Fixed Points of Functions Analytic in the Unit Disk

Carl C. Cowen

IUPUI

(Indiana University Purdue University Indianapolis)

Conference on Complex Analysis, University of Illinois, May 22, 2010

Fixed Points of Functions Analytic in the Unit Disk

Carl C. Cowen

IUPUI

(Indiana University Purdue University Indianapolis)

Conference on Complex Analysis, University of Illinois, May 22, 2010

Some of this is joint work with Christian Pommerenke (1982).

Let φ be an analytic function that maps the unit disk $\mathbb{D}$ into itself.

Today, I want to consider the fixed points of φ

and especially values of the derivative of φ at these fixed points.

The Schwarz-Pick Lemma implies φ has at most one fixed point in $\mathbb{D}$

and some maps have none!

If φ is a continuous map of $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}}$, then φ must have a fixed point in $\overline{\mathbb{D}}$.

Only assume φ is analytic on $\mathbb{D}$, open disk!

Definition

Suppose φ is an analytic map of $\mathbb{D}$ into itself.

If $|b| < 1$, we say b *is a fixed point of* φ if $\varphi(b) = b$.

If $|b| = 1$, we say b *is a fixed point of* φ if $\lim_{r \to 1^-} \varphi(rb) = b$.

Julia-Caratheordory Theorem implies

If b is a fixed point of φ with $|b| = 1$, then $\lim_{r \to 1^-} \varphi'(rb)$ exists (call it $\varphi'(b)$) and $0 < \varphi'(b) \leq \infty$.

Denjoy-Wolff Theorem (1926)

If φ is an analytic map of $\mathbb{D}$ into itself, not the identity map, there is a unique fixed point, a, of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

This distinguished fixed point will be called the *Denjoy-Wolff point of* φ.

The Schwarz-Pick Lemma implies φ has at most one fixed point in $\mathbb{D}$ and if φ has a fixed point in $\mathbb{D}$, it must be the Denjoy-Wolff point.

Examples

(1) $\varphi(z) = (z+1/2)/(1+z/2)$ is an automorphism of $\mathbb{D}$ fixing 1 and -1.
The Denjoy-Wolff point is $a = 1$ because $\varphi'(1) = 1/3$ (and $\varphi'(-1) = 3$)

(2) $\varphi(z) = z/(2-z^2)$ maps $\mathbb{D}$ into itself and fixes 0, 1, and -1.
The Denjoy-Wolff point is $a = 0$; $\varphi'(0) = 1/2$ (and $\varphi'(\pm 1) = 3$)

(3) $\varphi(z) = (2z^3+1)/(2+z^3)$ is an inner function fixing fixing 1 and -1
with Denjoy-Wolff point $a = 1$ because $\varphi'(1) = 1$ (and $\varphi'(-1) = 9$)

(4) Inner function $\varphi(z) = \exp(z+1)/(z-1)$ has a fixed point in $\mathbb{D}$,
Denjoy-Wolff point $a \approx .21365$, and infinitely many fixed points on $\partial\mathbb{D}$

Today, discuss questions about fixed points of analytic maps of $\mathbb{D}$ into $\mathbb{D}$ and the values of the derivative at the fixed points.

Examples of maps with several fixed points in the closed disk and Schwarz-Pick and Denjoy-Wolff say that all but one of these fixed points satisfies $\varphi'(b) > 1$.

Questions about φ, analytic and $\varphi(\mathbb{D}) \subset \mathbb{D}$:

- How many fixed points can φ have? with φ univalent?
- How small can $\varphi'(b)$ be? with φ univalent?

Definition: If φ is an analytic map of the unit disk into itself, the fixed point set of φ is the set

$$F = \left\{ z \in \overline{\mathbb{D}} : \lim_{r \to 1^-} \varphi(rz) = z \right\}$$

Theorem:

If φ is an analytic function that maps the unit disk into itself, then the part of the fixed point set of φ in $\partial\mathbb{D}$ has linear measure zero.

Example:

Let K be a compact set of measure zero in $\partial\mathbb{D}$. There is a function φ analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and the fixed point set of φ is $\{0\} \cup K$.

Theorem:

If φ is a univalent analytic function that maps the unit disk into itself, then the fixed point set of φ has capacity zero.

Example:

Let K be a compact set of capacity zero in $\partial\mathbb{D}$ and a be a point of $\partial\mathbb{D} \setminus K$. There is a function φ analytic and univalent in $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and the fixed point set of φ is $\{a\} \cup K$.

In the Theorem below, we assume that the Denjoy-Wolff point, $a = b_0$ has been normalized so that $b_0 = 0$ or $b_0 = 1$.

Theorem:

Let φ be an analytic function with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and suppose $b_0, b_1, \cdots, b_n$ are fixed points of φ.

(i) If $b_0 = 0$ then

$$\sum_{j=1}^{n} \frac{1}{\varphi'(b_j) - 1} \leq \operatorname{Re} \frac{1 + \varphi'(0)}{1 - \varphi'(0)}$$

Theorem (cont'd):

(ii) If $b_0 = 1$ and $0 < \varphi'(1) < 1$ then

$$\sum_{j=1}^{n} \frac{1}{\varphi'(b_j) - 1} \leq \frac{\varphi'(1)}{1 - \varphi'(1)}$$

(iii) If $b_0 = 1$ and $\varphi'(1) = 1$ then

$$\sum_{j=1}^{n} \frac{|1 - b_j|^2}{\varphi'(b_j) - 1} \leq 2\,\mathrm{Re}\left(\frac{1}{\varphi(0)} - 1\right)$$

Moreover, equality holds if and only if φ is a Blaschke product of order $n + 1$ in case (i) or order n in cases (ii) and (iii).

Note that, if φ has infinitely many fixed points, then the appropriate inequality holds for any choice of finitely many fixed points.

In particular, only countably many of the fixed points of φ can have finite angular derivative.

If $b_0, b_1, b_2, \cdots$ are the countably many fixed points for which $\varphi'(b_j) < \infty$, then the corresponding infinite sum converges and the appropriate inequality holds.

About the proof:

First, notice that $\varphi'(b) = 1$ at a fixed point b is a fixed point with "multiplicity greater than 1" in the sense that

b fixed means b is a root of $\varphi(z) - z = 0$ and $\varphi'(b) - 1 = 0$ means it is a root of $\varphi(z) - z = 0$ with multiplicity greater than 1.

The proof is based on the following fact:

If $r(z)$ is a rational function of order n with distinct fixed points $p_1, p_2, \cdots, p_n, p_{n+1}$ in the sphere, then

$$\sum_{j=1}^{n+1} \frac{1}{1 - r'(p_j)} = 1$$

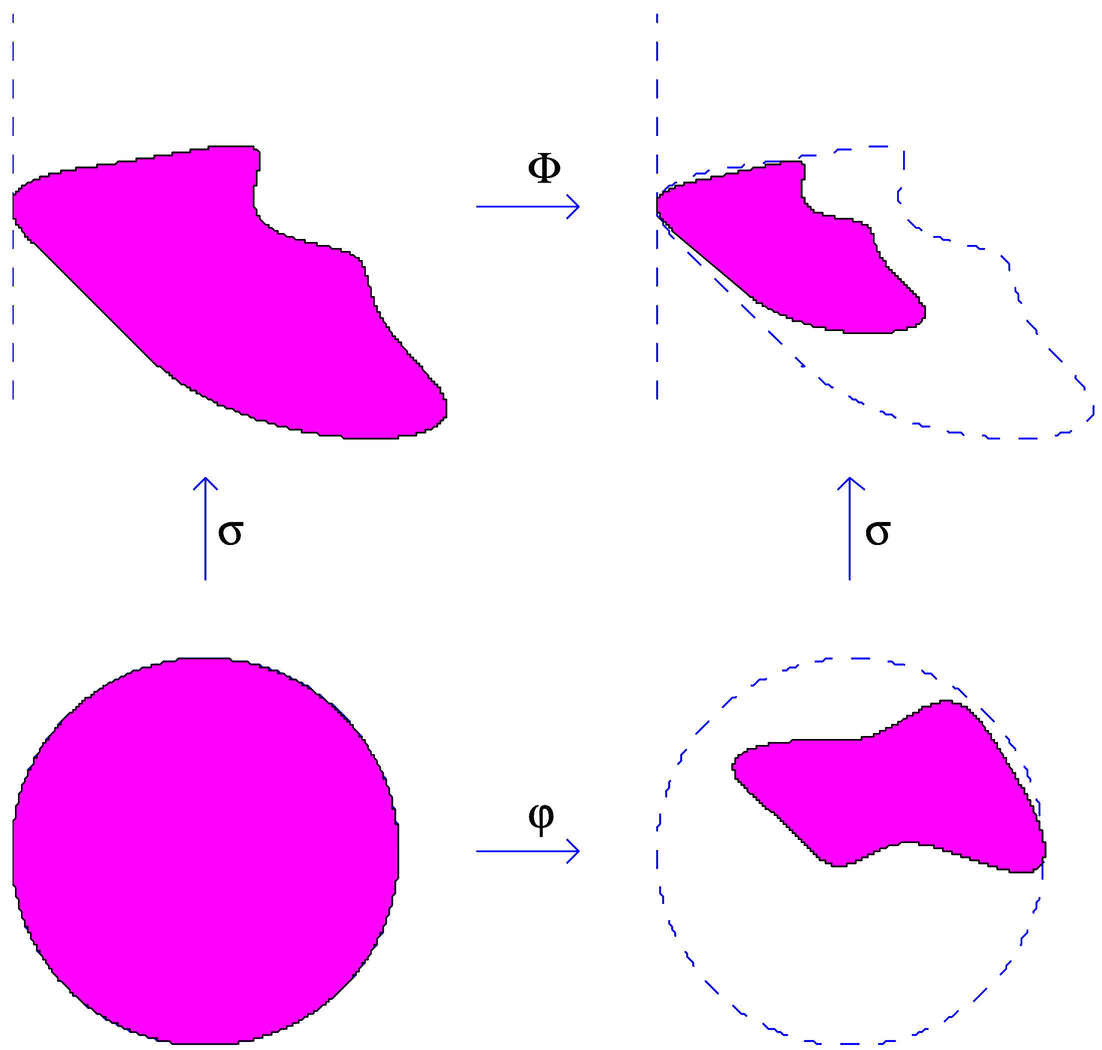
The inequalities for univalent maps seem related to iteration of the maps.

Five different types of maps of $\mathbb{D}$ into itself from the perspective of iteration,
classified by the behavior of the map near the Denjoy-Wolff point, a

In one of these types, $\varphi'(a) = 0$, (e.g., $\varphi(z) = (z^2 + z^3)/2$ with $a = 0$),
a model for iteration doesn't seem useful for studying these inequalities.

In the other four types, when $\varphi'(a) \neq 0$, the map φ can be intertwined with
a linear fractional map and classified by the possible type of intertwining:
σ intertwines Φ and φ in the equality $\Phi \circ \sigma = \sigma \circ \varphi$

We want to do this with Φ linear fractional and σ univalent near a, so that σ is, locally, a change of variables. Using the notion of fundamental set, this linear fractional model becomes essentially unique [Cowen, 1981]



A linear fractional model in which φ maps $\mathbb{D}$ into itself with $a = 1$ and

$\varphi'(1) = \dfrac{1}{2}$, σ maps $\mathbb{D}$ into the right half plane, and $\Phi(w) = \dfrac{1}{2}\,w$

Linear Fractional Models:

- φ maps $\mathbb{D}$ into itself with $\varphi'(a) \neq 0$ (φ not an elliptic automorphism)
- Φ is a linear fractional automorphism of Ω onto itself
- σ is a map of $\mathbb{D}$ into Ω with $\quad \Phi \circ \sigma = \sigma \circ \varphi$

I. (plane dilation) $|a| < 1,\ \Omega = \mathbb{C},\ \sigma(a) = 0,\ \Phi(w) = \varphi'(a)w$

II. (half-plane dilation) $|a| = 1$ with $\varphi'(a) < 1,\ \Omega = \{w : \operatorname{Re} w > 0\},$

$\sigma(a) = 0,\ \Phi(w) = \varphi'(a)w$

III. (plane translation) $|a| = 1$ with $\varphi'(a) = 1,\ \Omega = \mathbb{C},\ \Phi(w) = w + 1$

IV. (half-plane translation) $|a| = 1$ with $\varphi'(a) = 1,\ \Omega = \{w : \operatorname{Im} w > 0\},$

(or $\Omega = \{w : \operatorname{Im} w < 0\}$), $\Phi(w) = w + 1$

Linear Fractional Models:

- φ maps $\mathbb{D}$ into itself with $\varphi'(a) \neq 0$ (φ not an elliptic automorphism)
- Φ is a linear fractional automorphism of Ω onto itself
- σ is a map of $\mathbb{D}$ into Ω with $\Phi \circ \sigma = \sigma \circ \varphi$

I. (plane dilation) $|a| < 1,\ \Omega = \mathbb{C},\ \sigma(a) = 0,\ \Phi(w) = \varphi'(a)w$

II. (half-plane dilation) $|a| = 1$ with $\varphi'(a) < 1,\ \Omega = \{w : \operatorname{Re} w > 0\},$

$\sigma(a) = 0,\ \Phi(w) = \varphi'(a)w$ (equivalent to strip translation)

III. (plane translation) $|a| = 1$ with $\varphi'(a) = 1,\ \Omega = \mathbb{C},\ \Phi(w) = w + 1$

$\{\varphi_k(0)\}$ is NOT an interpolating sequence (i.e. $\{\varphi_k(0)\}$ close together)

IV. (half-plane translation) $|a| = 1$ with $\varphi'(a) = 1,\ \Omega = \{w : \operatorname{Im} w > 0\},$

(or $\Omega = \{w : \operatorname{Im} w < 0\}$), $\Phi(w) = w + 1$

$\{\varphi_k(0)\}$ IS an interpolating sequence (i.e. $\{\varphi_k(0)\}$ far apart)

The constructions involved give σ is univalent if and only if φ is univalent.

Note that the iterates of φ, say $\varphi_2 = \varphi \circ \varphi$, $\varphi_3 = \varphi \circ \varphi_2$, etc., satisfy

$\Phi_k \circ \sigma = \sigma \circ \varphi_k$

This discrete semi-group for Φ is embedded in the continuous semigroup

$\Phi_t(w) = \varphi'(a)^t w$ or $\Phi_t(w) = w + t$.

For φ univalent, whether φ is embeddable in a continuous semi-group is a geometric condition on $\sigma(\mathbb{D})$: whether $\varphi_t(z) = \sigma^{-1}(\Phi_t(\sigma(z)))$ is defined for all z depends on whether $\sigma(\mathbb{D})$ always contains $\varphi'(a)^t w$ or $w + t$ if it contains w.

Assume that the Denjoy-Wolff point, $a = b_0$ is $b_0 = 0$ or $b_0 = 1$.

Theorem:

Let φ be a univalent analytic function with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and suppose $b_0, b_1, \cdots, b_n$ are fixed points of φ.

(i) If $b_0 = 0$ then

$$\sum_{j=1}^{n} (\log \varphi'(b_j))^{-1} \leq 2\mathrm{Re}\, B^{-1}$$

where $B = \lim_{r \to 1^-} \log\left(\frac{\varphi(rb_1)}{\varphi'(0)rb_1}\right)$ and $\lim_{z \to 0} \log\left(\frac{\varphi(z)}{z}\right) = \mathrm{Log}\, \varphi'(0)$.

(ii) If $b_0 = 1$ and $0 < \varphi'(1) < 1$ then

$$\sum_{j=1}^{n} (\log \varphi'(b_j))^{-1} \leq -(\log \varphi'(1))^{-1}$$

Moreover, equality holds if and only if φ is embeddable in a semigroup and $\varphi(\mathbb{D})$ is $\mathbb{D}$ with n (in case (i)) or $n-1$ (in case (ii)) analytic arcs removed.

Theorem (cont'd):

(iii) If $b_0 = 1$ and $\varphi'(1) = 1$ then

$$\sum_{j=1}^{n} c_j^2 (\log \varphi'(b_j))^{-1} \leq 2 \log \frac{1 - |\varphi(0)|^2}{|\varphi'(0)|}$$

where $c_j = \lim_{r \to 1^-} \operatorname{Im} \left(\log \left(\frac{1}{b_j} \frac{\varphi(rb_j) - \varphi(0)}{1 - \varphi(0)\overline{\varphi(rb_j)}} \frac{1 - \varphi(0)\overline{\varphi(r)}}{\varphi(r) - \varphi(0)} \right) \right)$

About the proof:

Proofs in the published version use Grunsky inequalities. The geometric arguments seemed to need extra hypotheses, but at least some of these have been proved by Poggi-Corradini.

The statement in case (iii) does not include an equality condition, and it is presumed that this inequality is not best possible.

Progress has been made in the past few years on these ideas.

For example, Milne Anderson and Alexandre Vasil'ev have proved (2008) a sharp inequality for $b_0 = 0$ (case (i)):

$$\prod_{j=1}^{n} \varphi'(b_j)^{2\alpha_j^2} \geq \frac{1}{|\varphi'(0)|}$$

where $\alpha_j \geq 0$ and $\sum_{j=1}^{n} \alpha_j = 1$ where equality holds only for the unique solution of a given complex differential equation with a given initial condition.

They use tools involving reduced moduli on digons and extremal partitions.

Also, Contreras, Díaz-Madrigal, and Pommerenke have proved (2006) a sharp inequality for $b_0 = 1$ (case (iii)) *with the assumption that* φ *is embeddable in a continuous semigroup*:

$$\sum_{j=1}^{n} \frac{1 - \operatorname{Re} b_j}{\log \varphi'(b_j)} \leq \operatorname{Re} \frac{1}{G(0)} = \operatorname{Re} \sigma'(0)$$

where G is the infinitesimal generator of the semi-group and σ is the map from the linear fractional model.

Note that case (iii), $\varphi'(b_0) = 1$, still does not have a sharp inequality to describe the general case.

References:

- C. C. Cowen, Iteration and the Solution of Functional Equations for Functions Analytic in the Unit Disk. *Trans. Amer. Math. Soc.* **265**(1981) 69–95.

- C. C. Cowen and Ch. Pommerenke, Inequalities for the Angular Derivative of an Analytic Function in the Unit Disk. *J. London Math. Soc.* **26**(1982) 271–289.

- M. D. Contreras, S. Díaz-Madrigal, and Ch. Pommerenke, On Boundary Critical Points for Semigroups of Analytic Functions. *Math. Scand.* **98**(2006) 125–142.

- J. M. Anderson and A. Vasil'ev, Lower Schwarz-Pick Estimates and Angular Derivatives. *Ann. Acad. Sci. Fennicae Math.* **33**(2008) 101–110.