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webpage for this course: www.iupui.edu/~ccowen/UCMLectures.html

> Questions – language – slow me down

Functional analysis ~ 100 yrs old

Questions had to do with interpreting differential operators as linear transformations on vector spaces of functions

Spaces needed structure connected to limit processes

Concrete functional analysis spaces of integrable functions, special classes of differential operators, integral ops

→ Banach Hilbert spaces
Bounded, unbounded closed ops, compact ops
Spectral theory – generalization of linearity
Multiplication ops as extension of diagonalizable matrices
Normal – self-adjoint

Shift operators as examples of asymmetric behavior possible on infinite dim'l spaces

Study of composition operators extends this process of development

In the next few weeks we will develop this subject beginning with examples of spaces and the analytic function theory that is relevant for the study of the topic

really about the interplay of the operator th. props connected to the analytic function theory props

Boundedness, compactness, norms examples connected linear fractional maps invertibility
Adjoints
Spectra for compact, non-compact, and weighted composOps
Finish with some examples of unsolved problems & future direction for research.

Assume background includes theory of analytic functions & basic functional analysis

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Classical Banach spaces ~~vector space~~ Banach space functions on a set X
If $\varphi : X \to X$ we can imagine composition operator
$C_\varphi f = f \circ \varphi$ for f in Banach space
formally linear: $(af + bg) \circ \varphi = a f \circ \varphi + b\, g \circ \varphi$

Is $f \circ \varphi$ in space? depends on φ & space

$\ell^p(N)$ functions $N \to \mathbb{C}$ p^{th} power integrable wrt counting measure

$x(k) = x_k$ $\varphi : N \to N$ $\varphi(k) = k+1$

$(C_\varphi x)(k) = x(k+1)$

$C_\varphi : (x_1, x_2, \ldots) \to (x_2, x_3, \ldots)$

backward shift mult 1 all shifts!

Classic operators M_{z^2} mult by z^2

$(M_{z^2} f)(z) = z^2 f(z)$

If $\varphi(z) = -z$ then $M_{z^2} C_\varphi = C_\varphi M_{z^2}$

$(M_{z^2} C_\varphi f)(z) = z^2 f(-z)$

$(C_\varphi M_{z^2} f) z = (-z)^2 f(-z)$

1964(?) Forelli showed all isometries of $H^p(D)$ for $p \neq 2$ are weighted composition operators

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Our context: Functional Banach spaces

A Banach space of functions on a set X is a <u>functional Banach space</u> if

1. vector space operations are pt wise ops
2. $f(x) = g(x)$ $\forall x \in X$ $\Rightarrow$ $f = g$ in space
3. $f(x) = f(y)$ for all f in space $\Rightarrow$ $x = y$ in X
4. $f \to f(x)$ is a bounded linear functional on the space

Denote linear functional in 4 by K_x

so $K_x(f) = f(x)$ or in Hilbert space $K_x \in$ space $\langle f, K_x \rangle = f(x)$

Examples

① $\ell^p(\mathbb{N})$ is functional Banach space

② $C([0,1])$ the set of continuous functions on $[0,1]$ with sup norm is functional Banach space

③ $L^1([0,1])$ is not a functional Banach space because $f \to f(\frac{1}{2})$ is not a bounded linear functional on L^1

Exercise 1 ③: Strategy is set of continuous functions on $[0,1]$ is a vector subspace of L^1 and we know what $f \to f(\frac{1}{2})$ means for cont. functions

Show with the induced norm from L^1 this is not bounded. Hahn-Banach implies that if L^1 were functional Banach space, it would be because this map on cont. functions has extension to L^1

In this course we will consider functional Banach spaces whose functions are analytic on the underlying set X: this is what we mean by "Banach space of analytic functions"

Actually will do Hilbert spaces of analytic functions.

④ Hardy Hilbert space; $H^2(\partial\mathbb{D})$ or $H^2(\mathbb{D})$ or H^2

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : f(z) = \sum a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2 < \infty\}$$

where for $f, g \in H^2$ $\langle f, g\rangle = \sum a_n \bar{b}_n$

Sometimes important to use

$$H^2(\partial D) = \{ f \text{ analytic in } D : \|f\|^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \}$$

$H^2(D)$ + $H^2(\partial D)$ describe same set & same norm

(but fin func H.S. only for D) moreover if $f \in H^2(D)$ then for almost all θ $0 \le \theta < 2\pi$

$$\lim_{r\to 1} f(re^{i\theta}) = \hat{f}(e^{i\theta}) \text{ exists}$$

and $\hat{f} \in L^2(\partial D)$ $\|\hat{f}\|_{L^2}^2 = \|f\|_{H^2(D)}^2$ so H^2 is a natural closed subspace of $L^2(\partial D)$.

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(5) Bergman Hilbert space : $A^2(\mathbb{D})$ $X = \mathbb{D}$

$$A^2(\mathbb{D}) = \{ f \text{ analytic on } \mathbb{D} : \|f\|^2 = \int_D |f(\zeta)|^2 \frac{d\zeta}{\pi} < \infty \}$$

where for $f, g \in A^2$ $\langle f, g \rangle = \int_D f(\zeta)\overline{g(\zeta)} \frac{d\zeta}{\pi}$.

Exercise 2 show A^2 is complete

Exercise 3 find expression for $f \in A^2$ like def of H^2

(6) Dirichlet space $X = \mathbb{D}$

Def f analytic in $\mathbb{D}$ $\int_D |f'(\zeta)|^2 \frac{d\zeta}{\pi} < \infty$

$$\|f\|^2 = \|f\|_{H^2} + \int |f'(\zeta)|^2 \frac{d\zeta}{\pi} \text{ or } \|f\|^2 = |f(0)|^2 + \int |f'(\zeta)|^2 \frac{d\zeta}{\pi}$$