Composition Operators on Hilbert Spaces of Analytic Functions

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At this time, there are several active areas, in the world and in España, in functional analysis and the theory connected with Hilbert space operators and these include both study of very abstract questions and the study of more concrete problems. It is well known and easy to prove that all separable, complex, infinite dimensional Hilbert spaces are isomorphic. Since this is the case, it is reasonable to ask why questions are set in different concrete Hilbert spaces like ℓ^2, $L^2([0,1])$, or the Hardy Hilbert space H^2. One answer is that different contexts suggest different tools and different approaches to problems and it may be that one context might provide more easily discoverable solutions to an abstract problem than other contexts do. This course will provide an introduction to the study of Hilbert space problems in a context closely connected to the study of analytic functions and their links to algebra and geometry, an area in which there is active interest in many parts of the world.

The study of composition operators lies at the interface of analytic function theory and operator theory and uses tools from both areas to solve problems that involve the connections between them. As a part of operator theory, research on composition with a fixed function acting on a space of analytic functions is of fairly recent origin, dating back to work of E. Nordgren in the mid 1960's. The first explicit reference to composition operators in the Mathematics Subject Classification Index appeared in 1990. This was a recognition that over the intervening years the literature had grown enough to deserve notice and it continues to grow today. There are themes that have developed so that it is possible to see important groups of papers as exploring the same theme. This course will try to synthesize some of the basic achievements in the area so that those who wish to learn about it can get an overview of the field as it exists today, or at least the view from where I stand. The goal will be to see the broad outlines of the foundations and the developing theory. Many interesting and seemingly basic problems remain open and it is my hope that the lectures will point out areas in which further exploration is desirable and serve to entice you into thinking about some of these problems.

One of the attractive features of this subject is that the prerequisites are minimal. The lectures will presume a basic background in analysis, both functional analysis and complex analysis. Rather than seeking the utmost generality, the theory will be developed in a context that is comfortable and illustrates the nature of the general results.

The approach is based on a philosophy that mathematics develops best from a base of well chosen examples and that its theorems describe and generalize what is true about the characteristic objects in a subject. Thus the lectures are about the concrete operator theory that arises when we study the operation of composition of analytic functions in the context of the classical spaces. In particular, we study the relationship between properties of the operator C_φ and properties of the function φ: the goal is to see the boundedness, spectrum, adjoint, etc., of C_φ as resulting from the particular geometric and analytic features of the symbol φ. The theory of multiplication operators, arising from the spectral theorem for normal operators, has developed and branched into the study of Toeplitz operators, subnormal operators and so on. I believe composition operators can similarly inform the development of operator theory because they are very diverse and occur naturally in a variety of problems. Composition operators have arisen in the study of commutants of multiplication operators and more general operators and play a role in the theory of dynamical systems. De Branges' original proof of the Bieberbach conjecture depended on composition operators (he called them substitution operators) on a space of analytic functions.

There will be opportunities for questions and some exercises will be posed.

Principal Reference: *Composition Operators on Spaces of Analytic Functions,* by C. C. Cowen and B. D. MacCluer, CRC Press, Boca Raton, 1995.

Other References:

Adjoints of rationally induced composition operators, by P. Bourdon and J. Shapiro, *Journal of Functional Analysis* **255**(2008)1995–2012.

Geometric properties of linear fractional maps, by C. Cowen, D. Crosby, D. Horine, R. Ortiz Albino, A. Richman, Y. Yeow, and B. Zerbe, *Indiana University Mathematics Journal* **55**(2006), 553–577.

A new class of operators and a description of adjoints of composition operators, by C. Cowen and E. Gallardo Gutiérrez, *Journal of Functional Analysis* **238**(2006)447–462.

Hermitian weighted composition operators and Bergman extremal functions, by C. Cowen, G. Gunatillake, and E. Ko, *Complex Analysis and Operator Theory*, to appear.

Theory of H^p Spaces, by P. Duren, Academic Press, New York, 1970.

Adjoints of composition operators with rational symbol, by C. Hammond, J. Moorhouse, and M. Robbins, *J. Mathematical Analysis and Applications* **341**(2008), 626–639.

Weighted Composition Operators, by G. Gunatillake, Thesis, Purdue University, 2005.

Function Theory in the Unit Ball of $\mathbf{C}^N$, by W. Rudin, Springer–Verlag, New York, 1980.

Composition Operators and Classical Function Theory, by J. Shapiro, Springer–Verlag, New York, 1993.

Basic outline for the course

1. **Introduction & related function theory.** Functional Banach spaces, description of the classical Hardy, Bergman, Dirichlet spaces, and their generalizations. Littlewood subordination, Schwarz–Pick, Julia–Carathéodory, and Denjoy–Wolff, theorems, the Denjoy–Wolff point, a, the model for iteration.

2. **Basic results.** Boundedness, norms, compactness. Properties of linear fractional maps and their composition operators, adjoints.

3. **Adjoints.** C_φ^* for φ rational and C_φ defined on H^2.

4. **Spectra.** C_φ invertible or Fredholm, spectra of composition operators with inner symbol, spectra for symbol with $|a| = 1$ and $\varphi'(a) \leq 1$. Spectra for symbol with $|a| < 1$ and $|\varphi'(a)| < 1$, C_φ compact and non-compact composition operators, weighted composition operators.

5. **Connections to Invariant Subpaces & other Problems.** Discussion of future research directions and unsolved problems.