Some Old Thoughts about Commutants

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(Indiana University Purdue University Indianapolis)

Special Session on Complex Analysis and Operator Theory

Southeastern Section AMS, University of South Florida, March 9, 2012

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continuing, joint work with Rebecca Wahl.

In this talk $\mathcal{H}$ will denote a Hilbert space of analytic functions on $\mathbb{D}$,

Usual spaces: f analytic in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy:} \quad H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman:} \quad A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Bergman } (\alpha > 0): \quad A^2_\alpha = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \beta_n > 0): \quad H^2(\beta) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty\}$$

Recall: For w in $\mathbb{D}$, the *reproducing kernel function for* $\mathcal{H}$ is K_w in $\mathcal{H}$ with

$$\langle f, K_w \rangle = f(w) \quad \text{for all} \quad f \in \mathcal{H}$$

For H^2, we have $K_w(z) = (1 - \overline{w}z)^{-1}$

For A^2, we have $K_w(z) = (1 - \overline{w}z)^{-2}$

In this talk, we will consider spaces $H^2(\beta_\kappa)$ for $\kappa \geq 1$ which are the weighted Hardy spaces with

$$K_w(z) = (1 - \overline{w}z)^{-\kappa}$$

The spaces $H^2(\beta_\kappa)$ include the usual Hardy and Bergman spaces and all the weighted Bergman spaces ($\alpha = \kappa + 2$).

Definition:

If φ is a bounded analytic function on the unit disk, the operator T_φ defined by $(T_\varphi f)(z) = \varphi(z)f(z)$ is called the *multiplication operator* or the *analytic Toeplitz operator* with symbol φ.

For spaces today, the Toeplitz operator is bounded and $\|T_\varphi\| = \|\varphi\|_\infty$

Beurling's Theorem (1949):

Let T_z be the operator of multiplication by z on $H^2(\mathbb{D})$. A closed subspace M of $H^2(\mathbb{D})$ is invariant for T_z if and only if there is an inner function ψ such that $M = \psi H^2(\mathbb{D})$.

This result is indicative of the interest in the operator T_z of multiplication by z on $H^2(\mathbb{D})$ and in analytic Toeplitz operators T_φ on Hilbert spaces of analytic functions more generally.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set

$\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$

For example, for T_z on H^2,

$$\{T_z\}' = \{T_\varphi : \varphi \in H^\infty\}$$

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By the 1970's, there was interest in the more general question,

For φ in H^∞ and T_φ an operator on H^2, what is $\{T_\varphi\}'$?

or more specifically,

For B a finite Blaschke product and T_B an operator on H^2, what is $\{T_B\}'$?

Deddens & Wong's 1973 paper used the fact that for B a finite Blaschke product, the operator T_B acting on H^2 is a pure isometry to show that

The operator S in $\mathcal{B}(H^2)$ is in $\{T_B\}'$ if and only if

S can be represented as a lower triangular block Toeplitz matrix with respect to the description of H^2 as $\bigoplus_{k=0}^{\infty} B^k\mathcal{W}$ where $\mathcal{W}$ is the wandering subspace $\mathcal{W} = (BH^2)^\perp$, that is,

$$S = \begin{pmatrix} A_0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ A_3 & A_2 & A_1 & A_0 & \cdots \\ & \vdots & & & \ddots \end{pmatrix}$$

Shortly thereafter, Thomson's papers and Cowen's papers computed $\{T_B\}'$ from a different perspective:

Fundamental Lemma:

For S a bounded operator on H^2 and φ in H^∞, equivalent are

- *S commutes with T_φ*
- *For all α in $\mathbb{D}$, $S^*K_\alpha \perp (\varphi - \varphi(\alpha))H^2$*

Proof: (Main calculation)

For α in $\mathbb{D}$, φ in H^∞, and $ST_\varphi = T_\varphi S$, if f is in H^2,

$$\begin{aligned} \langle (\varphi - \varphi(\alpha))f, S^*K_\alpha\rangle &= \langle ST_\varphi f, K_\alpha\rangle - \varphi(\alpha)\langle Sf, K_\alpha\rangle \\ &= \langle T_\varphi Sf, K_\alpha\rangle - \varphi(\alpha)\langle Sf, K_\alpha\rangle \\ &= \varphi(\alpha)(Sf)(\alpha) - \varphi(\alpha)(Sf)(\alpha) = 0 \end{aligned}$$

For B a finite Blaschke product of order n, except for $n(n-1)$ points of the disk for which $B(\alpha)=B(\beta)$ and $B'(\beta)=0$,

$$\left(\left((B-B(\alpha))\,H^2\right)^{\perp} = \text{ span } \{K_{\beta_1}, K_{\beta_2}, \cdots, K_{\beta_n}\}$$

where the points $\alpha=\beta_1,\ \beta_2,\ \cdots,\ \beta_n$ are the n distinct points of $\mathbb{D}$ for which $B(\beta_j)=B(\alpha)$.

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Observation:

For the study of commutants of Toeplitz operators, it is more important that a Blaschke product B is an n-to-1 map of $\mathbb{D}$ onto itself than the fact that T_B is a pure isometry on H^2.

Of course, the points $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ depend on α, so we might write them as $\alpha = \beta_1(\alpha), \beta_2(\alpha), \cdots, \beta_n(\alpha)$.

In fact (!), if B is a finite Blaschke product of order n and α is a point of the disk that is *NOT* one of the $n(n-1)$ points of the disk for which

$B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

the maps $\alpha \mapsto \beta_j(\alpha)$ *are just the* n *branches of the analytic function* $B^{-1} \circ B$ *that is defined and arbitrarily continuable on the disk with the* $n(n-1)$ *exceptional points removed.*

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Theorem: (Cowen, 1974)

For B *a finite Blaschke product, the branches of* $B^{-1} \circ B$ *form a group whose normal subgroups are associated with compositional factorizations of* B *into compositions of two Blaschke products.*

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Theorem: ~~(Cowen, 1974)~~ (Ritt, 1922, '23)

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With this in mind, we can rewrite the ‘Fundamental Lemma’ as

Fundamental Lemma(2):

Let B be a finite Blaschke product. Let F be the set

$F = \{\alpha \in \mathbb{D} : B(\alpha) = B(\beta) \text{ for some } \beta \text{ with } B'(\beta) = 0\}.$

If S is a bounded operator on H^2, then S is in $\{T_B\}'$ if and only if

$$S^* K_\alpha = \sum_{j=1}^{n} c_j(\alpha) K_{\beta_j(\alpha)} \text{ for each } \alpha \text{ in } \mathbb{D} \setminus F.$$

We use this to write Sf as a function of α in the disk.

Let W be the Riemann surface for $B^{-1} \circ B$.

Theorem: (Cowen, 1978). *Let B, F, and W be as above.*

If S is a bounded operator on H^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in H^2,

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{1}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and ψ_1, ψ_2, $\cdots$, ψ_n is a basis for $\left((B - B(\alpha_0))\, H^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \;\; \text{has a zero of order } m \text{ at } \alpha_0 \tag{2}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has properties (2) at each zero of B', then (1) defines a bounded linear operator on H^2 with S in $\{T_B\}'$.

In 2006, Cowen and Gallardo-Gutiérrez, in connection with their study of adjoints of composition operators, developed a formal class of operators called 'multiple-valued weighted composition operators'. The operators S in $\{T_B\}'$ are just such operators.

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In the past few years, Douglas, Sun, and Zheng, and Douglas, Putinar, and Wang, and others have used related tools to study problems concerning commutants of T_B on the Bergman space, such as consideration of the reducing subspaces of T_B.

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Observation:

The class of 'multiple-valued weighted composition operators', an extension of classes of algebras of operators generated by multiplication and composition operators, appear to be useful in the study of certain kinds of problems in operator theory, including questions related to commutants.

Theorem: (Cowen, 1978).

If S is a bounded operator on H^2 such that $ST_B = T_B S$, then for all f in H^∞, Sf is in H^∞, also.

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If S is a bounded operator on H^2 such that $ST_B = T_BS$, then for all f in H^∞, Sf is in H^∞, also.

Theorem: (C. & Wahl, 2012).

If S is a bounded operator on A^2 such that $ST_B = T_BS$, then for all f in H^∞, Sf is in H^∞, also.

Theorem: (C. & Wahl, 2012). *Let B, F, and W be as above.*

If S is a bounded operator on A^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in A^2,

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha)f(\beta(\alpha)) \tag{3}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and ψ_1, ψ_2, $\cdots$, ψ_n is a basis for $\left((B - B(\alpha_0))\, H^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \tag{4}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has properties (4) at each zero of B', then (3) defines a bounded linear operator on A^2 with S in $\{T_B\}'$.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

Corollary:

If P is a bounded operator acting on H^2 such that $P^2 = P$ and $T_BP = PT_B$, then P is a bounded an operator acting on A^2 such that $P^2 = P$ and $T_BP = PT_B$.

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`http://www.math.iupui.edu/~ccowen/Downloads.html`

If $B(z) = z^2 \left(\frac{z - .5}{1 - .5z} \right)^2$, the group $B^{-1} \circ B$ is isomorphic to D_4.

D_4 has several normal subgroups, and most give trivial factorizations of B into the composition of a Blaschke product of order 1 and one of order 4. However, there is a normal subgroup that "finds" the non-trivial decomposition of B as $B = J_1 \circ J_2$ where $J_1(z) = z^2$ and $J_2(z) = z\frac{z - .5}{1 - .5z}$