Unitary Equivalence of One-parameter Groups of Toeplitz and Composition Operators

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AMS Richmond, November 7, 2010

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Joint work with Eva Gallardo Gutiérrez, U. Zaragoza, Spain

The Hardy Hilbert space on the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is:

$$H^2 = \{f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \ \text{ with } \ \|f\|^2 = \sum |a_n|^2 < \infty\}$$

where for f and g in H^2, we have $\langle f, g\rangle = \sum a_n \overline{b_n}$

and we will consider two types of operators on H^2:

For ψ an analytic map of $\mathbb{D}$ into the complex plane,

the *analytic Toeplitz operator* T_ψ is

$$(T_\psi f)(z) = \psi(z) f(z) \quad \text{for } f \text{ in } H^2$$

and, for φ an analytic map of $\mathbb{D}$ into itself,

the *composition operator* C_φ is

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{for } f \text{ in } H^2$$

For example, if φ is defined by

$$\varphi(z) = \frac{3z+1}{z+3}$$

then φ is an automorphism of the disk $\mathbb{D}$ with $\varphi(\pm 1) = \pm 1$ and $\varphi'(1) = \frac{1}{2}$

The spectrum of C_φ is the annulus

$$\sigma(C_\varphi) = \{\lambda : \frac{1}{\sqrt{2}} \leq |\lambda| \leq \sqrt{2}\}$$

and each λ with

$$\frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}$$

is an eigenvalue of infinite multiplicity for C_φ.

Similarly, if ψ is defined by

$$\psi(z) = \left(\frac{1-z}{1+z}\right)^{(i\log 2)/\pi}$$

then ψ is the covering map of the disk $\mathbb{D}$ onto the annulus

$$\psi(\mathbb{D}) = \{\lambda : \frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}\}$$

The spectrum of the Toeplitz operator $T_{\overline{\psi}} = T_\psi^*$ is the annulus

$$\psi(\mathbb{D}) = \{\lambda : \frac{1}{\sqrt{2}} \leq |\lambda| \leq \sqrt{2}\}$$

and each λ with

$$\frac{1}{\sqrt{2}} < |\lambda| < \sqrt{2}$$

is an eigenvalue of infinite multiplicity for T_ψ^*.

Both of these operators are part of one-parameter groups of operators:

$$\varphi_t(z) = \frac{(1+e^{-t})z + (1-e^{-t})}{(1-e^{-t})z + (1+e^{-t})}$$

with $C_{\varphi_s}C_{\varphi_t} = C_{\varphi_s \circ \varphi_t} = C_{\varphi_{s+t}} \quad \text{for } -\infty < s, t < \infty$

and

$$\psi_t(z) = \left(\frac{1-z}{1+z}\right)^{it/\pi}$$

with $T^*_{\psi_s}T^*_{\psi_t} = T^*_{\psi_s\psi_t} = T^*_{\psi_{s+t}} \quad \text{for } -\infty < s, t < \infty$

and each λ with

$$e^{-t/2} < |\lambda| < e^{t/2}$$

is an eigenvalue of infinite multiplicity for each of C_{φ_t} and $T^*_{\psi_t}$.

IDEA!!

If there were a connection (e.g. similarity or unitary equivalence)

between these operators,

then the eigenvectors for each of these operators should correspond

to the eigenvectors for the same eigenvalue for the other operator!

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BUT there are too many eigenvectors!

TRY infinitesimal generators!

The infinitesimal generator of a (semi)group A_t of operators

is an operator G such that for each f (in the domain of G)

$$Gf = \left.\frac{d}{dt}\right|_{t=0} A_t f$$

and analogous to the ideas from solution of first order linear elementary differential equations, we imagine that

$$A_t \text{ ``=''} \; e^{tG}$$

The infinitesimal generator of the group of composition operators is

$$\begin{aligned}\left(\left.\frac{d}{dt}\right|_{t=0} C_{\varphi_t} f\right)(z) &= \left.\frac{d}{dt}\right|_{t=0} f(\varphi_t(z)) \\ &= f'(\varphi_t(z)) \left.\frac{2e^{-t}(1-z^2)}{[(1-e^{-t})z + (1+e^{-t})]^2}\right|_{t=0} \\ &= f'(z)\frac{1-z^2}{2}\end{aligned}$$

so G is the differential operator

$$(Gf)(z) = \frac{1}{2}(1-z^2)f'(z)$$

A similar calculation gives the infinitesimal generator, H, of the group $T^*_{\psi_t}$.

Eigenvectors for the same eigenvalues of G and H should also be connected!

GOOD NEWS!!

The corresponding eigenspaces are 1-dimensional!

Let's try to match them up!

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Let's try to match them up!

For $-1/2 < \operatorname{Re}\lambda < 1/2$, the eigenvectors of G and H are multiples of

$$w_\lambda = \left(\frac{1-z}{1+z}\right)^{-\lambda} \quad \text{and} \quad v_\lambda = \left(1 - \frac{-i\sin\left(\lambda\frac{\pi}{2}\right)}{\cos\left(\lambda\frac{\pi}{2}\right)} z\right)^{-1}$$

and, for each G and H,

the eigenvectors corresponding to $-1/2 < \lambda < 1/2$ have dense span in H^2

If G and H are to correspond to each other,

for $-1/2 < \lambda,\ \mu < 1/2$,

the relationship between w_λ and w_μ should be analogous to

the relationship between v_λ and v_μ.

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BAD NEWS!!

Nasty computations give:

$$2\langle v_\lambda, v_\mu \rangle = \langle w_\lambda, w_\mu \rangle + 1$$

Not a good correspondence!

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Not a good correspondence!

AND it can't be fixed by multiplying the vectors by a constant!

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When does a unitary operator take a pair of one dimensional subspaces onto another (specific) pair??

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Lemma.

Let w_λ, w_μ, v_λ, and v_μ be non-zero vectors in $\mathcal{H}$, and let $M_\lambda = \text{span}\{w_\lambda\}$, $M_\mu = \text{span}\{w_\mu\}$, $N_\lambda = \text{span}\{v_\lambda\}$, and $N_\mu = \text{span}\{v_\mu\}$. There is a unitary operator U on $\mathcal{H}$ such that $UM_\lambda = N_\lambda$ and $UM_\mu = N_\mu$ if and only if

$$\frac{|\langle w_\lambda, w_\mu\rangle|}{\|w_\lambda\|\|w_\mu\|} = \frac{|\langle v_\lambda, v_\mu\rangle|}{\|v_\lambda\|\|v_\mu\|}$$

Lemma.

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Theorem.

There is no unitary operator U on H^2 such that $U^* C_{\varphi_t} U = T^*_{\psi_t}$ for every real number t.

Proof:

Such a unitary U would take eigenspaces M_λ and M_μ onto the eigenspaces N_λ and N_μ, but for $\lambda = 0$ and $\mu = 1/4$, the angles don't match.

If G and H are to correspond to each other,

for $-1/2 < \lambda,\ \mu < 1/2$,

the relationship between w_λ and w_μ should be analogous to

the relationship between v_λ and v_μ.

IDEA!!

Could the relationship

$$2\langle v_\lambda, v_\mu \rangle = \langle w_\lambda, w_\mu \rangle + 1$$

come from invariant subspaces??

Lemma.

For D bounded operator on Hilbert space $\mathcal{H}$ and M an invariant subspace, then $M^{\perp}$ is an invariant subspace for D^*.

$$D = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \qquad D^* = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix}$$

Lemma.

For D bounded operator on Hilbert space $\mathcal{H}$ and M an invariant subspace,

then $M^\perp$ is an invariant subspace for D^*.

Furthermore, if r is an eigenvector for D with eigenvalue λ and $r = p + q$

where p is in M and q is in $M^\perp$,

then either $q = 0$ or q is an eigenvector for the eigenvalue λ for the

compression of D to $M^\perp$, which is the adjoint of the restriction

of D^* to its invariant subspace $M^\perp$.

$$\begin{pmatrix} \lambda p \\ \lambda q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix} = \lambda r = Dr = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} Ap + Bq \\ Cq \end{pmatrix}$$

$$\text{so } \lambda q = Cq$$

Let [1] denote subspace of H^2 spanned by constants so that $H^2 = [1] \oplus zH^2$.

For each t, the subspace zH^2 is an invariant subspace for T_{ψ_t} and for $C^*_{\varphi_t}$.

Letting $x_\lambda = w_\lambda - 1$ and $u_\lambda = v_\lambda - 1$, each in zH^2, work above means that

x_λ and u_λ are each eigenvectors of the compressions of C_{φ_t} and $T^*_{\psi_t}$ to zH^2

and they are eigenvectors of the compressions of G and H to zH^2

that correspond to the same eigenvalues.

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zH^2 is an invariant subspace for T_{ψ_t} and for $C^*_{\varphi_t}$ for each t.

Letting $x_\lambda = w_\lambda - 1$ and $u_\lambda = v_\lambda - 1$, each in zH^2, this means that

x_λ and u_λ are each eigenvectors of the compressions of C_{φ_t} and $T^*_{\psi_t}$ to zH^2

and they are eigenvectors of the compressions of G and H to zH^2

that correspond to the same eigenvalues.

A MIRACLE:

$$2\langle u_\lambda, u_\mu\rangle = \langle x_\lambda, x_\mu\rangle$$

Theorem.

(1) The operator U defined by

$$U(x_\lambda) = \sqrt{2}u_\lambda$$

can be extended to a unitary operator of zH^2 onto itself.

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can be extended to a unitary operator of zH^2 onto itself.

(2) For each real number t,

$$U\, C_{\varphi_t}^*\big|_{zH^2} = T_{\psi_t}\big|_{zH^2}\, U$$

so the operators $C_{\varphi_t}^*\big|_{zH^2}$ and $T_{\psi_t}\big|_{zH^2}$ are unitarily equivalent.

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so the operators $C^*_{\varphi_t}\big|_{zH^2}$ and $T_{\psi_t}\big|_{zH^2}$ are unitarily equivalent.

That is, there is a unitary operator on zH^2 that shows the restrictions of $C^*_{\varphi_t}$ and T_{ψ_t} to zH^2 are unitarily equivalent for each t.

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Slides posted on webpage:

`www.math.iupui.edu/˜ccowen/Richmond1011.pdf`

Paper posted on webpage:

`www.math.iupui.edu/˜ccowen/Downloads.html`