Hermitian Weighted Composition Operators and Bergman Extremal Functions

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Joint work with Gajath Gunatillake, Univ. Sharjah

and with Eungil Ko, Ewha Women's University

We consider Hilbert spaces of functions analytic on $\mathbb{D}$, the unit disk in $\mathbb{C}$.

For a weight sequence, $\beta(j) > 0$ with $\beta(0) = 1$, let

$$H^2(\beta) = \{f(z) = \sum_0^\infty a_n z^n : \sum_0^\infty |a_n|^2 \beta(j)^2 < \infty\}$$

For example, the usual Hardy Hilbert space is the case $\beta(j) \equiv 1$ which is also described as $H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int_0^{2\pi} |f_r|^2 \frac{d\theta}{2\pi} < \infty\}$.

Also, for $\kappa > 1$, the standard weight Bergman spaces are

$$A^2_{\kappa-2} = \{f \text{ analytic in } \mathbb{D} : \int_{\mathbb{D}} |f(z)|^2 (\kappa - 1)(1 - |z|^2)^{\kappa-2} \frac{dA}{\pi} < \infty\}$$

For example, the usual Bergman Hilbert space is the case $\kappa = 2$, which is $A^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \int_D |f(z)|^2 \frac{dA}{\pi} < \infty\}$.

If $\mathcal{H}$ is a Hilbert space of analytic functions, for each α in the disk, the *reproducing kernel function* K_α satisfies $\langle f, K_\alpha \rangle = f(\alpha)$ for all f in $\mathcal{H}$.

In this talk, we will restrict attention to the weighted Hardy spaces $H^2(\beta_\kappa)$ for which $\kappa \geq 1$ and the kernels are given by

$$K_\alpha(z) = (1 - \overline{\alpha} z)^{-\kappa}$$

This includes the usual Hardy ($\kappa = 1$) and Bergman ($\kappa = 2$) spaces and the standard weight Bergman spaces because $A^2_{\kappa-2}(\mathbb{D})$ and $H^2(\beta_\kappa)$ consist of the same functions.

Let φ and ψ be analytic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$.

The *composition operator* C_φ is the operator on $H^2(\beta_\kappa)$ given by

$$(C_\varphi f)(z) = f(\varphi(z))$$

for z in $\mathbb{D}$ and the *weighted composition operator* $W_{\psi,\varphi}$ is the operator on $H^2(\beta_\kappa)$ given by

$$(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z))$$

for z in $\mathbb{D}$.

Goal: relate the function–theoretic properties of φ and ψ to the operator–theoretic properties of $W_{\psi,\varphi}$

Further Goal: use results from the function–theoretic realm to enhance operator–theoretic understanding and vice versa

Since $H^2(\beta_\kappa)$ contains the constants,

if $W_{\psi,\varphi}$ is bounded, then $\psi = W_{\psi,\varphi}(1)$ is in $H^2(\beta_\kappa)$.

Clearly, if ψ is in $H^\infty(\mathbb{D})$, then for any φ mapping the unit disk into itself, $W_{\psi,\varphi}$ is bounded on $H^2(\beta_\kappa)$ and

$$\|W_{\psi,\varphi}\| \leq \|\psi\|_\infty \|C_\varphi\|$$

BUT, it is not necessary for ψ to be bounded for $W_{\psi,\varphi}$ to be bounded.

As usual, we have a simple formula for $W_{\psi,\varphi}^*$ acting on kernel functions:

$$\langle f, W_{\psi,\varphi}^* K_\alpha \rangle = \langle W_{\psi,\varphi} f, K_\alpha \rangle = \psi(\alpha) f(\varphi(\alpha)) = \langle f, \overline{\psi(\alpha)} K_{\varphi(\alpha)} \rangle$$

so $W_{\psi,\varphi}^* K_\alpha = \overline{\psi(\alpha)} K_{\varphi(\alpha)}$

Theorem.

For $\kappa \geq 1$,

$W_{\psi,\varphi}$ *is a bounded Hermitian weighted composition operator on* $H^2(\beta_\kappa)$,

if and only if

$$\psi(z) = c(1 - \overline{a_0}z)^{-\kappa} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0}z}$$

where $c = \psi(0)$ *and* $a_1 = \varphi'(0)$ *are real numbers*

and a_1 *and* $a_0 = \varphi(0)$ *are such that* φ *maps the unit disk into itself.*

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Proposition. *Let* a_1 *be real.*

$\varphi(z) = a_0 + a_1 z/(1 - \overline{a_0}z)$ *maps the unit disk into itself if and only if*

$$|a_0| < 1 \qquad \text{and} \qquad -1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2$$

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The three cases,

$$a_1 = -1 + |a_0|^2, \quad -1 + |a_0|^2 < a_1 < (1 - |a_0|)^2, \quad \text{and} \;\; a_1 = (1 - |a_0|)^2$$

are quite different from each other.

Cases 1 & 2:

If $|a_0| < 1$ and $a_1 < (1 - |a_0|)^2$, then φ has a fixed point in $\mathbb{D}$ and the analysis and the spectral theory follow directly from consideration of the point spectra of these operators.

Case 3:

When $a_1 = (1 - |a_0|)^2$, $a_0 \neq 0$,

the map φ has a fixed point on the unit circle, (none in the disk),

but is not an automorphism of the disk, and $W_{\psi,\varphi}$ is not compact.

By normalizing, WLOG, we may assume $0 < a_0 < 1$.

Writing $t = a_0/(1 - a_0)$, each such $W_{\psi,\varphi}$ is a multiple of $A_t = W_{\psi_t,\varphi_t}$ where

$$\psi_t = (1 + t - tz)^{-\kappa}$$

and

$$\varphi_t = (t + (1 - t)z)/(1 + t - tz)$$

Then for $0 \leq t < \infty$, A_t is a semigroup of Hermitian weighted composition operators. (And(!!) for $\operatorname{Re} t > 0$, A_t is a semigroup of normal operators.)

Theorem.

For $\kappa \geq 1$ *and* $0 \leq t < \infty$, *let* $A_t = W_{\psi_t,\varphi_t}$ *where*

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \textit{and} \quad \varphi_t = (t + (1-t)z)/(1 + t - tz)$$

The A_t *form a strongly continuous semigroup of Hermitian weighted composition operators on* $H^2(\beta_\kappa)$. *If* Δ *is the infinitesimal generator of this semigroup,* $\mathcal{D}_A = \{f \in H^2(\beta_\kappa) : (z-1)^2 f' \in H^2(\beta_\kappa)\}$ *is the domain of* Δ *and* $\quad \Delta(f)(z) = (z-1)^2 f'(z) + \kappa(z-1)f(z) \quad$ *for* f *in* $\mathcal{D}_A$.

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Corollary.

For $\kappa \geq 1$ *and for* $t > 0$, *the operator* A_t *on* $H^2(\beta_\kappa)$ *has no eigenvalues.*

Proof: There are no non-zero functions in $H^2(\beta_\kappa)$ that satisfy

$$(z-1)^2 f' + \kappa(z-1)f = \lambda f(z)$$

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For $\kappa \geq 1$ *and* $0 \leq t < \infty$, *let* $A_t = W_{\psi_t,\varphi_t}$ *where*

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t = (t + (1 - t)z)/(1 + t - tz)$$

For each t, *the operator* A_t *is a cyclic Hermitian weighted composition operator on* $H^2(\beta_\kappa)$. *Indeed, the vector* 1 *is a cyclic vector for* A_t.

If μ *is the absolutely continuous probability measure given by*

$$d\mu = \frac{(\ln(1/x))^{\kappa-1}}{\Gamma(\kappa)}\, dx$$

the operator U *given by* $U(\psi_t) = x^t$ *for* $0 \leq t < \infty$, *is a unitary map of* $H^2(\beta_\kappa)$ *onto* $L^2([0,1],\mu)$ *and satisfies* $UA_t = M_{x^t}U$.

In particular, for each $t > 0$, *these operators satisfy* $\|A_t\| = 1$ *and have spectrum* $\sigma(A_t) = [0,1]$.

We define subspaces H_c of $H^2(\beta_\kappa) = A^2_{\kappa-2}$ as follows:

Let $H_0 = H^2(\beta_\kappa)$. For $c < 0$, define the subspace H_c by

$$H_c = \text{ closure } \{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\}$$

For $0 \leq t$ and $c \leq 0$, the subspace H_c is invariant for A_t.

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For $0 \le \delta \le 1$ define the subspace L_δ of $L^2([0,1],\mu)$ by

$$L_\delta = \{f \in L^2([0,1],\mu) : f(x) = 0 \text{ for } \delta < x \le 1\}$$

These are spectral subspaces of the multiplication operators M_{x^t}

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Theorem.

If U gives unitary equivalence from A_t on $H^2(\beta_\kappa)$ to M_{x^t} on $L^2([0,1],\mu)$,

$$\textit{then} \quad U^* L_\delta = H_{(\ln \delta)/2} \quad \textit{or equivalently} \quad U H_c = L_{e^{2c}}$$

Suppose N is a subspace of $H^2(\beta_\kappa)$ that is invariant for the operator of multiplication by z.

If there is f in N with $f(0) \neq 0$ and G is a function of N so that

$$\|G\| = 1 \quad \text{and} \quad G(0) = \sup\{\operatorname{Re} f(0) : f \in N \text{ and } \|f\| = 1\}$$

then we say G *solves the extremal problem for the invariant subspace* N.

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Subspaces H_c are spectral subspaces for A_t, but more interestingly, they are invariant subspaces for M_z on $H^2(\beta_\kappa)$ generated by atomic inner functions!

The unitary equivalence between the subspaces H_c in $H^2(\beta_\kappa)$ and L_δ in $L^2([0,1],\mu)$ gives an opportunity to compute the extremal functions for L_δ and translate the answer back to H_c!!

Our computation requires the use of the *incomplete Gamma function*

$$\Gamma(a, w) = \int_w^\infty t^{a-1} e^{-t}\, dt$$

where a is a complex parameter and w is a real parameter. An alternate definition in which both a and w are complex parameters is

$$\Gamma(a, w) = e^{-w} w^a \int_0^\infty e^{-wu} (1+u)^{a-1}\, du$$

Theorem.

For $c < 0$, if H_c is the invariant subspace of $H^2(\beta_\kappa)$ defined by

$$H_c = \text{closure}\{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\}$$

then the extremal function for H_c is

$$G_c(z) = \frac{\Gamma(\kappa, -2c/(1-z))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa, -2c)}}$$

Theorem.

For $0 < r < 1$*, let* P_r *be the orthogonal projection onto the subspace* $H_{(\ln r)/2}$ *in* $H^2(\beta_\kappa)$*. If* u *is any point of the open unit disk, then for* $K_u(z) = (1 - \overline{u}z)^{-\kappa}$

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1 - \overline{u}z)^\kappa} \Gamma\left(\kappa, -\frac{(\ln r)(1 - \overline{u}z)}{(1 - \overline{u})(1 - z)}\right)$$

This gives the kernel functions for the invariant subspaces H_c in $H^2(\beta_\kappa)$, including for the usual Bergman space ($\kappa = 2$). This result generalizes the formula for the usual Bergman space computed in a different way by W. Yang in his thesis.

http://www.math.iupui.edu/˜ccowen/Downloads.html