

# Commutants of Finite Blaschke Product Multiplication Operators on Bergman Spaces

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(Indiana University Purdue University Indianapolis)

Wabash Modern Analysis Seminar, April 12, 2014

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joint work with Rebecca Wahl (Butler University)

In this talk  $\mathcal{H}$  will denote a Hilbert space of analytic functions on  $\mathbb{D}$ ,

Usual spaces:  $f$  analytic in  $\mathbb{D}$ , with  $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy: } H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman: } A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Bergman } (\gamma > -1): A_{\gamma}^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\gamma} \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \omega_n > 0): H^2(\omega) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \omega_n^2 < \infty\}$$

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the *reproducing kernel function* for  $\mathcal{H}$  is  $K_\alpha$  in  $\mathcal{H}$  with

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For  $H^2$ , we have  $K_\alpha(z) = (1 - \bar{\alpha}z)^{-1}$

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In this talk, we will consider spaces  $H_\kappa^2$  for  $\kappa \geq 1$  which are the weighted Hardy spaces with

$$K_\alpha(z) = (1 - \bar{\alpha}z)^{-\kappa}$$

The spaces  $H_\kappa^2$  include the usual Hardy and Bergman spaces and all the weighted Bergman spaces ( $\gamma = \kappa + 2$ ).

Conversation with Axler made it clear that right generality is to consider Hilbert spaces,  $\mathcal{H}$ , of functions analytic on  $\mathbb{D}$  that satisfy:

- (I) The constant function  $1(z) \equiv 1$  for  $z$  in  $\mathbb{D}$  is in  $\mathcal{H}$  and  $\|1\| = 1$
- (II) For  $\alpha$  in  $\mathbb{D}$ , the linear functional  $f \mapsto f(\alpha)$  is continuous on  $\mathcal{H}$
- (III) For  $\psi$  in  $H^\infty$ , operator  $T_\psi$  given by  $(T_\psi f)(z) = \psi(z)f(z)$  is in  $\mathcal{B}(\mathcal{H})$ .
- (IV) For  $\alpha$  in  $\mathbb{D}$  and  $f$  in  $\mathcal{H}$  with  $f(\alpha) = 0$ , then  $f/(z - \alpha)$  is also in  $\mathcal{H}$ .

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- Conditions (I) & (III) say  $\mathcal{H}$  and its multiplier algebra contain  $H^\infty$
- Condition (II) says  $\mathcal{H}$  has kernel functions & its multiplier algebra is  $H^\infty$
- For  $\psi$  in  $H^\infty$ , the operator  $T_\psi$  in condition (III) is called  
an *analytic multiplication operator* or an *analytic Toeplitz operator*  
and conditions imply  $\|T_\psi\| = \|\psi\|_\infty$  and this means  $\|\psi\| \leq \|\psi\|_\infty$

The Hardy space  $H^2$ , the Bergman space  $A^2$ , and the standard weight Bergman spaces  $H^2_\kappa$  satisfy Conditions (I), (II), (III), and (IV).

The usual Dirichlet space, and many weighted Dirichlet spaces, do not satisfy all the conditions: not all  $H^\infty$  functions are in Dirichlet space!

The Hardy space  $H^2$ , the Bergman space  $A^2$ , and the standard weight Bergman spaces  $H_{\kappa}^2$  satisfy Conditions (I), (II), (III), and (IV).

Consequence: if  $f$  is in  $\mathcal{H}$ ,  $\psi$  is bounded analytic function, and  $\alpha$  is in  $\mathbb{D}$ ,

$$\langle f, T_{\psi}^* K_{\alpha} \rangle = \langle T_{\psi} f, K_{\alpha} \rangle = \psi(\alpha) f(\alpha) = \psi(\alpha) \langle f, K_{\alpha} \rangle = \langle f, \overline{\psi(\alpha)} K_{\alpha} \rangle$$

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Since  $f$  is arbitrary, this means  $T_{\psi}^* K_{\alpha} = \overline{\psi(\alpha)} K_{\alpha}$

and every kernel function is an eigenvector for  $T_{\psi}^*$ .

The spectrum of  $T_{\psi}$  is the closure of  $\psi(\mathbb{D})$ , there no eigenvalues for  $T_{\psi}$ , but the complex conjugate of  $\psi(\mathbb{D})$  consists of eigenvalues of  $T_{\psi}^*$ .

**Definition:**

An *inner function* is a bounded analytic function,  $\psi$ , on  $\mathbb{D}$  such that

$$\lim_{r \rightarrow 1^-} |\psi(re^{i\theta})| = 1 \quad \text{a. e. } d\theta$$

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A function  $B$  is a *Blaschke product of order  $n$*  if it can be written as

$$B(z) = \mu \left( \frac{\zeta_1 - z}{1 - \overline{\zeta_1}z} \right) \left( \frac{\zeta_2 - z}{1 - \overline{\zeta_2}z} \right) \cdots \left( \frac{\zeta_n - z}{1 - \overline{\zeta_n}z} \right)$$

where  $|\mu| = 1$  and  $\zeta_1, \zeta_2, \dots, \zeta_n$  are points of  $\mathbb{D}$ .

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For  $\psi$ , a non-constant inner function, the multiplication operator  $T_\psi$  is a pure isometry on  $H^2$  but is *not* isometric on the Bergman spaces.



## Beurling's Theorem (1949):

Let  $T_z$  be the operator of multiplication by  $z$  on  $H^2(\mathbb{D})$ . A closed subspace  $M$  of  $H^2(\mathbb{D})$  is invariant for  $T_z$  if and only if there is an inner function  $\psi$  such that  $M = \psi H^2(\mathbb{D})$ .

This result is indicative of the interest in the operator  $T_z$  of multiplication by  $z$  on  $H^2(\mathbb{D})$  and in analytic Toeplitz operators  $T_\psi$  on Hilbert spaces of analytic functions more generally.

## Definition:

If  $A$  is a bounded operator on a space  $\mathcal{H}$ , the *commutant of  $A$*  is the set

$$\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$$

For example, for  $T_z$  on  $H^2$ ,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \end{pmatrix}$$

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This means that  $a_{0j} = 0$  for  $j \geq 1$  and  $a_{i,j} = a_{i+1,j+1}$  for  $i, j \geq 0$

In particular, the matrix is lower triangular and is constant along diagonals:

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & & \cdots \end{pmatrix}$$

This is  $T_\psi$  for  $\psi(z) = \sum_{j=0}^{\infty} a_j z^j$  where  $\|\psi\|_\infty = \|T_\psi\|$ .

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We have seen for  $T_z$  on  $H^2$ ,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

By the 1970's, there was interest in the more general question,

For  $\psi$  in  $H^\infty$  and  $T_\psi$  an operator on  $H^2$ , what is  $\{T_\psi\}'$ ?

or more specifically,

For  $B$  a finite Blaschke product and  $T_B$  operating on  $H^2$ , what is  $\{T_B\}'$ ?

Deddens & Wong's 1973 paper used the fact that for  $B$  a finite Blaschke product, the operator  $T_B$  acting on  $H^2$  is a pure isometry to show that

*The operator  $S$  in  $\mathcal{B}(H^2)$  is in  $\{T_B\}'$  if and only if*

*$S$  can be represented as a lower triangular block Toeplitz matrix with respect to the description of  $H^2$  as  $\bigoplus_{k=0}^{\infty} B^k \mathcal{W}$  where  $\mathcal{W}$  is the wandering subspace  $\mathcal{W} = (BH^2)^\perp$ , that is,*

$$S = \begin{pmatrix} A_0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ A_3 & A_2 & A_1 & A_0 & \cdots \\ \vdots & & & & \ddots \end{pmatrix}$$

Shortly thereafter, Thomson's papers and Cowen's papers computed  $\{T_B\}'$  from a different perspective:

**Fundamental Lemma:**

*For  $S$  a bounded operator on  $H^2$  and  $\psi$  in  $H^\infty$ , these • are equivalent*

- *$S$  commutes with  $T_\psi$*
- *For all  $\alpha$  in  $\mathbb{D}$ ,  $S^*K_\alpha \perp (\psi - \psi(\alpha))H^2$*

**Proof:** (Main calculation)

For  $\alpha$  in  $\mathbb{D}$ ,  $\psi$  in  $H^\infty$ , and  $ST_\psi = T_\psi S$ , if  $f$  is in  $H^2$ ,

$$\begin{aligned}
 \langle (\psi - \psi(\alpha))f, S^*K_\alpha \rangle &= \langle ST_\psi f, K_\alpha \rangle - \psi(\alpha)\langle Sf, K_\alpha \rangle \\
 &= \langle T_\psi Sf, K_\alpha \rangle - \psi(\alpha)\langle Sf, K_\alpha \rangle = \langle Sf, T_\psi^* K_\alpha \rangle - \psi(\alpha)\langle Sf, K_\alpha \rangle \\
 &= \psi(\alpha)(Sf)(\alpha) - \psi(\alpha)(Sf)(\alpha) = 0
 \end{aligned}$$

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

That is, maybe there is a small set  $\mathcal{S}$  of  $H^\infty$  functions so that for each  $\psi$  in  $H^\infty$ , there is  $\varphi$  in  $\mathcal{S}$  so that  $\{T_\psi\}' = \{T_\varphi\}'$ .



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It became clear that, inner functions and covering maps should be part of any such set  $\mathcal{S}$  because Toeplitz operators associated with many other  $H^\infty$  functions have commutants the same as inner function or covering map Toeplitz operators.

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For example, the Fundamental Lemma, immediately implies

*If  $\varphi$  and  $\psi$  are in  $H^\infty$  and there is an analytic function  $g$*

*so that  $\varphi = g \circ \psi$ , then  $\{T_\varphi\}' \supset \{T_\psi\}'$ .*

So a natural question is: “If  $\varphi = g \circ \psi$ , when does  $\{T_\varphi\}' = \{T_\psi\}'$ ?”

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

**Theorem:** [C., 1978]

*If  $\psi$  is a bounded analytic function on the disk  $\mathbb{D}$*

*and  $\alpha_0$  is a point of the disk so that the inner factor of  $\psi - \psi(\alpha_0)$*

*is a finite Blaschke product,*

*then there is a finite Blaschke product  $B$  so that*

$$\{T_\psi\}' = \{T_B\}'$$

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*then there is a finite Blaschke product  $B$  so that*

$$\{T_\psi\}' = \{T_B\}'$$

In fact, the Blaschke product  $B$  is the “largest” inner function for which there is bounded function  $g$  so that  $\psi = g \circ B$ .

For  $B$  a finite Blaschke product of order  $n$ , except for  $n(n - 1)$  points of the disk for which  $B(\alpha) = B(\beta)$  and  $B'(\beta) = 0$ ,

$$\left( (B - B(\alpha)) H^2 \right)^\perp = \text{span} \{ K_{\beta_1}, K_{\beta_2}, \dots, K_{\beta_n} \}$$

where the points  $\alpha = \beta_1, \beta_2, \dots, \beta_n$  are the  $n$  distinct points of  $\mathbb{D}$  for which  $B(\beta_j) = B(\alpha)$ .

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The important fact behind this work is that the kernel functions  $K_\alpha$ ,  $K_\alpha(z) = (1 - \bar{\alpha}z)^{-1}$  in  $H^2$  and  $K_\alpha(z) = (1 - \bar{\alpha}z)^{-2}$  in  $A^2$ , depend conjugate analytically on  $\alpha$ , so if  $A$  is a linear operator so that  $AK_\alpha$  is always in  $\left( (B - B(\alpha)) H^2 \right)^\perp$ , then

$$AK_\alpha = \sum_j c_j K_{\beta_j}$$

where the  $c_j$ 's and the  $K_{\beta_j}$ 's are conjugate analytic in  $\alpha$

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### **Observation:**

For the study of commutants of Toeplitz operators, it is more important that a Blaschke product  $B$  is an  $n$ -to-1 map of  $\mathbb{D}$  onto itself than the fact that  $T_B$  is a pure isometry on  $H^2$ .

Of course, since the points  $\alpha = \beta_1, \beta_2, \dots, \beta_n$  depend on  $\alpha$ , we may write them as  $\alpha = \beta_1(\alpha), \beta_2(\alpha), \dots, \beta_n(\alpha)$ .

In fact (!), if  $B$  is a finite Blaschke product of order  $n$  and  $\alpha$  is a point of the disk that is *NOT* one of the  $n(n - 1)$  points of the disk for which

$$B(\alpha) = B(\beta) \text{ and } B'(\beta) = 0,$$

*the maps  $\alpha \mapsto \beta_j(\alpha)$  are just the  $n$  branches of the analytic function  $B^{-1} \circ B$  that is defined and arbitrarily continuable on the disk with the  $n(n - 1)$  exceptional points removed.*



Of course, since the points  $\alpha = \beta_1, \beta_2, \dots, \beta_n$  depend on  $\alpha$ , we may write them as  $\alpha = \beta_1(\alpha), \beta_2(\alpha), \dots, \beta_n(\alpha)$ .

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**Theorem:** (Cowen, 1974)

*For  $B$  a finite Blaschke product, the branches of  $B^{-1} \circ B$  form a group whose normal subgroups are associated with compositional factorizations of  $B$  into compositions of two Blaschke products.*

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**Theorem:** ~~(Cowen, 1974)~~ (Ritt, 1922, '23)

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Recall the

**Fundamental Lemma:**

*For  $S$  a bounded operator on  $H^2$  and  $\psi$  in  $H^\infty$ , these  $\bullet$  are equivalent*

- $\bullet$   $S$  commutes with  $T_\psi$*
- $\bullet$  For all  $\alpha$  in  $\mathbb{D}$ ,  $S^*K_\alpha \perp (\psi - \psi(\alpha))H^2$*

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Use ideas about,  $W$ , the Riemann surface for  $B^{-1} \circ B$  to rewrite this as:

### **Fundamental Lemma(2):**

*Let  $B$  be a finite Blaschke product. Let  $F$  be the set*

$$F = \{\alpha \in \mathbb{D} : B(\alpha) = B(\beta) \text{ for some } \beta \text{ with } B'(\beta) = 0\}.$$

*If  $S$  is a bounded operator on  $H^2$ , then  $S$  is in  $\{T_B\}'$  if and only if*

$$S^*K_\alpha = \sum_{j=1}^n c_j(\alpha)K_{\beta_j(\alpha)} \text{ for each } \alpha \text{ in } \mathbb{D} \setminus F.$$

We use this to write  $Sf$  as a function of  $\alpha$  in the disk.

**Theorem:** (Cowen, 1978). *Let  $B$ ,  $F$ , and  $W$  be as above.*

*If  $S$  is a bounded operator on  $H^2$  that commutes with  $T_B$ , then there is a bounded analytic function  $G$  on the Riemann surface  $W$  so that for  $f$  in  $H^2$ ,*

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha)f(\beta(\alpha)) \quad (1)$$

*where the sum is taken over the  $n$  branches of  $B^{-1} \circ B$  at  $\alpha$ . Moreover, if  $\alpha_0$  is a zero of order  $m$  of  $B'$ , and  $\psi_1, \psi_2, \dots, \psi_n$  is a basis for  $((B - B(\alpha_0))H^2)^\perp$ , then  $G$  has the property that*

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \quad (2)$$

*for  $j = 1, 2, \dots, n$ .*

*Conversely, if  $G$  is a bounded analytic function on  $W$  that has properties (2) at each zero of  $B'$ , then (1) defines a bounded linear operator on  $H^2$  with  $S$  in  $\{T_B\}'$ .*

In 2006, Cowen and Gallardo-Gutiérrez, in connection with their study of adjoints of composition operators, developed a formal class of operators called ‘multiple-valued weighted composition operators’. The operators  $S$  in  $\{T_B\}'$  are just such operators.

In 2006, Cowen and Gallardo-Gutiérrez, in connection with their study of adjoints of composition operators, developed a formal class of operators called ‘multiple-valued weighted composition operators’. The operators  $S$  in  $\{T_B\}'$  are just such operators.

In the past few years, Douglas, Sun, and Zheng, and Douglas, Putinar, and Wang, and others have used related tools to study problems concerning commutants of  $T_B$  on the Bergman space, such as consideration of the reducing subspaces of  $T_B$ .

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### **Observation:**

The class of ‘multiple-valued weighted composition operators’, an extension of classes of algebras of operators generated by multiplication and composition operators, appear to be useful in the study of certain kinds of problems in operator theory, including questions related to commutants.



**Theorem:** (C. & Wahl, 2012). *Let  $B$ ,  $F$ , and  $W$  be as above.*

*If  $S$  is a bounded operator on  $A^2$  that commutes with  $T_B$ , then there is a bounded analytic function  $G$  on the Riemann surface  $W$  so that for  $f$  in  $A^2$ ,*

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha)f(\beta(\alpha)) \quad (3)$$

*where the sum is taken over the  $n$  branches of  $B^{-1} \circ B$  at  $\alpha$ . Moreover, if  $\alpha_0$  is a zero of order  $m$  of  $B'$ , and  $\psi_1, \psi_2, \dots, \psi_n$  is a basis for  $((B - B(\alpha_0))A^2)^\perp$ , then  $G$  has the property that*

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \quad (4)$$

*for  $j = 1, 2, \dots, n$ .*

*Conversely, if  $G$  is a bounded analytic function on  $W$  that has properties (4) at each zero of  $B'$ , then (3) defines a bounded linear operator on  $A^2$  with  $S$  in  $\{T_B\}'$ .*

**Theorem:** (Cowen, 1978).

*If  $B$  is a finite Blaschke product and  $S$  is a bounded operator on  $H^2$*

*such that  $ST_B = T_B S$ ,*

*then for all  $f$  in  $H^\infty$ ,  $Sf$  is also in  $H^\infty$ .*

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Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha)) \beta'(\alpha) f(\beta(\alpha)) \quad (3)$$

First,  $f$  is assumed to be bounded on the disk:

so  $|f(\beta(\alpha))|$  is bounded by  $\|f\|_\infty$ .

**Theorem:** (C. & Wahl, 2012).

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Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha)) \beta'(\alpha) f(\beta(\alpha)) \quad (3)$$

Second,  $B$  is an  $n$ -to-1 map of the Riemann sphere to itself, so it has  $n$  poles outside the closed unit disk. In particular,  $B$  is analytic in a disk strictly larger than  $\mathbb{D}$  and the  $\beta'(\alpha)$  are bounded in a disk larger than  $\mathbb{D}$ .

**Theorem:** (C. & Wahl, 2012).

*If  $B$  is a finite Blaschke product and  $S$  is a bounded operator on  $A^2$*

*such that  $ST_B = T_B S$ ,*

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Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha)) \beta'(\alpha) f(\beta(\alpha)) \quad (3)$$

Third,  $B'$  has  $2n - 2$  zeros on the Riemann sphere,  $n - 1$  in  $\mathbb{D}$  and the other  $n - 1$  are reflections of these outside the closed unit disk.

In particular,  $B'$  is analytic and non-zero in an annulus strictly containing the unit circle.

This means  $(B'(\alpha))^{-1}$  is bounded near the unit circle.

**Theorem:** (C. & Wahl, 2012).

*If  $B$  is a finite Blaschke product and  $S$  is a bounded operator on  $A^2$*

*such that  $ST_B = T_B S$ ,*

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Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha)) \beta'(\alpha) f(\beta(\alpha)) \quad (3)$$

Finally, the sum appears to depend on all  $n$  of the branches of  $B^{-1} \circ B$  simultaneously.

Of course, it does, but using bounded analytic functions as multipliers, we can eliminate all but one term in the sum (3).

This allows us to show that each term of the sum  $G((\beta, \alpha))$  is bounded separately and there are  $n$  bounded terms in the sum.

**Theorem:** (C. & Wahl, 2012).

*If  $B$  is a finite Blaschke product and  $S$  is a bounded operator on  $A^2$*

*such that  $ST_B = T_B S$ ,*

*then for all  $f$  in  $H^\infty$ ,  $Sf$  is also in  $H^\infty$ .*

**Corollary:**

*The commutants of  $T_B$  as an operator on  $H^2$  and of  $T_B$  as an operator on  $A^2$  are ‘the same’.*



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**Corollary:**

*The commutants of  $T_B$  as an operator on  $H^2$  and of  $T_B$  as an operator on  $A^2$  are ‘the same’.*

The bounded analytic functions on the disk are dense in both  $H^2$  and  $A^2$ .

Since these functions are mapped in the same way as vectors in  $H^2$  and  $A^2$ , the operators agree on all vectors common to  $H^2$  and  $A^2$ .

These ideas apply in the same way to the weighted Bergman spaces as well.

**Theorem:** (C. & Wahl, 2012).

*If  $B$  is a finite Blaschke product and  $S$  is a bounded operator on  $A^2$*

*such that  $ST_B = T_B S$ ,*

*then for all  $f$  in  $H^\infty$ ,  $Sf$  is also in  $H^\infty$ .*

**Corollary:**

*The commutants of  $T_B$  as an operator on  $H^2$  and of  $T_B$  as an operator on  $A^2$  are ‘the same’.*

**Corollary:**

*If  $\psi$  is a bounded analytic function on the disk  $\mathbb{D}$*

*and  $\alpha_0$  is a point of the disk so that the inner factor of  $\psi - \psi(\alpha_0)$*

*is a finite Blaschke product, there is finite Blaschke product  $B$  with*

$$\{T_\psi\}' = \{T_B\}' \quad \text{as operators on } A^2$$

**Theorem:** (C. & Wahl, 2012).

*If  $B$  is a finite Blaschke product and  $S$  is a bounded operator on  $A^2$*

*such that  $ST_B = T_B S$ ,*

*then for all  $f$  in  $H^\infty$ ,  $Sf$  is also in  $H^\infty$ .*

**Corollary:**

*The commutants of  $T_B$  as an operator on  $H^2$  and of  $T_B$  as an operator on  $A^2$  are ‘the same’.*

**Corollary:**

*If  $P$  is a bounded operator acting on  $H^2$  such that  $P^2 = P$  and*

*$T_B P = P T_B$ , then  $P$  is a bounded an operator acting on  $A^2$  such that*

*$P^2 = P$  and  $T_B P = P T_B$ .*

The result

**Corollary:**

*If  $P$  is a bounded operator acting on  $H^2$  such that  $P^2 = P$  and  $T_B P = P T_B$ , then  $P$  is a bounded operator acting on  $A^2$  such that  $P^2 = P$  and  $T_B P = P T_B$ .*

leads to some obvious, but still unsolved problems: “Which of the projections that commute with  $T_B$  on the Bergman space are self-adjoint?”

It is easy to see that many more self-adjoint projections commute with  $T_B$  on  $H^2$  than on  $A^2$  because multiplication by  $B$  is an isometry in  $H^2$ , but not on  $A^2$ .

The question “What is  $\{T_B, T_B^*\}$ ’?” is largely unstudied!

**Thank You!**

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If  $B(z) = z^2 \left( \frac{z - .5}{1 - .5z} \right)^2$ , the group  $B^{-1} \circ B$  is isomorphic to  $D_4$ .

$D_4$  has several normal subgroups, and most give trivial factorizations of  $B$  into the composition of a Blaschke product of order 1 and one of order 4.

However, there is a normal subgroup that “finds” the non-trivial

decomposition of  $B$  as  $B = J_1 \circ J_2$  where  $J_1(z) = z^2$  and  $J_2(z) = z \frac{z - .5}{1 - .5z}$