The Theory of Composition Operators on Analytic Spaces: Where Has It Been and Where Is It Going?

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Let Ω be a domain in $\mathbf{C}$ or $\mathbf{C}^N$ and suppose $\mathcal{H}$ is a Hilbert space or Banach space of analytic functions on Ω.

$$H^p(\mathrm{D}) = \{f \text{ analytic in D} : \sup_{0<r<1} \int_0^{2\pi} |f_r|^p \, \frac{d\theta}{2\pi} < \infty\}$$

$$A^p(\mathrm{D}) = \{f \text{ analytic in D} : \int_D |f(z)|^p \, \frac{dA}{\pi} < \infty\}$$

$$H^p(\mathrm{B}_N) = \{f \text{ analytic in } \mathrm{B}_N : \sup_{0<r<1} \int_{\partial B_N} |f_r|^p \, d\sigma_N < \infty\}$$

$$A^p(\mathrm{B}_N) = \{f \text{ analytic in } \mathrm{B}_N : \int_{B_N} |f(z)|^p \, d\nu_N < \infty\}$$

For weights $\beta(n) > 0$,

$$H^2(\beta, \mathrm{D}) = \{f = \sum_0^\infty a_n z^n : \sum_0^\infty |a_n|^2 \beta(n)^2 < \infty\}$$

If φ is an analytic map of Ω into itself and

$\mathcal{H}$ is a Hilbert or Banach space of analytic functions on Ω,

the *composition operator* C_φ is the operator on $\mathcal{H}$ given by

$$C_\varphi f = f \circ \varphi$$

Goal: relate the function–theoretic properties of φ to the operator–theoretic properties of C_φ.

Theorem. *If φ is an analytic map of the disk into itself, and $1 \leq p < \infty$, then C_φ is bounded on $H^p(\mathbf{D})$ and*

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/p} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/p}$$

- Usually, finding the exact norm of a composition operator is very difficult – and not very interesting.

If $\varphi(z) = sz + t$, with $|t| < 1, |s| + |t| \leq 1$, then on $H^2(\mathrm{D})$,

$$\|C_\varphi\| = \sqrt{\frac{2}{1 + |s|^2 - |t|^2 + \sqrt{(1 - |s|^2 + |t|^2)^2 - 4|t|^2}}}$$

- On medium sized spaces, all composition operators are bounded

- For small spaces, if z is in $\mathcal{H}$, then $C_\varphi z = \varphi$ is in $\mathcal{H}$, and this restricts φ.

- For large spaces, many bad functions are in $\mathcal{H}$, so even composing with a nice function can make things much worse.

If $\beta(n)$ is a weight sequence that decays rapidly, then $H^2(\beta)$ is a large space. If $n^A\beta(n) \to 0$ for all $A > 0$, and $0 < r < 1$, then for the automorphism

$$\varphi(z) = \frac{z+r}{1+rz}$$

C_φ is not bounded on $H^2(\beta)$.

Conjecture (MacCluer and Cowen, 1996):

If $\beta(n)$ is monotone decreasing, then C_φ is bounded on $H^2(\beta)$ for all automorphisms φ of the disk

if and only if there exists a positive integer n so that $(1-z)^{-n}$ is not in $H^2(\beta)$.

The general principle for compactness is C_φ is compact if and only if $\varphi(\Omega)$ is far enough from the boundary of Ω.

For example, if the closure of $\varphi(\mathrm{D})$ is contained in D, then C_φ is compact on $H^2(\mathrm{D})$.

Theorem (Shapiro and Taylor, 1973).

If φ is an analytic map of the disk into itself, and

$$\int_0^{2\pi} \frac{1}{1-|\varphi(e^{i\theta})|}\, d\theta \;<\; \infty$$

then C_φ is compact, indeed Hilbert - Schmidt, on $H^2(D)$.

Corollary.

If φ maps the disk into the interior of a polygon inscribed in the unit circle, then C_φ is compact on $H^2(D)$.

A standard tool for proving boundedness or compactness is Carleson measures. Often, boundedness corresponds to 'big O' and compactness to 'little O' conditions.

The most precise versions of the arguments for compactness use counting function arguments:

Theorem (Shapiro, 1987).

If φ is an analytic map of the disk into itself, then for C_φ on $H^2(D)$,

$$\|C_\varphi\|_e^2 = \lim_{|w|\to 1^-} \frac{N_\varphi(w)}{-\log|w|}$$

where $N_\varphi(w) = -\sum_j \log|z_j|$ for $\varphi(z_j) = w$.

The structure of a composition operator is related to the nature of the fixed points of φ.

We will say b in the closed disk is a fixed point of φ if

$$\lim_{r \to 1^-} \varphi(rb) = b$$

If b is a fixed point of φ in the closed disk, then

$$\lim_{r \to 1^-} \varphi'(rb)$$

exists and we denote it by $\varphi'(b)$.

Theorem (Denjoy, Wolff, 1926).

If φ is an analytic map of the disk into itself, not an automorphism, then there is a unique fixed point a in the closed disk for which $|\varphi'(a)| \leq 1$. Moreover,

$$\lim_{n \to \infty} \varphi_n(z) = a$$

for all z in the open disk, uniformly on compact sets.

The point a of the theorem above will be called the Denjoy-Wolff point of φ.

Model for iteration of analytic functions mapping the unit disk into itself.

Maps of the disk into itself are like linear fractional maps.

Let φ *be an analytic map of the unit disk D into itself, not an automorphism of the disk.*

Suppose that either φ *does not have a fixed point in D or that* $\varphi'(a) \neq 0$ *for the fixed point* a *in D.*

Then there is a domain Δ*, either the plane or a half-plane, an automorphism* Φ *of* Δ *onto* Δ*, and a mapping* σ *of D into* Δ *such that*

$$\sigma \circ \varphi = \Phi \circ \sigma$$

Four distinct cases in the model:

If φ has a fixed point in D:

- *(plane/dilation)* $\Delta = \mathbf{C},\ \Phi(z) = \alpha z$

$$\varphi(z) = \frac{z}{2-z} \qquad a = 0 \qquad \alpha = \varphi'(a) = \frac{1}{2}$$

If φ has no fixed points in D:

- *(half-plane/dilation)* $\Delta = \{\mathrm{Re}\ z > 0\},\ \Phi(z) = \alpha z$

$$\varphi(z) = \frac{1}{3}z + \frac{2}{3} \qquad a = 1 \qquad \alpha = \varphi'(a) = \frac{1}{3}$$

- *(plane/translation)* $\Delta = \mathbf{C},\ \Phi(z) = z + 1$

$$\varphi(z) = \frac{1+z}{3-z} \qquad a = 1 \qquad \varphi'(a) = 1$$

- *(half-plane/translation)*

  $\Delta = \{\mathrm{Im}\ z > 0\},\ \Phi(z) = z \pm 1$

$$\varphi(z) = \frac{(1+i)z - i}{iz + 1 - i} \qquad a = 1 \qquad \varphi'(a) = 1$$

Some applications of the model:

- Better understanding of iteration of the function φ, including questions about embeddability of the discrete semi-group of iterates of φ into a continuous semi-group
- Determination of the functions ψ mapping the disk into the disk that satisfy

$$\psi \circ \varphi = \varphi \circ \psi$$

- Determination of the eigenvectors and eigenvalues of composition operators on spaces of analytic functions on the disk
- Determination of the spectrum of composition operators on spaces of analytic functions on the disk

Spectra of C_φ:

C_φ is invertible if and only if φ is an automorphism

C_φ is compact (or power compact) implies $|a| < 1$ and

$$\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\varphi'(a)^n : n = 1, 2, \cdots\}$$

$|a| < 1$

$|a| = 1$, $\varphi'(a) < 1$

$|a| = 1$, $\varphi'(a) = 1$, half-plane translation

$|a| = 1$, $\varphi'(a) = 1$, plane translation

Explain the circular symmetry of the spectra of C_φ:

(Cowen, 1983)

If φ is a map of the disk into itself with $|a| = 1$ and $\varphi'(a) < 1$, then on $H^2(D)$,

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(a)^{-1/2}\}$$

Moreover, $C_\varphi \approx e^{i\theta}C_\varphi$

There is no circular symmetry in the case φ has the model translation on the plane, but there appears to be circular symmetry in the half-plane translation case, and there is some circular symmetry when the fixed point is in the open disk.

Conjecture (Cowen, 1994):

If φ is a map of the disk into itself, not an automorphism, with $|a| < 1$ and essential spectral radius of C_φ is not zero, then there is an invariant subspace $\mathcal{K}$ for C_φ so that

$$C_\varphi|_{\mathcal{K}} \approx e^{i\theta} C_\varphi|_{\mathcal{K}}$$

for θ real.

Theorem (Wahl, 1998)

If

$$\varphi(z) = \frac{z^2}{2 - z}$$

then there is an invariant subspace $\mathcal{K}$ for C_φ so that

$$C_\varphi|_{\mathcal{K}} \approx e^{i\theta} C_\varphi|_{\mathcal{K}}$$

for θ real.

Some omitted topics:

- Adjoints of C_φ.
- Topology of the set of composition operators.
- Cyclicity, hypercyclicity, etc. of C_φ and C_φ^*.
- Normality, subnormality, hyponormality of C_φ and C_φ^*.
- Similarity and unitary equivalence of C_φ and C_ψ.

Composition operators in several variables

Still many mysteries in several variables... even boundedness is problematic.

Wogen (1988) gave necessary and sufficient conditions for a smooth map to give a bounded operator on $H^2(\mathrm{B}_N)$.

For example,

$$\varphi(z_1, z_2) = (\frac{5}{9} + \frac{5}{9}z_1 - \frac{1}{9}z_1^2 + \frac{1}{6}z_2^2, \frac{1}{5}z_2^2)$$

is a map of $\mathbf{B}_2$ into $\mathbf{B}_2$ that gives an unbounded composition operator on $H^2(\mathbf{B}_2)$.

On the other hand, some things carry over to several variables.

Theorem (MacCluer, 1984)

If C_φ is compact on $H^2(B_N)$, then φ has an attractive fixed point a in B_N.

Moreover, the spectrum of C_φ is

$$\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\textit{all products of eigenvalues of } \varphi'(a)\}$$

If φ is an analytic map of B_N into itself, $\varphi(0) = 0$, and φ is not unitary on a slice, then

$$\sigma(C_\varphi) \supset \{\lambda : |\lambda| \leq \rho\}$$

where ρ is computed in terms of the essential spectral radius of C_φ and a constant depending on the local behavior of φ.

Some broad areas for investigation:

What can you say about the spectrum of C_φ if φ does not have a fixed point in B_N?

What effect do degeneracies of φ have on the structure of C_φ?

For example, if $\varphi(\mathrm{B}_N) \subset \mathrm{B}_N \cap \{(w_1, 0)\}$ and C_φ is bounded, what is the structure of C_φ?

For example, if

$$\varphi(z_1, z_2) = (2z_1z_2, 0) \quad \text{or } \varphi(z_1, z_2) = (z_1^2 + z_2^2, 0)$$

then C_φ is unbounded, but if

$$\varphi(z_1, z_2) = (z_1z_2, 0)$$

then C_φ is compact.

Similarly, what if $\varphi(z_1, z_2) = (z_1\psi(z_1, z_2), z_2)$ which is unitary on the slice $z_1 = 0$,

or what if $\varphi(z_1, z_2) = (\psi_1(z_1), \psi_2(z_2))$?

We need a better understanding of maps of the ball into the ball, for example, it would be useful to have a substitute for the "Model for Iteration" for several variables.

What is a 'nice' class of functions of B_N into itself?

Given φ, can we find a 'nice' map that is 'like' φ?

Some specific questions (one variable):

See also `http://www.math.purdue.edu/~cowen`

- If $|a| = 1$, must $\sigma(C_\varphi)$ be connected?
- Find $\sigma(C_\varphi)$! If $|a| = 1$ and $\varphi'(a) = 1$, we only know special cases.
- When are C_φ and C_φ^* subnormal? hyponormal?
- How can you compute $\|C_\varphi\|$?
- Is there a useful description of C_φ^*?
- When are two composition operators unitarily equivalent? similar?, quasi-similar?
- On which spaces is C_φ bounded for φ an automorphism?
- Which operators commute with C_φ?
- If φ has $|a| < 1$, is there an invariant subspace on which C_φ is similar to rotates of itself?

Some specific questions (several variables):

- Describe boundedness; give necessary conditions, and give sufficient conditions (not necessarily the same).
- Describe compactness.
- Find relationship between degeneracies of φ and the structure of C_φ.
- Find a class of simple maps for which C_φ can be understood and large enough that every map is 'like' one of them.
- Find spectra of C_φ.