|=1$ $\quad |u|<1$

Change variables by $\quad e^{it} = \lambda \frac{e^{i\theta}+u}{1+\bar{u}e^{i\theta}}$ $\quad e^{i\theta} = \varphi^{-1}(e^{it}) = \bar{\lambda}\frac{e^{it}-\lambda u}{1-\bar{\lambda}\bar{u}e^{it}}$

$$ie^{i\theta}\frac{d\theta}{2\pi} = \int \frac{1-|u|^2}{(1-\bar{u}e^{it})^2}\, ie^{it}\frac{dt}{2\pi}$$

$$\frac{d\theta}{2\pi} = \frac{1-|u|^2}{|1-\bar{u}e^{it}|^2}\frac{dt}{2\pi}$$

$$|f(\psi(e^{i\theta}))|^2\frac{d\theta}{2\pi} = \int |f(e^{it})|^2 \frac{1-|u|^2}{|1-\bar{u}e^{it}|^2}\frac{dt}{2\pi}$$

$$\int |f(\psi(e^{i\theta}))|^2 \le \frac{1-|u|^2}{(1-|u|)^2}\int |f(e^{it})|^2\frac{dt}{2\pi}$$

so $\|f\circ\psi\|^2 \le \frac{1+|u|}{1-|u|}\|f\|^2 = \frac{1+|\psi(0)|}{1-|\psi(0)|}\|f\|^2$

Cor $\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{1}{2}} \le \|C_\varphi\| \le \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/2}$

Proof: $\psi(z) = \frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}$ $\quad \varphi_0 = \psi\circ\varphi \quad \varphi_0(0)=0$

$\varphi = \psi^{-1}\circ\varphi = \psi\circ\varphi_0$ so $C_\varphi = C_{\varphi_0}C_\psi$

~~Cor spectral radius of C_φ is 1 for $|a|\le 1$~~

Bergman space and proof of the similar

~~$\|C_\varphi\| = \frac{1+|\varphi(0)|}{1-|\varphi(0)|}$~~

Dirichlet space There are unbounded C_φ's!

$C_\varphi z = \varphi$

Function Theory:

non-constant)

Linear Fractional maps $\varphi(z)=\frac{\alpha z+\beta}{\gamma z+\delta}$ (for $\alpha\delta-\beta\gamma\neq 0$)

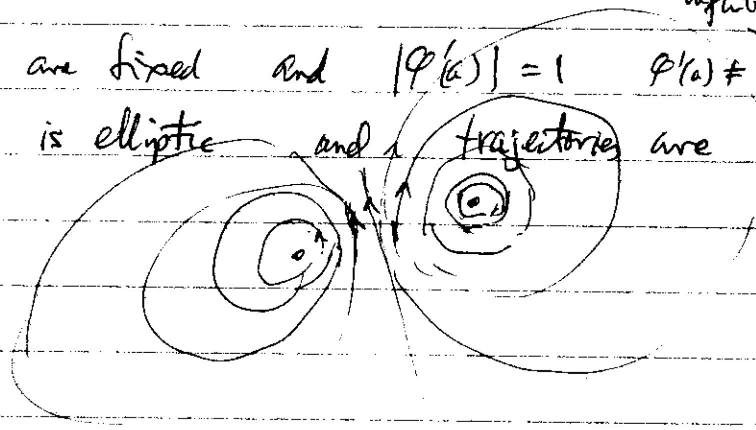
are 1-1 maps of unit sphere to itself and each has exactly two fixed points (counting multiplicity) - quadratic equation

and the derivatives at the fixed points are reciprocals

LFT's take circles to circles (straight is a circle through infinity)

If $a+b$ are fixed and $|\varphi'(a)|=1$ $\varphi'(a)\neq 1$

then φ is elliptic and trajectories are



if $a+b$ are fixed $|\varphi'(a)|<1$ then $|\varphi'(b)|>1$

$\varphi(z)=z$ has a "double root" if $\varphi(z)-z=0$ has a double root

i.e. $\left.\frac{d}{dz}\right|_a(\varphi(z)-z)=0$

i.e. $\varphi'(a)=1$

φ maps D into D and $\varphi(c)=c$ on ∂D then $\varphi'(c)>0$

φ is an automorphism of the disk onto itself then

and one fixed point in inside circle

elliptic

parabolic $\varphi(a)$.

two fixed points on circle