Composition Operators on
Spaces of Analytic Functions, II

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Today, we want to consider the spectral theory of composition operators.

We finished yesterday with the introduction of the linear fractional model for iteration and Koenigs' construction which gives the model in the plane dilation case.

We begin with the spectrum for compact composition operators, as was historically first.

Recall that if A is an operator, the spectrum of A is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \;\text{ does not have a continuous inverse}\}$$

Suppose φ is an analytic map of $\mathbb{D}$ into itself and C_φ is compact on $H^2(\mathbb{D})$.

The compactness of C_φ implies that the Denjoy-Wolff point, a, is in $\mathbb{D}$.

Let $\psi(z) = \dfrac{a - z}{1 - \overline{a}z}$ It is easy to check that ψ is an automorphism of $\mathbb{D}$ such that $\psi(a) = 0$, $\psi(0) = a$, and $\psi(\psi(z)) = z$, so $\psi = \psi^{-1}$

The map $\psi \circ \varphi \circ \psi$ is therefore a map of $\mathbb{D}$ into itself such that

$\psi \circ \varphi \circ \psi(0) = 0$. Moreover,

$$C_{\psi\circ\varphi\circ\psi} = C_\psi C_\varphi C_\psi = (C_\psi)^{-1} C_\varphi C_\psi$$

so that $C_{\psi\circ\varphi\circ\psi}$ is similar to C_φ.

This means that $C_{\psi\circ\varphi\circ\psi}$ is also compact and $\sigma(C_{\psi\circ\varphi\circ\psi}) = \sigma(C_\varphi)$.

Thus, with no loss of generality, assume Denjoy-Wolff point φ is $a = 0$

Theorem (Caughran-Schwartz, 1975)

Let φ be analytic map on $\mathbb{D}$ with D.W. point a and C_φ compact on H^2. Then $|a| < 1$ and the spectrum of C_φ is

$$\sigma(C_\varphi) = \{0, 1\} \cup \{\varphi'(a)^n : n = 1, 2, 3, \cdots\}$$

Moreover, each of the eigenspaces is one dimensional and, if $\varphi'(a) \neq 0$, for each non-negative integer n, the eigenspace corresponding to $\varphi'(a)^n$ is spanned by σ^n, where σ is the Koenigs function for φ.

Proof:

Without loss of generality, $\varphi(0) = 0$.

The monomials $1, z, z^2, \cdots$ form an orthonormal basis for H^2 and we will consider the matrix for C_φ with respect to this basis.

Proof (cont'd):

Without loss of generality, $\varphi(0) = 0$.

Since $C_\varphi 1 = 1 \circ \varphi = 1$, the column of the matrix for C_φ corresponding to the basis vector 1 is $(1, 0, 0, \cdots)$. Similarly, column of the the matrix for C_φ corresponding to the basis vector z^k is the vector of Taylor coefficients of $C_\varphi z^k = \varphi^k$ which is $(0, 0, \cdots 0, \varphi'(0)^k, k\varphi'(0)^{k-1}\varphi''(0)/2, \cdots)$

In particular, the matrix for C_φ is lower triangular which means the matrix for C_φ^* is upper triangular

Triangularity of C_φ^* implies, for any positive integer n, as a block matrix

$$C_\varphi^* \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is $n \times n$ upper triangular and the lower left is a 0 matrix

Proof (cont'd):

The compactness of C_φ^* implies, for sufficiently large n, as a block matrix

$$C_\varphi^* \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

and $\|D\|$ can be made as small as we like.

As a consequence, each of the non-zero eigenvalues of C_φ^* is an eigenvalue of an upper left corner, A, for sufficiently large n, and every eigenvalue of such an A is an eigenvalue of C_φ^*.

Note that

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \overline{\varphi'(0)} & * & * & \cdots & * \\ 0 & 0 & \overline{\varphi'(0)}^2 & * & \cdots & * \\ 0 & 0 & 0 & \overline{\varphi'(0)}^3 & \cdots & * \\ & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \overline{\varphi'(0)}^n \end{pmatrix}$$

We see that this means that the eigenvalues of A are 1, $\overline{\varphi'(0)}$, $\overline{\varphi'(0)}^2$,$\cdots$, and $\overline{\varphi'(0)}^n$, each of multiplicity one, and that therefore, the non-zero eigenvalues of C_φ^* are $\{\overline{\varphi'(0)}^k\}_{k=0}^\infty$, each with multiplicity one.

The spectral theory of compact operators therefore implies that the non-zero eigenvalues of C_φ are $\{\varphi'(0)^k\}_{k=0}^\infty$, each with multiplicity one.

Since each of the numbers $\{\varphi'(0)^k\}_{k=0}^{\infty}$, is a non-zero eigenvalue with multiplicity one, for each of the numbers $\varphi'(0)^k$ there must be a one-dimensional subspace of $H^2(\mathbb{D})$ (!!!) consisting of eigenvectors of C_φ.

Koenigs' Theorem says that the only solutions of $f \circ \varphi = \varphi'(0)^k f$ are multiples of σ^k, where σ is the Koenigs function for φ. This means that if C_φ is compact on H^2, then for every positive integer k, the function σ^k is in H^2 and that these vectors span the eigenspaces of C_φ.

Noting that 0 is always a point of the spectrum of a compact operator on an infinite dimensional space, that the constant functions are eigenvectors for the eigenvalue 1, and that 0 is never an eigenvalue of C_φ because $\varphi(\mathbb{D})$ is an open set, we see that the proof is complete. ■

The eigenvalue equation for C_φ, $f \circ \varphi = \lambda f$, is called Schroeder's functional equation. Koenigs solved Schroeder's functional equation for functions analytic in the disk with Denjoy-Wolff point in $\mathbb{D}$. However, as we saw in the proof of the theorem, it is not enough to have a solution of Schroeder's equation, one must also show it is in H^2 to have a eigenvector of C_φ.

We will split the general problem into two pieces: find solutions of Schroeder's equation and then decide which, if any, are in H^2 (or other space of interest).

The linear fractional model for iteration is helpful in this effort. We have

$$\sigma \circ \varphi = \Phi \circ \sigma$$

for Φ a specific linear fractional map when φ is analytic map on $\mathbb{D}$ and the Denjoy-Wolff point satisfies $|a| = 1$.

Specifically, when $|a| = 1$ and $s = \varphi'(a) < 1$, σ maps $\mathbb{D}$ into the right half plane and $\Phi(w) = sw$. When $|a| = 1$, σ maps $\mathbb{D}$ into the upper or lower half plane or into the whole plane, depending on the nature of the iterates of φ, and $\Phi(w) = w + 1$.

Suppose F solves Schroeder's equation for Φ for some number λ, that is, $F \circ \Phi = \lambda F$. Using $\sigma \circ \varphi = \Phi \circ \sigma$ and choosing $f = F \circ \sigma$, we see

$$f \circ \varphi = (F \circ \sigma) \circ \varphi = F \circ (\sigma \circ \varphi) = F \circ (\Phi \circ \sigma) = (F \circ \Phi) \circ \sigma = \lambda F \circ \sigma = \lambda f$$

For $\Phi(w) = sw$ on the right half plane, using the branch of logarithm with $\log(1) = 0$ and writing $F(w) = e^{r \log w}$, we get $F \circ \Phi = s^r F$ for any r.

For $\Phi(w) = w + 1$, writing $F(w) = e^{rw}$, we get $F \circ \Phi = e^r F$ for any r.

Theorem

Let φ be analytic map on $\mathbb{D}$ with Denjoy-Wolff point a and $|a| = 1$. Then for each non-zero number λ, Schroeder's equation has an infinite dimensional subspace of solutions.

Proof:

The remark above shows that for every complex number r, either s^r (if $\varphi'(a) = s < 1$) or e^r (if $\varphi'(a) = 1$) are solutions of Schroeder's Equation.

Notice that each non-zero complex number λ can be written as $\lambda = s^r$ or $\lambda = e^r$ for infinitely many different r's. Since the corresponding functions $f(z) = e^{r \log \sigma(z)}$ or $f(z) = e^{r\sigma(z)}$ are linearly independent for different r's, the Theorem is proved. ∎

Koenigs' Theorem gives the solutions of Schroeder's equation when $|a| < 1$.

Theorem

Let φ be automorphism of $\mathbb{D}$ and a in $\overline{\mathbb{D}}$ the fixed point with $|\varphi'(a)| \leq 1$.

- *If $|a| < 1$ (φ is elliptic), then $\sigma(C_\varphi) = \overline{\{\varphi'(a)^n\}_{n=0}^{\infty}}$*
- *If $|a| = 1$ and $\varphi'(a) = 1$ (φ is parabolic), then $\sigma(C_\varphi) = \partial\mathbb{D}$*
- *If $|a| = 1$ and $\varphi'(a) < 1$ (φ is hyperbolic), then*

$$\sigma(C_\varphi) = \{\lambda : \sqrt{\varphi'(a)} \leq |\lambda| \leq \frac{1}{\sqrt{\varphi'(a)}}\}$$

Proof:

If $|a| < 1$, so that φ is an elliptic automorphism, the automorphism $\psi(z) = (a - z)/(1 - \overline{a}z)$ gives $\zeta = \psi^{-1} \circ \varphi \circ \psi$ so that C_ζ is similar to C_φ and $\zeta(z) = \gamma z$ with $\gamma = \varphi'(a)$ and $|\gamma| = 1$. For each non-negative integer n, the function z^n is an eigenvector for the eigenvalue γ^n, so $\sigma(C_\zeta) = \sigma(C_\varphi)$ is a closed set that includes γ^n for all positive integers n.

Proof (cont'd):

If γ is root of unity, the spectrum includes the finite set of powers of $\gamma = \varphi'(a)$. If γ is not a root of unity, the set $\{\gamma^n\}_{n=0}^{\infty}$ is dense in the unit circle and the spectrum includes the unit circle. Since the matrix for C_ζ is diagonal, it is easy to see that each of these containments is actually equality.

If $|a| = 1$ and $\varphi'(a) = 1$, so that φ is an parabolic automorphism, and it has a half-plane translation model. Without loss of generality, $a = 1$. For example, $\varphi(z) = ((1+i)z - i)\,/\,((1-i) + iz)$ is such a parabolic automorphism. The result above shows that the eigenfunctions are exponentials composed with the map σ.

Proof (cont'd):

In the case of the map φ above, the eigenvectors are

$$f(z) = e^{r\frac{z+1}{z-1}}$$

and for each complex number r, the corresponding eigenvalues are e^{-2ir}. These functions are in $H^2(\mathbb{D})$ if and only if $r \geq 0$. This means the spectrum of C_φ includes the unit circle, and since the spectral radius of C_φ and C_φ^{-1} are 1, the spectrum is actually equal to the unit circle.

Finally, if $|a| = 1$ and $\varphi'(a) = s < 1$, so that φ is hyperbolic, then without loss of generality, $a = 1$ and the other fixed point of φ is -1. In this case,

$$\varphi(z) = \frac{(1+s)z + 1 - s}{1 + s + (1-s)z}$$

and it has a half-plane dilation model.

Proof (cont'd):

The result above shows the eigenfunctions are powers of the map σ which, in this case, means the eigenfunctions are

$$f(z) = \left(\frac{1-z}{1+z}\right)^r$$

and for each complex number r, the corresponding eigenvalue is s^r. Now the eigenfunctions are in $H^2(\mathbb{D})$ if and only if $-\frac{1}{2} < \operatorname{Re} r < \frac{1}{2}$ so this, together with spectral radius calculations, shows the spectrum is

$$\sigma(C_\varphi) = \{\lambda : \sqrt{s} \leq |\lambda| \leq \frac{1}{\sqrt{s}}\}$$

■

Just as in the automorphism cases, for φ a linear fractional map that is not an automorphism, all the computations needed to determine the spectrum can be carried out explicitly using the linear fractional models.

Examples

(1) (plane dilation) $\varphi(z) = (1+i)z/2, \quad a = 0, \quad \varphi'(a) = (1+i)/2,$

C_φ compact

$$\sigma(C_\varphi) = \{0\} \cup \{\left(\frac{1+i}{2}\right)^n : n = 0, 1, 2, \cdots\}$$

(2) (plane dilation) $\varphi(z) = -z/2 + 1/2, \quad a = 1/3, \quad \varphi'(a) = -1/2,$

C_φ not compact ($\varphi(-1) = 1$), but $C_\varphi^2 = C_{\varphi\circ\varphi}$ is compact

$$\sigma(C_\varphi) = \{0\} \cup \{\left(-\frac{1}{2}\right)^n : n = 0, 1, 2, \cdots\}$$

Examples (cont'd)

(3) (plane dilation) $\varphi(z) = z/(2-z), \quad a = 0, \quad \varphi'(a) = 1/2,$

but also $\varphi(1) = 1$ and $\varphi'(1) = 2$, so C_φ not compact

$$\sigma(C_\varphi) = \{1\} \cup \{\lambda : |\lambda| \leq \frac{1}{\sqrt{2}}\}$$

(4) (half-plane dilation) $\varphi(z) = z/3 + 2/3, \quad a = 1, \quad \varphi'(a) = 1/3,$

so C_φ not compact

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \frac{1}{\sqrt{\varphi'(a)}}\} = \{\lambda : |\lambda| \leq \sqrt{3}\}$$

(5) (plane translation) $\varphi(z) = \dfrac{(2-t)z + t}{-tz + 2 + t}$ for $\operatorname{Re} t > 0, \quad a = 1, \quad \varphi'(1) = 1$

$$\sigma(C_\varphi) = \{e^{\beta t} : \beta \leq 0\} \cup \{0\}$$

The examples from the linear fractional maps give an indication of how the spectra vary depending on the case from the model for iteration – this dependence appears to persist throughout the study of composition operators on spaces of analytic functions.

By far the easiest case to handle is the half-plane dilation case.

Theorem

If φ is an analytic mapping of the unit disk to itself with Denjoy–Wolff point a on the unit circle and $\varphi'(a) < 1$, then for real θ the operator C_φ on $H^2(\mathbb{D})$ is similar to the operator $e^{i\theta}C_\varphi$.

Thus, if λ is in the spectrum of C_φ then for real θ, $e^{i\theta}\lambda$ is also.

(half-plane dilation)

Theorem

If φ, not an inner function, is analytic in a neighborhood of the closed unit disk, maps the disk to itself, and has Denjoy–Wolff point a on the unit circle with $\varphi'(a) < 1$, then for C_φ acting on the Hardy space $H^2(\mathbb{D})$,

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(a)^{-1/2}\}$$

For $\varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2}$, the number λ is always an eigenvalue of infinite multiplicity for C_φ

(plane dilation)

Theorem

Let φ, not an inner function, be analytic in a neighborhood of the closed disk with $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(a) = a$ for some point a with $|a| < 1$.

If C_φ is the associated composition operator on $H^2(\mathbb{D})$, then

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \rho\} \cup \left\{\varphi'(a)^k : k = 1, 2, \ldots\right\} \cup \{1\}$$

where ρ is the essential spectral radius of C_φ.

Theorem

If φ is an inner function with a fixed point in the disk, not an automorphism, then C_φ acting on $H^2(\mathbb{D})$ is similar to an isometry whose unitary part is the identity on a one dimensional space and whose purely isometric part is a unilateral shift of infinite multiplicity. Moreover,

$$\sigma(C_\varphi) = \sigma_e(C_\varphi) = \{\lambda : |\lambda| \leq 1\}$$

Theorem

Let φ be an inner function, not an automorphism, with Denjoy–Wolff point a on the unit circle. Then on $H^2(\mathbb{D})$

$$\sigma(C_\varphi) = \sigma_e(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(a)^{-1/2}\}$$

In the half-plane translation case, we have some information about the spectrum:

Theorem

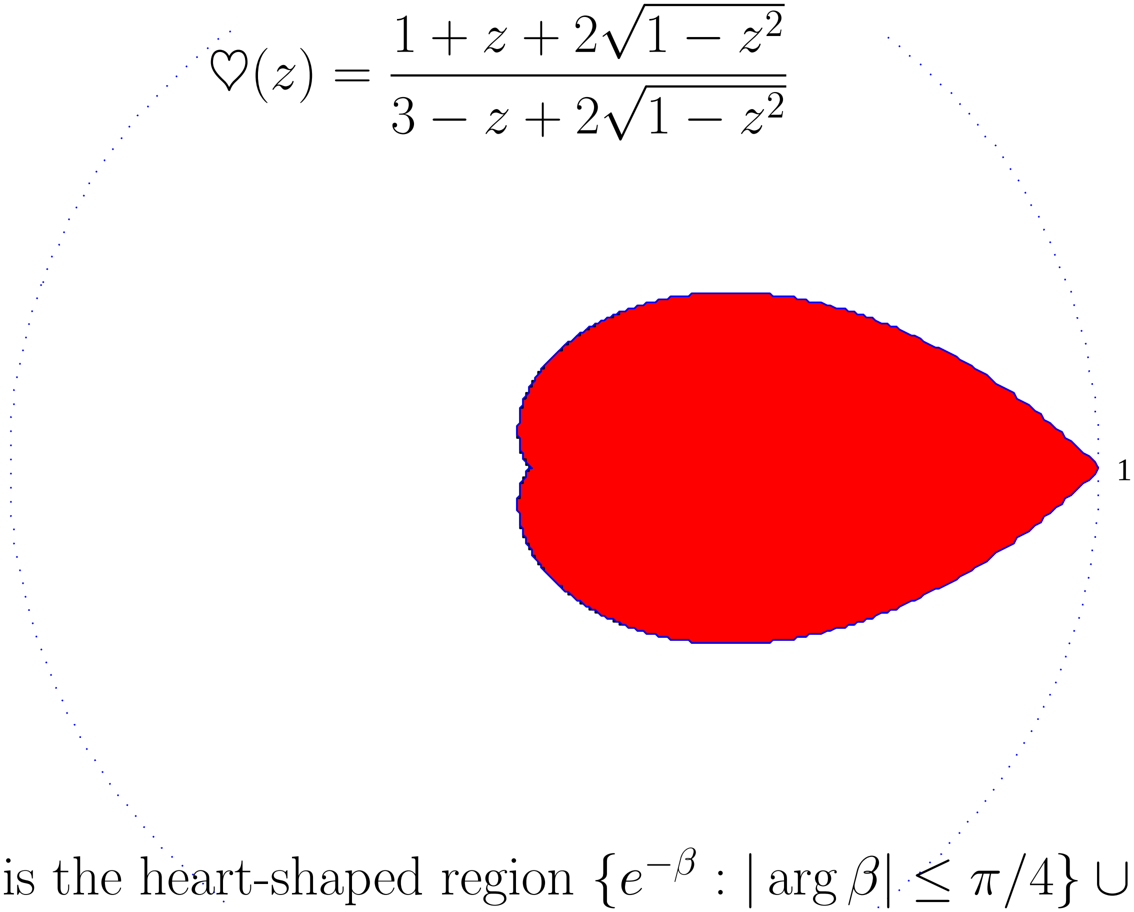
If φ is an analytic mapping of the unit disk with a halfplane/translation model for iteration, then the spectrum and essential spectrum of C_φ on $H^2(D)$ contain the unit circle. Moreover, if λ is an eigenvalue of C_φ, then $e^{i\theta}\lambda$ is also an eigenvalue of C_φ for each positive number θ.

Problem

If φ is in the half-plane translation case, not an automorphism, is $\sigma(C_\varphi)$ always $\{\lambda : |\lambda| \leq 1\}$?

In the plane translation case, we have no information about the spectrum, only a few examples:

$$\heartsuit(z) = \frac{1 + z + 2\sqrt{1 - z^2}}{3 - z + 2\sqrt{1 - z^2}}$$



We find $\sigma(C_\heartsuit)$ is the heart-shaped region $\{e^{-\beta} : |\arg\beta| \leq \pi/4\} \cup \{0\}$

In the plane translation case, the only examples for which we know the spectra are symbols that belong to a semigroup of analytic functions, and the spectrum is computed using semigroup theory.

Problem

If φ is in the plane translation case, is $\sigma(C_\varphi)$ always a union of spirals joining 0 and 1?

Problem

Find the spectrum of C_φ for a function φ in the plane translation case that is not inner, linear fractional, or a member of a semigroup of analytic functions.