Sometimes important to use

$$H^2(\partial D) = \{ f \text{ analytic in } D : \|f\|^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \}$$

$H^2(D)$ + $H^2(\partial D)$ describe same set + same norm (but function H.S. only for D) moreover if $f \in H^2(D)$ then for almost all θ $0 \le \theta < 2\pi$

$$\lim_{r \to 1} f(re^{i\theta}) = \hat{f}(e^{i\theta}) \text{ exists}$$

and $\hat{f} \in L^2(\partial D)$ $\|\hat{f}\|^2_{L^2} = \|f\|^2_{H^2(D)}$ so H^2 is a natural closed subspace of $L^2(\partial D)$.

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⑤ Bergman Hilbert space : $A^2(\mathbb{D})$ $X = \mathbb{D}$

$$A^2(\mathbb{D}) = \{ f \text{ analytic on } \mathbb{D} : \|f\|^2 = \int_D |f(\zeta)|^2 \frac{d\zeta}{\pi} < \infty \}$$

where for $f, g \in A^2$ $\langle f, g \rangle = \int_D f(\zeta) \overline{g(\zeta)} \frac{d\zeta}{\pi}$.

Exercise 2 show A^2 is complete

Exercise 3 find expression for $f \in A^2$ like def of H^2

⑥ Dirichlet space $X = \mathbb{D}$

$\mathcal{D}$ = { f analytic in $\mathbb{D}$: $\int_D |f'(\zeta)|^2 \frac{d\zeta}{\pi} < \infty$ }

$$\|f\|^2 = \|f\|_{H^2} + \int |f'(\zeta)|^2 \frac{d\zeta}{\pi} \text{ or } \|f\|^2 = |f(0)|^2 + \int |f'(\zeta)|^2 \frac{d\zeta}{\pi}$$

(5) Generalization for $X = \mathbb{B}_N$ ball of radius 1 in $\mathbb{C}^N$
or $X = \mathbb{D}^N$ the polydisk

Def If H is a Hilbert space of analytic functions on Ω in $\mathbb{C}$ or $\mathbb{C}^n$ and $\varphi: \Omega \to \Omega$ analytic
then the comp op $(C_\varphi f)(z) = f(\varphi(z))$

this defines C_φ as a linear transformation,
we need to prove C_φ is bdd operator!

Goal Relate properties of C_φ to properties of φ.

Backtrack Show H^2 is functional Hilbert space
for $\alpha \in D$ K_α is function in H^2 so that
$f(\alpha) = \langle f, K_\alpha \rangle$ for $f \in H^2$.

$f(\alpha) = \sum a_n \alpha^n$ where $f(z) = \sum a_n z^n$

and $K_\alpha(z) = \sum b_n z^n$
$\langle f, K_\alpha \rangle = \sum a_n \overline{b_n}$

so $\sum a_n \alpha^n = \langle f, K_\alpha \rangle = \sum a_n \overline{b_n}$
so $\overline{b_n} = \alpha^n$ or so $b_n = \overline{\alpha}^n$ and
$K_\alpha(z) = \sum_{n=0}^{\infty} \overline{\alpha}^n z^n = \frac{1}{1 - \overline{\alpha} z}$

Notice that $\sum \bar{\alpha}^n z^n$ converges for $\alpha \in D$
and $K_\alpha(z) = \sum \bar{\alpha}^n z^n = 1/(1-\bar{\alpha} z)$

Notice that $\|K_\alpha\|^2 = \langle K_\alpha, K_\alpha \rangle$
$= K_\alpha(\alpha) = \sum \bar{\alpha}^n \alpha^n = \sum |\alpha|^{2n}$
$\|K_\alpha\|^2 = K_\alpha(\alpha) = \frac{1}{1-|\alpha|^2} < \infty$ $= \frac{1}{1-|\alpha|^2} \Rightarrow H^2$ is functional Hilbert space!

Exercise 3 is to think about $A^2(\mathbb{D})$ in this way!

Def For $\varphi : D \to D$ analytic
define $C_\varphi : H^2 \to H^2$ onto
by $(C_\varphi f)(z) = f(\varphi(z))$

Questions: Which φ give C_φ bdd?
What does C_φ^* do?
When is C_φ invertible?

Example Some comp ops not bdd on Dirichlet space:

e.g. $\varphi(z) = e^{\frac{z+1}{z-1}}$
maps $\mathbb{D}$ ∞-to-1 onto $\mathbb{D} \setminus \{0\}$

so $C_\varphi z = \varphi \notin \mathcal{D}$