

Unitary Equivalence of One-parameter Groups of Toeplitz and Composition Operators

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Joint work with Eva Gallardo Gutiérrez, U. Zaragoza, Spain

The Hardy Hilbert space on the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is:

$$H^2 = \left\{ f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2 = \sum |a_n|^2 < \infty \right\}$$

where for f and g in H^2 , we have $\langle f, g \rangle = \sum a_n \bar{b}_n$

and we will consider two types of operators on H^2 :

For ψ an analytic map of \mathbb{D} into the complex plane,

the *analytic Toeplitz operator* T_ψ is

$$(T_\psi f)(z) = \psi(z)f(z) \quad \text{for } f \text{ in } H^2$$

and, for φ an analytic map of \mathbb{D} into itself,

the *composition operator* C_φ is

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{for } f \text{ in } H^2$$

For example, if φ is defined by

$$\varphi(z) = \frac{3z + 1}{z + 3}$$

then φ is an automorphism of the disk \mathbb{D} with $\varphi(\pm 1) = \pm 1$ and $\varphi'(1) = \frac{1}{2}$

The spectrum of C_φ is the annulus

$$\sigma(C_\varphi) = \{\lambda : \frac{1}{\sqrt{2}} \leq |\lambda| \leq \sqrt{2}\}$$

and each λ with

$$\frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}$$

is an eigenvalue of infinite multiplicity for C_φ .

Similarly, if ψ is defined by

$$\psi(z) = \left(\frac{1-z}{1+z} \right)^{(i \log 2)/\pi}$$

then ψ is the covering map of the disk \mathbb{D} onto the annulus

$$\psi(\mathbb{D}) = \left\{ \lambda : \frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2} \right\}$$

The spectrum of the Toeplitz operator $T_{\psi} = T_{\psi}^*$ is the annulus

$$\psi(\mathbb{D}) = \left\{ \lambda : \frac{1}{\sqrt{2}} \leq |\lambda| \leq \sqrt{2} \right\}$$

and each λ with

$$\frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}$$

is an eigenvalue of infinite multiplicity for T_{ψ}^* .

Both of these operators are part of one-parameter groups of operators:

$$\varphi_t(z) = \frac{(1 + e^{-t})z + (1 - e^{-t})}{(1 - e^{-t})z + (1 + e^{-t})}$$

with $C_{\varphi_s}C_{\varphi_t} = C_{\varphi_s \circ \varphi_t} = C_{\varphi_{s+t}}$ for $-\infty < s, t < \infty$

and

$$\psi_t(z) = \left(\frac{1 - z}{1 + z} \right)^{it/\pi}$$

with $T_{\psi_s}^* T_{\psi_t}^* = T_{\psi_s \psi_t}^* = T_{\psi_{s+t}}^*$ for $-\infty < s, t < \infty$

and each λ with

$$e^{-t/2} < |\lambda| < e^{t/2}$$

is an eigenvalue of infinite multiplicity for each of C_{φ_t} and $T_{\psi_t}^*$.

IDEA!!

If there were a connection (e.g. similarity or unitary equivalence)

between these operators,

then the eigenvectors for each of these operators should correspond

to the eigenvectors for the same eigenvalue for the other operator!

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TRY Infinitesimal generators!

The infinitesimal generator of a (semi)group A_t of operators
is an operator G such that for each f (in the domain of G)

$$Gf = \left. \frac{d}{dt} \right|_{t=0} A_t f$$

and analogous to the ideas from solution of first order linear elementary differential equations, we imagine that

$$A_t \text{ “=” } e^{tG}$$

The infinitesimal generator of the group of composition operators is

$$\begin{aligned}
 \left(\frac{d}{dt} \Big|_{t=0} C_{\varphi_t} f \right) (z) &= \frac{d}{dt} \Big|_{t=0} f(\varphi_t(z)) \\
 &= f'(\varphi_t(z)) \frac{2(1 + e^{-t})e^{-t}(1 - z^2)}{[(1 - e^{-t})z + (1 + e^{-t})]^2} \Big|_{t=0} \\
 &= f'(z) \frac{1 - z^2}{2}
 \end{aligned}$$

so G is the differential operator

$$(Gf)(z) = \frac{1}{2}(1 - z^2)f'(z)$$

A similar calculation gives the infinitesimal generator, H , of the group $T_{\psi_t}^*$.

Eigenvectors for the same eigenvalues of G and H should also be connected!

GOOD NEWS!!

The corresponding eigenspaces are 1-dimensional!

Let's try to match them up!

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For $-1/2 < \operatorname{Re} \lambda < 1/2$, the eigenvectors of G and H are multiples of

$$w_\lambda = \left(\frac{1-z}{1+z} \right)^{-\lambda} \quad \text{and} \quad v_\lambda = \left(1 - \frac{-i \sin(\lambda \frac{\pi}{2})}{\cos(\lambda \frac{\pi}{2})} z \right)^{-1}$$

and, for each G and H ,

the eigenvectors corresponding to $-1/2 < \lambda < 1/2$ have dense span in H^2

If G and H are to correspond to each other,

for $-1/2 < \lambda, \mu < 1/2$,

the relationship between w_λ and w_μ should be analogous to

the relationship between v_λ and v_μ .

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BAD NEWS!!

Nasty computations give:

$$2\langle v_\lambda, v_\mu \rangle = \langle w_\lambda, w_\mu \rangle + 1$$

Not a good correspondence!

Let $[1]$ denote subspace of H^2 spanned by constants so that $H^2 = [1] \oplus zH^2$.

For each t , the subspace zH^2 is an invariant subspace for T_{ψ_t} and for $C_{\varphi_t}^*$.

Letting $x_\lambda = w_\lambda - 1$ and $u_\lambda = v_\lambda - 1$, each in zH^2 , work above means that

x_λ and u_λ are each eigenvectors of the compressions of C_{φ_t} and $T_{\psi_t}^*$ to zH^2

and they are eigenvectors of the compressions of G and H to zH^2

that correspond to the same eigenvalues.

Let $[1]$ denote subspace of H^2 spanned by constants so that $H^2 = [1] \oplus zH^2$.

zH^2 is an invariant subspace for T_{ψ_t} and for $C_{\varphi_t}^*$ for each t .

Letting $x_\lambda = w_\lambda - 1$ and $u_\lambda = v_\lambda - 1$, each in zH^2 , this means that

x_λ and u_λ are each eigenvectors of the compressions of C_{φ_t} and $T_{\psi_t}^*$ to zH^2

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that correspond to the same eigenvalues.

A MIRACLE:

$$2\langle u_\lambda, u_\mu \rangle = \langle x_\lambda, x_\mu \rangle$$

Theorem.

(1) *The operator U defined by*

$$U(x_\lambda) = \sqrt{2}u_\lambda$$

can be extended to a unitary operator of zH^2 onto itself.

(2) *For each real number t ,*

$$U C_{\varphi_t}^*|_{zH^2} = T_{\psi_t}|_{zH^2} U$$

so the operators $C_{\varphi_t}^|_{zH^2}$ and $T_{\psi_t}|_{zH^2}$ are unitarily equivalent.*

That is, there is a unitary operator on zH^2 that shows the restrictions of

$C_{\varphi_t}^*$ and T_{ψ_t} to zH^2 are unitarily equivalent for each t .

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Slides posted on webpage:

www.math.iupui.edu/~ccowen/ICMS10.html