The Use of Technology in Teaching the Mathematics of Our Courses

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How can you explain this to your colleagues who don't use technology??

NONSENSE!! We *all* use technology!! We must *choose* which to use.

Since my first years of teaching, my *favorite* technology has been chalk on a slate blackboard!

Introduction of chalk and blackboard technology in 1801 revolutionized teaching!

However, this technology has limitations – and it is about the use of computer technology, as an add-on to the more traditional technology, that I want to speak today.

I use computer technology (usually MATLAB) to do a better job of:

- Responding spontaneously to student questions and suggestions $\cdots$
- Modeling recovery from errors for my students $\cdots$
- Illustrating interplay between ideas w/o computational impediments $\cdots$
- Motivating development of the mathematics and mastery of the ideas $\cdots$

Most of my examples today come from Linear Algebra

I like to teach linear algebra and I have a lot of ideas about how to teach it

and

It is especially amenable to the techniques I want to talk about

But, I also use special technology when the session includes

Students presenting their work to the class (ELMO (opaque projector)

Algorithms for RSA Cryptosystems (ISETL)

Complicated diagrams or movies: modeling biological phenomena $\cdots$

In the old days, $\cdots$

I would carefully prepare calculations and examples for class, being ready with the correct arithmetic for solving the examples correctly

This means, at least as far as calculations go, I was stuck to the script!

Now, if students ask “What if?” questions, we can pursue them easily because we can create examples quickly and the machine can do the calculations effortlessly.

Perhaps more importantly, this spontaneity allows for natural reaction to student errors.

I might ask for suggestions for solution of the problem, but if they offered a wrong suggestion, I would apologize and explain why it would not work.

Now, $\cdots$

we can pursue a suggestion (*any suggestion!*)

are we done? (*Exit!*)

are we stuck?

reflect on what went wrong,

develop a new suggestion,

Repeat!

In addition to errors being a problem,

I believe students' ideas about errors interfere with their learning!

Students bring to our classes ideas about themselves and about us

– as to our expertise in mathematics.

Often, students believe that someone who knows mathematics well (like us) sees how to answer every question immediately and never makes conceptual or strategic errors in solving problems.

In particular, they sometimes infer that they are 'no good' at mathematics because they do not see how to answer every question immediately.

We *know* we do *not* always see the way to the solution of every problem and that we make errors – but we know how to recover from them!

When we use the strategy, $\cdots$

pursue a suggestion

are we done? (*Exit!*)

are we stuck?

reflect on what went wrong,

develop a new suggestion,

We are modeling how to recover from errors!!

And, the technology has made this easier to do!

Use of the computer technology allows me to bring in the theory more directly.

One example is in solution of systems of equations

– the first part of the basic linear algebra course is about this problem.

It is easy to get students to learn elimination, but harder to get students to learn the theorems about solutions of systems.

Consider the system: $\begin{cases} 2a - b + c - e = 4 \\ -2a + 2b - c + 2d + 2e = 4 \\ -a - c + d + e = 3 \\ -a + b - c + 3d + 2e = 3 \end{cases}$

```
>> A = [ 2 -1 1 0 -1; -2  2 -1 2 2; -1 0 -1 1 1; -1 1 -1 3  2 ]

A =

     2    -1     1     0    -1

    -2     2    -1     2     2

    -1     0    -1     1     1

    -1     1    -1     3     2

>> A\[4; 4; 3; 1]

ans = [   0.8750;      0.2500;      0;   1.3750;  0  ]
```

```
>> check = A*ans

check =   1.5000

          1.5000

          0.5000

          3.5000
```

But this is not b!

What went wrong? What can we do about it?

Change the system: $\begin{cases} 2a - b + c - e = -2 \\ -2a + 2b - c + 2d + 2e = 3 \\ -a - c + d + e = 0 \\ -a + b - c + 3d + 2e = 1 \end{cases}$

```
>> A = [ 2 -1 1 0 -1; -2  2 -1 2 2; -1 0 -1 1 1; -1 1 -1 3  2 ]

A =

     2    -1     1     0    -1

    -2     2    -1     2     2

    -1     0    -1     1     1

    -1     1    -1     3     2

>> A\[-2; 3; 0; 1]

ans = [   -0.2500;  1.5000; 0;  -0.2500;  0  ]
```

```
>> check = A*ans

check =  -2.0000
          3.0000
         -0.0000
          1.0000
```

This is c !!!

So $a = -0.25$, $b = 1.5$, $c = 0$, $d = -0.25$, and $e = 0$ is a solution!

What next??

```
>> null(A)

ans =

    0.5936    0.1163
    0.3266   -0.1991
   -0.6533    0.3982
   -0.2669   -0.3154
    0.2072    0.8299
```

So the vectors $v_1 = [0.5936; 0.3266; -0.6533; -0.2669; 0.2072]$

and $v_2 = [0.1163; -0.1991; 0.3982; -0.3154; 0.8299]$

are a basis for the nullspace of A

What does this mean?

So the conclusion is that there are infinitely many solutions, and

$$\begin{aligned}
a &= -.25 + 0.5936s + 0.1163t \\
b &= 1.5 + 0.3266s - 0.1991t \\
c &= -0.6533s + 0.3982t \\
d &= -.25 - 0.2669s - 0.3154t \\
e &= 0.2072s + 0.8299t
\end{aligned}$$

is the general solution.

Exercise: L.A. class assignment: Find the nullspace of the 4×5 matrix B.

John's answer was

$$\mathcal{N}(B) = \text{span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 0 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 4 \\ -3 \\ 11 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 2 \\ -4 \\ 13 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -2 \\ 1 \\ -4 \end{pmatrix} \right\}$$

Mary's answer was

$$\mathcal{N}(B) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 4 \\ -1 \\ 5 \end{pmatrix} \right\}$$

Are their answers consistent with each other? (Explain your answer!)

One strategy is to find out whether each vector in John's set is in Mary's subspace and whether each vector in Mary's set is in John's subspace.

This in not difficult, requiring only the solution of 7 linear systems, or if reformulated as matrix equations, to solve two matrix equations. This is fairly easy to do by machine, but to do that by hand is quite tedious.

Even so, there is a more efficient, although more sophisticated, strategy if you have a tool like MATLAB!

```
>> J1=[-2; -2; 0; 2; -6]

J1 =    -2

        -2

         0

         2

        -6          :

                    :

>> J = [J1 J2 J3 J4]

J =     -2     1     3     0

        -2     5     5    -2

         0     4     2    -2

         2    -3    -4     1

        -6    11    13    -4
```

```
>> M = [M1 M2 M3]

M =

     1    -2    -1

     1     0     3

     0     2     4

    -1     1    -1

     3    -2     5

>> J\M1

ans =

           0

     -0.2000

      0.4000

           0
```

```
>> J*ans

ans =    1.0000

         1.0000

         0.0000

        -1.0000

         3.0000

>> M1

M1 =

     1

     1

     0

    -1

     3
```

```
>> J=[-2 1 3 0; -2 5 5 -2; 0 4 2 -2; 2 -3 -4 1; -6 11 13 -4]

J =

    -2     1     3     0
    -2     5     5    -2
     0     4     2    -2
     2    -3    -4     1
    -6    11    13    -4


>> rank(J)

              ans = 2
```

Thus, the space John has identified is a 2-dimensional subspace of $\mathbb{R}^5$.

```
>> M = [1 -2 -1; 1 0 3; 0 2 4; -1 1 -1; 3 -2 5]

M =

     1    -2    -1

     1     0     3

     0     2     4

    -1     1    -1

     3    -2     5

>> rank(M)

                ans =  2
```

The space Mary has identified is also a 2-dimensional subspace of $\mathbb{R}^5$!

```
>> rank([J M])

ans =    2
```

Finally, I use the technology to motivate the mathematics and to prepare the students for upcoming developments.

In the syllabus, I announce that one of the goals of the course is to enable the students to read the documentation for linear algebra software packages.

I do not usually have to tell the students about the commands `null` or `orth` in MATLAB: they come to expect that MATLAB has commands for everything, and someone discovers `null` and spreads the word before I tell them about it!

On the other hand, they don't like the answers!

For example, if C is the matrix

$$\begin{pmatrix} -2 & 1 & 3 & 0 \\ -2 & 5 & 5 & -2 \\ 0 & 4 & 2 & -2 \\ 2 & -3 & -4 & 1 \end{pmatrix}$$

By hand, they might find a basis for $\mathcal{N}(C)$ to be $u = (3, 0, 2, 2)$ and $v = (5, -2, 4, 0)$, a *nice* answer.

But using the machine,

```
>> null(C)


ans =

    0.7558    0.1482
   -0.2703    0.4637
    0.5940   -0.0558
    0.0533    0.8717
```

an *ugly* answer! Why does MATLAB always give *ugly* answers??

Matlab always finds an orthonormal basis... and this motivates inner products and Gram-Schmidt; it makes the students want to learn those things so that they understand the machine's answer.

To summarize, I believe the technology helps me teach more effectively:

- Responding spontaneously to student questions and suggestions ⋯
- Modeling recovery from errors for my students ⋯
- Illustrating interplay between ideas w/o computational impediments ⋯
- Motivating development of the mathematics and mastery of the ideas ⋯

`http://www.math.iupui.edu/~ccowen/Downloads.html`