Invariant Subspaces for Composition Operators

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If φ is analytic map of $\mathbb{D}$ into itself,

and $\mathcal{H}$ is Hilbert space of analytic functions on $\mathbb{D}$,

then composition operator C_φ on $\mathcal{H}$ is operator

$$C_\varphi f = f \circ \varphi \quad \text{for} \quad f \in \mathcal{H}$$

Usual spaces: f analytic in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy:} \quad H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman:} \quad A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Bergman } (\alpha > 0)\text{:} \quad \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \beta_n > 0)\text{:} \quad H^2(\beta) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{\beta_n^2} < \infty\}$$

Recall: for w in $\mathbb{D}$, the *reproducing kernel function for* $\mathcal{H}$ is K_w in $\mathcal{H}$ with

$$\langle f, K_w \rangle = f(w) \quad \text{for all} \quad f \in \mathcal{H}$$

For H^2, we have $K_w(z) = (1 - \overline{w}z)^{-1}$

For A^2, we have $K_w(z) = (1 - \overline{w}z)^{-2}$

In this talk, we will consider spaces $H^2(\beta_\kappa)$ for $\kappa \geq 1$ which are the weighted Hardy spaces with

$$K_w(z) = (1 - \overline{w}z)^{-\kappa}$$

Spaces $H^2(\beta_\kappa)$ include the usual Hardy and Bergman spaces and all of the weighted Bergman spaces $(\alpha = \kappa + 2)$.

On all of these spaces, for any φ analytic map of $\mathbb{D}$ into itself, the composition operator C_φ is a bounded operator and for all α in $\mathbb{D}$

$$C_\varphi^* K_w = K_{\varphi(w)}$$

For A a bounded operator on $\mathcal{H}$, a (closed) subspace M will be called a (non-trivial) *invariant subspace of* A if $M \neq 0$ and $M \neq \mathcal{H}$ and

$$v \in M \quad \text{implies} \quad Av \in M$$

In finite dimensional spaces, every operator has invariant subspaces,

and understanding the structure of the invariant subspaces

has been critical in understanding the structure of the operators.

Want the same for operators on infinite dimensional spaces!

Invariant Subspace Problem:

Does every bounded operator have a (non-trivial) invariant subspace?

No! in general, for Banach spaces! (C.J. Read and others 1984–. . .)

Still open for Hilbert space!

But,

for Hilbert space operator for which lattice of invariant subspaces is known,

we feel we have a basic understanding of the structure of the operator!!

Goal today:

Outline three sets of ideas about invariant subspaces of composition ops

and thereby persuade you that now is good time to think about the topic!

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A Digression!

One of the first times that invariant subspaces were mentioned in connection with composition operators was in a paper [NRW] of Nordgren, Rosenthal, and Wintrobe: "Invertible Composition Operators on H^p"

J. Func. Anal. **73**(1987), 324–344.

Definition:

An operator U is called *universal* if for every operator T, some multiple of T is similar to the restriction of U to one of its invariant subspaces.

Caradus (1969) showed that

an operator is universal if it is onto and has an infinite dimensional kernel

and

[NRW] showed that for $\varphi(z) = \dfrac{2z-1}{2-z}$ the operator $C_\varphi - I$ is universal.

End of Digression!

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for Hilbert space operator for which lattice of invariant subspaces is known,

we feel we have a basic understanding of the structure of the operator!!

Beurling's Theorem (1949)

Let S be the operator of multiplication by z on $H^2(\mathbb{D})$. A closed subspace M of $H^2(\mathbb{D})$ is invariant for S if and only if there is an inner function ψ such that $M = \psi H^2(\mathbb{D})$.

First Example: A complete lattice!

Theorem (Montes-Rodríguez, Ponce-Escudero, & Shkarin, 2010)

For $\operatorname{Re} a > 0$, let

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}$$

A closed subspace M of $H^2(\mathbb{D})$ is invariant for C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

$$M = \text{ closed span}\{e^{t\frac{z+1}{z-1}} : t \in F\}$$

The relevance of the functions $e^{t\frac{z+1}{z-1}}$ is that they are eigenvectors for C_{φ_a}:

$$C_{\varphi_a}\left(e^{t\frac{z+1}{z-1}}\right) = e^{-at}e^{t\frac{z+1}{z-1}}$$

In other words, each of the invariant subspaces for C_{φ_a} is the closed span of a collection of eigenvectors.

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A closed subspace M of $H^2(\mathbb{D})$ is invariant for C_{φ_a} if and only if there is a closed set F of $[0, \infty)$ such that

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Corollary

If $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$,

then the lattice of invariant subspaces of C_{φ_a} and C_{φ_b} are the same.

Corollary

If $\operatorname{Re} a > 0$, then C_{φ_a} has no non-trivial reducing subspaces.

Their proof is based on two quite different ideas.

First, suppose $\mathcal{A}$ is a Banach algebra. If τ is in $\mathcal{A}$, we say τ is cyclic if the algebra generated by τ is dense in $\mathcal{A}$. Let M_τ be the operator on $\mathcal{A}$ of multiplication by τ, that is, $M_\tau\omega = \tau\omega$ for ω in $\mathcal{A}$.

Proposition

If τ be a cyclic element in the Banach algebra $\mathcal{A}$,

then the invariant subspaces of M_τ are the closed ideals of $\mathcal{A}$.

Let $W^{1,2}[0,\infty)$ be the Sobolev space with inner product

$$\langle f, g\rangle_{1,2} = \frac{1}{2}\int_0^\infty f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\,dt$$

where f and g functions in $L^2[0,\infty)$ that are absolutely continuous on each bounded subinterval of $[0,\infty)$ and whose derivatives are in $L^2[0,\infty)$.

Second, they give a unitary equivalence between $H^2(\mathbb{D})$ and $W^{1,2}[0,\infty)$ and they show that $W^{1,2}[0,\infty)$ is a Banach algebra.

Then they show that the unitary equivalence of the spaces carries adjoints of the composition operators to operators of multiplication by cyclic elements of the Banach algebra to which they can apply the Proposition.

■

Second Example:

Invariant subspaces with application to function theory

Let φ be an analytic map of $\mathbb{D}$ into itself and ψ be analytic on $\mathbb{D}$.

Weighted composition operator $W_{\psi,\varphi}$ is the operator on $H^2(\beta_\kappa)$ given by

$$(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$$

for z in $\mathbb{D}$.

Since $H^2(\beta_\kappa)$ contains the constants,

if $W_{\psi,\varphi}$ is bounded, then $\psi = W_{\psi,\varphi}(1)$ is in $H^2(\beta_\kappa)$.

Clearly, if ψ is in $H^\infty(\mathbb{D})$, then $W_{\psi,\varphi}$ is bounded on $H^2(\beta_\kappa)$ and

$$\|W_{\psi,\varphi}\| \leq \|\psi\|_\infty \|C_\varphi\|$$

BUT, it is not necessary for ψ to be bounded for $W_{\psi,\varphi}$ to be bounded.

Theorem. (Ko & C. for $H^2(\mathbb{D})$; Gunatillake, Ko, & C. for $H^2(\beta_\kappa)$)

For $\kappa \geq 1$,

$W_{\psi,\varphi}$ is a bounded Hermitian weighted composition operator on $H^2(\beta_\kappa)$,

if and only if

$$\psi(z) = c(1 - \overline{a_0}z)^{-\kappa} \qquad \textit{and} \qquad \varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a_0}z}$$

where $c = \psi(0)$ and $a_1 = \varphi'(0)$ are real numbers

and a_1 and $a_0 = \varphi(0)$ are such that φ maps the unit disk into itself.

Without loss of generality, $0 < a_0 < 1$, and then the most interesting case occurs when $a_1 = (1 - a_0)^2$ which means $\varphi(1) = \varphi'(1) = 1$.

Writing $t = a_0/(1 - a_0)$, each such $W_{\psi,\varphi}$ is a multiple of $A_t = W_{\psi_t,\varphi_t}$ where

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \text{and} \quad \varphi_t = (t + (1 - t)z)/(1 + t - tz)$$

Theorem.

For $\kappa \geq 1$ *and* $0 \leq t < \infty$, *let* $A_t = W_{\psi_t,\varphi_t}$ *where*

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \textit{and} \quad \varphi_t = (t + (1-t)z)/(1 + t - tz)$$

The A_t *form a strongly continuous semigroup of Hermitian weighted composition operators on* $H^2(\beta_\kappa)$. *If* Δ *is the infinitesimal generator of this semigroup,* $\mathcal{D}_A = \{f \in H^2(\beta_\kappa) : (z-1)^2 f' \in H^2(\beta_\kappa)\}$ *is the domain of* Δ *and* $\Delta(f)(z) = (z-1)^2 f'(z) + \kappa(z-1)f(z)$ *for* f *in* $\mathcal{D}_A$.

Theorem.

For $\kappa \geq 1$ and $0 \leq t < \infty$, let $A_t = W_{\psi_t,\varphi_t}$ where

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The A_t form a strongly continuous semigroup of Hermitian weighted composition operators on $H^2(\beta_\kappa)$. If Δ is the infinitesimal generator of this semigroup, $\mathcal{D}_A = \{f \in H^2(\beta_\kappa) : (z-1)^2 f' \in H^2(\beta_\kappa)\}$ is the domain of Δ and $\quad \Delta(f)(z) = (z-1)^2 f'(z) + \kappa(z-1)f(z) \quad$ for f in $\mathcal{D}_A$.

Corollary.

For $\kappa \geq 1$ and for $t > 0$, the operator A_t on $H^2(\beta_\kappa)$ has no eigenvalues.

Proof: There are no non-zero functions in $H^2(\beta_\kappa)$ that satisfy

$$(z-1)^2 f' + \kappa(z-1)f = \lambda f(z)$$

Theorem.

For $\kappa \geq 1$ *and* $0 \leq t < \infty$, *let* $A_t = W_{\psi_t,\varphi_t}$ *where*

$$\psi_t = (1 + t - tz)^{-\kappa} \quad \textit{and} \quad \varphi_t = (t + (1-t)z)/(1 + t - tz)$$

For each t, *the operator* A_t *is a cyclic Hermitian weighted composition operator on* $H^2(\beta_\kappa)$. *Indeed, the vector* 1 *is a cyclic vector for* A_t.

If μ *is the absolutely continuous probability measure given by*

$$d\mu = \frac{(\ln(1/x))^{\kappa-1}}{\Gamma(\kappa)}\,dx$$

the operator U *given by* $U(\psi_t) = x^t$ *for* $0 \leq t < \infty$, *is a unitary map of* $H^2(\beta_\kappa)$ *onto* $L^2([0,1],\mu)$ *and satisfies* $UA_t = M_{x^t}U$.

In particular, for each $t > 0$, *these operators satisfy* $\|A_t\| = 1$ *and have spectrum* $\sigma(A_t) = [0,1]$.

We define subspaces H_c of $H^2(\beta_\kappa) = A^2_{\kappa-2}$ as follows:

Let $H_0 = H^2(\beta_\kappa)$. For $c < 0$, define the subspace H_c by

$$H_c = \text{ closure } \{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\}$$

For $0 \leq t$ and $c \leq 0$, the subspace H_c is invariant for A_t.

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For $0 \leq \delta \leq 1$ define the subspace L_δ of $L^2([0,1], \mu)$ by

$$L_\delta = \{f \in L^2([0,1], \mu) : f(x) = 0 \text{ for } \delta < x \leq 1\}$$

These are spectral subspaces of the multiplication operators M_{x^t}

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Theorem.

If U gives unitary equivalence from A_t on $H^2(\beta_\kappa)$ to M_{x^t} on $L^2([0,1],\mu)$,

$$\textit{then} \quad U^* L_\delta = H_{(\ln\delta)/2} \quad \textit{or equivalently} \quad U H_c = L_{e^{2c}}$$

Suppose N is a subspace of $H^2(\beta_\kappa)$ that is invariant for the operator of multiplication by z.

If there is f in N with $f(0) \neq 0$ and G is a function of N so that

$$\|G\| = 1 \quad \text{and} \quad G(0) = \sup\{\operatorname{Re} f(0) : f \in N \text{ and } \|f\| = 1\}$$

then we say G *solves the extremal problem for the invariant subspace* N.

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Subspaces H_c are spectral subspaces for A_t, but more interestingly, they are invariant subspaces for M_z on $H^2(\beta_\kappa)$ generated by atomic inner functions!

The unitary equivalence between the subspaces H_c in $H^2(\beta_\kappa)$ and L_δ in $L^2([0,1],\mu)$ gives an opportunity to compute the extremal functions for L_δ and translate the answer back to H_c!!

Our computation requires the use of the *incomplete Gamma function*

$$\Gamma(a,w) = \int_w^\infty t^{a-1} e^{-t}\, dt$$

where a is a complex parameter and w is a real parameter. An alternate definition in which both a and w are complex parameters is

$$\Gamma(a,w) = e^{-w} w^a \int_0^\infty e^{-wu}(1+u)^{a-1}\, du$$

Theorem.

For $c < 0$, if H_c is the invariant subspace of $H^2(\beta_\kappa)$ defined by

$$H_c = \text{closure}\{e^{c\frac{1+z}{1-z}} f : f \in H^2(\beta_\kappa)\}$$

then the extremal function for H_c is

$$G_c(z) = \frac{\Gamma(\kappa, -2c/(1-z))}{\sqrt{\Gamma(\kappa)}\sqrt{\Gamma(\kappa,-2c)}}$$

Theorem.

For $0 < r < 1$, *let* P_r *be the orthogonal projection onto the subspace* $H_{(\ln r)/2}$ *in* $H^2(\beta_\kappa)$. *If* u *is any point of the open unit disk, then for* $K_u(z) = (1 - \overline{u}z)^{-\kappa}$

$$(P_r K_u)(z) = \frac{1}{\Gamma(\kappa)(1 - \overline{u}z)^\kappa} \Gamma\left(\kappa, -\frac{(\ln r)(1 - \overline{u}z)}{(1 - \overline{u})(1 - z)}\right)$$

This gives the kernel functions for the invariant subspaces H_c in $H^2(\beta_\kappa)$, including for the usual Bergman space ($\kappa = 2$).

This result generalizes the formula for the usual Bergman space computed in a different way by W. Yang in his thesis.

Third Example:

Common invariant subspaces for C_φ and S, multiplication by z

(Wahl & C.)

Let φ be analytic map of $\mathbb{D}$ into itself.

We say b *is a fixed point of* φ if $\varphi(b) = b$ (when $|b| < 1$)

or $\lim_{r \to 1^-} \varphi(rb) = b$ (when $|b| = 1$).

Julia-Carathéordory Theorem implies

If b is a fixed point of φ with $|b| = 1$, then $\lim_{r \to 1^-} \varphi'(rb)$ exists

(call it $\varphi'(b)$) and $0 < \varphi'(b) \leq \infty$.

Denjoy-Wolff Theorem (1926)

If φ is an analytic map of $\mathbb{D}$ into itself (not an elliptic automorphism of $\mathbb{D}$), there is a unique fixed point, a, of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

Moreover, the sequence of iterates, φ_n, converges to a uniformly on compact subsets of $\mathbb{D}$, so for all points, $\lim_{n\to\infty} \varphi_n(z) = a$.

Analytic self-maps (not elliptic automorphisms) of $\mathbb{D}$ divide into distinct classes based on linear fractional models for iteration:

- (Plane/Dilation): $|a| < 1$ and $0 < |\varphi'(a)| < 1$
- (Half-Plane/Dilation): $|a| = 1$ and $0 < \varphi'(a) < 1$
- (Half-Plane/Translation): $|a| = 1$, $\varphi'(a) = 1$, and $\varphi_n(z)$ interpolating
- (Plane/Translation): $|a| = 1$, $\varphi'(a) = 1$, and $\varphi_n(z)$ not interpolating
- (no LF model): $|a| < 1$ and $\varphi'(a) = 0$

Without loss of generality, if a, the Denjoy-Wolff point of φ, is in $\mathbb{D}$, we can assume $a = 0$, and if $|a| = 1$, we can assume $a = 1$.

For simplicity, we will assume that the Hilbert space is $H^2(\mathbb{D})$, and when weighted composition operators $W_{\psi,\varphi}$ are discussed, that ψ is in $H^\infty(\mathbb{D})$.

Theorem:

If φ is an analytic map of $\mathbb{D}$ into itself, ψ is in H^∞, and M is an invariant subspace for C_φ and S, then M is an invariant subspace for $W_{\psi,\varphi}$.

Conversely, if ψ^{-1} is in H^∞ and M is an invariant subspace for $W_{\psi,\varphi}$ and S, then M invariant for C_φ.

Theorem:

If φ is an analytic map of the unit disk into itself with $\varphi(1) = 1$ and $\varphi'(1) \leq 1$, then $e^{\alpha\frac{z+1}{z-1}}H^2(\mathbb{D})$ is an invariant subspace for C_φ whenever $\alpha > 0$.

Outline of Proof:

Use Julia's Lemma to prove the following:

Let φ be an analytic map of the unit disk into itself such that $\varphi(1) = 1$ and $\varphi'(1) \leq 1$. Then, for z in $\mathbb{D}$,

$$\mathrm{Re}\left(\frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1}\right) \leq 0$$

For g in H^2,

$$\begin{aligned} C_\varphi(e^{\alpha\frac{z+1}{z-1}}g(z)) &= e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}\left(g\circ\varphi\right)(z) \\ &= \left(e^{\alpha\frac{z+1}{z-1}}e^{\alpha\left(\frac{\varphi(z)+1}{\varphi(z)-1}-\frac{z+1}{z-1}\right)}\right)\left(g\circ\varphi\right)(z) \\ &= e^{\alpha\frac{z+1}{z-1}}\left(e^{\alpha\left(\frac{\varphi(z)+1}{\varphi(z)-1}-\frac{z+1}{z-1}\right)}\left(g\circ\varphi\right)(z)\right) \end{aligned}$$

■

Theorem:

If φ is an analytic map of the unit disk into itself and $M = e^{\alpha\frac{z+1}{z-1}}H^2(\mathbb{D})$ is an invariant subspace for C_φ for some $\alpha > 0$, then $\varphi(1) = 1$ and $\varphi'(1) \leq 1$.

Outline of Proof:

$e^{\alpha\frac{z+1}{z-1}}$ in M implies $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}$ is in M which means $\lim_{r\to 1^-} e^{\alpha\frac{\varphi(r)+1}{\varphi(r)-1}} = 0$

which means $\varphi(1) = 1$.

Using the Julia-Carathéordory Theorem, we see $e^{\alpha\frac{\varphi(z)+1}{\varphi(z)-1}}$ in $e^{\alpha\frac{z+1}{z-1}}H^2(\mathbb{D})$ implies $\varphi'(1) \leq 1$.

■

Corollary:

The subspace $M = e^{\alpha\frac{z+1}{z-1}}H^2(\mathbb{D})$ is invariant for C_φ for $\alpha > 0$ if and only if 1 is the Denjoy-Wolff point of φ.

For $|\lambda| = 1$ and z_j, $j = 1, 2, \ldots$, points in $\mathbb{D}$ satisfying $\sum_j (1 - |z_j|) < \infty$, the function

$$B(z) = \lambda \prod_j \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \overline{z_j} z}$$

is a Blaschke product. The zero set, $\{z_j\}$, for B is denoted by $Z(B)$.

Lemma:

Let C_φ be a composition operator on $H^2(\mathbb{D})$. Then $BH^2(\mathbb{D})$ is C_φ-invariant if and only if $z_j \in Z(B)$ implies $\varphi_n(z_j) \in Z(B)$ for all non-negative integers j and n and if $w \in Z(B)$, then multiplicity $\varphi(w) \geq$ multiplicity w.

Outline of Proof:

$BH^2(\mathbb{D})$ invariant for C_φ insures there is g in H^2 such that $C_\varphi B = Bg$. Thus, $C_\varphi B = 0$ whenever $B = 0$, so z_j in $Z(B)$ implies $C_\varphi B(z_j) = B(\varphi(z_j)) = 0$, that is, $\varphi(z_j)$ is in $Z(B)$ also. ∎

Theorem:

Let C_φ be a composition operator on $H^2(\mathbb{D})$ with $\varphi(a) = a$ for a in $\mathbb{D}$.

If $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial, then (i) $a \in Z(B)$ and

(ii) for every $z_j \in Z(B)$, there exists an integer n_j such that $\varphi_{n_j}(z_j) = a$.

Outline of Proof:

If w is in $Z(B)$, then $\varphi_k(w)$ is in $Z(B)$ for all k, but $\lim_{k\to\infty} \varphi_k(w) = a$.

If there were infinitely many points $\varphi_k(w)$, then $B \equiv 0$, so there are only

finitely many and there is n so that $\varphi_n(w) = a$. This means a is in $Z(B)$.

∎

Theorem:

Let C_φ be a composition operator on $H^2(\mathbb{D})$ with $\varphi(a) = a$ for a in $\mathbb{D}$.

If $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial, then (i) $a \in Z(B)$ and

(ii) for every $z_j \in Z(B)$, there exists an integer n_j such that $\varphi_{n_j}(z_j) = a$.

Corollary:

Let φ be a univalent analytic function mapping the unit disk into itself with $\varphi(a) = a$ for some a in $\mathbb{D}$ and C_φ be the composition operator on $H^2(\mathbb{D})$.

Then the subspaces $\left(\frac{z-a}{1-\overline{a}z}\right)^k H^2(\mathbb{D})$ are the only non-trivial Blaschke product induced subspaces invariant for both C_φ and S.

Outline of Proof:

If B were a Blaschke product with a zero $w \neq a$ with $BH^2(\mathbb{D})$ invariant, then $\varphi_n(w) \in Z(B)$ for all n, but $\varphi_n(w) \neq a$ for any n because φ is univalent. This means $B \equiv 0$. Thus, the only zero of B is a. ∎

Theorem:

Let C_φ be a composition operator on $H^2(\mathbb{D})$ and suppose the Denjoy-Wolff point of φ is $a = 1$. If $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial, and $w \in Z(B)$, then the infinite set $\{\varphi_n(w) : n \in \mathbb{N}\} \subset Z(B)$.

Corollary:

If C_φ is a composition operator on $H^2(\mathbb{D})$ and the Denjoy-Wolff point of φ is $a = 1$, then there are no finite Blaschke products B so that $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial.

Theorem:

Let C_φ be a composition operator on $H^2(\mathbb{D})$ and suppose the Denjoy-Wolff point of φ is $a = 1$. If $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial, and $w \in Z(B)$, then the infinite set $\{\varphi_n(w) : n \in \mathbb{N}\} \subset Z(B)$.

Corollary:

If C_φ is a composition operator on $H^2(\mathbb{D})$, the Denjoy-Wolff point of φ is $a = 1$, and

either • $\varphi'(1) < 1$

or • $\varphi'(1) = 1$ and φ is in the half-plane translation case,

then for any finite Blaschke product B_0, there is an infinite Blaschke product B so that B_0 divides B and $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial.

Outline of Proof:

In these cases, if w is a zero of B_0, then $\{\varphi_n(w)\}$ is a Blaschke sequence. ■

Theorem:

Let C_φ be a composition operator on $H^2(\mathbb{D})$ and suppose the Denjoy-Wolff point of φ is $a = 1$. If $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial, and $w \in Z(B)$, then the infinite set $\{\varphi_n(w) : n \in \mathbb{N}\} \subset Z(B)$.

Example:

If $\varphi(z) = \dfrac{1}{2-z}$, which has $\varphi(1) = \varphi'(1) = 1$ but φ is in the plane translation case, then there is no Blaschke product for which $BH^2(\mathbb{D})$ is C_φ-invariant and non-trivial.

Outline of Proof:

If w is any point of $\mathbb{D}$, then $\{\varphi_n(w)\}$ is a *NOT* a Blaschke sequence. ■

There is space between our results: If J is a singular inner function whose singular measure has no atom, then our work says nothing about possible non-trivial spaces of the form $JH^2(\mathbb{D})$ that are C_φ-invariant!

¡Muchas Gracias!

http://www.math.iupui.edu/~ccowen/Downloads.html