Bridging Complex and Functional Analysis

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Linear algebra: Euclidean spaces: $\mathbb{R}^n$, $\mathbb{C}^n$

Problems:

Classify $n \times n$ matrices up to similarity: Jordan Canonical Form

For a given matrix A,

which matrices B satisfy $AB = BA$,

and what subspaces M satisfy $AM \subset M$?

Such a subspace is said to be "an invariant subspace for A"

An important example(!):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The analysis of differential equations necessitated extension

to infinite dimensional spaces:

Hilbert spaces are infinite dimensional Euclidean spaces: $\mathbb{C}^n$ expands to ℓ^2

$$v = (a_0, a_1, a_2, \cdots) \text{ with } \|v\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \text{ and } \langle v, w\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

It is now convenient to insist that $\|Ax\| \leq K\|x\|$ so that the function $x \mapsto Ax$

is continuous: the best value for $K \equiv \|A\|$.

Problems:

Classify operators up to similarity.

For a given operator A,

which operators B satisfy $AB = BA$,

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It is now convenient to insist that $\|Ax\| \leq K\|x\|$ so that the function $x \mapsto Ax$,

a linear operator, is continuous: the best value for $K \equiv \|A\|$.

Problems:

Classify operators up to similarity. (unsolved!)

For a given operator A,

which operators B satisfy $AB = BA$, (unsolved!)

and what subspaces M satisfy $AM \subset M$? (unsolved!)

An important example(!):

$$\text{On } \ell^2 = \{v = (a_0, a_1, a_2, \cdots) : \|v\|^2 = \sum |a_n|^2 < \infty\}$$

the *unilateral shift operator* is:

$$Sv = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}$$

An easy example of an operator that is not self-adjoint, normal, or compact, types of operators with much better understood structure than generic ones.

Let $A : \mathcal{H} \mapsto \mathcal{H}$ be a linear transformation and $\mathcal{H}$ a Hilbert space.

Some terminology:

- A is *bounded* (continuous) if there is K so $\|Ax\| \leq K\|x\|$ for all $x \in \mathcal{H}$
- A is *self-adjoint* if $A = A^*$, a generalization of real-symmetric matrices.
- A is *normal* if $AA^* = A^*A$
- A is *compact* if $A = \lim_{n\to\infty} B_n$ where the B_n have finite dimensional ranges.

The range of a compact operator is "small" in the sense that

the range contains no closed, infinite dimensional subspaces!

In 1949, Beurling published a theorem that gives a complete characterization of ALL the closed subspaces that are invariant for the unilateral shift.

This breakthrough was based on Beurling's understanding of how this interesting operator is connected to complex analysis!!

Defining the Hardy space on the unit disk, $\mathbb{D}$, by

$$H^2(D) = \{f \text{ analytic on } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \|f\|^2 = \sum |a_n|^2 < \infty\}$$

We see $\ell^2 \leftrightarrow H^2$ and $S \leftrightarrow T_z$ where $T_z(f) = zf$

The operators T_ψ, for ψ a bounded analytic function on $\mathbb{D}$ are defined by

$$T_\psi f = \psi f$$

and these operators are continuous with

$$\|T_\psi\| = \|\psi\|_\infty = \sup\{|\psi(z)| : |z| < 1\}$$

Beurling's Theorem. A closed subspace M of H^2 is invariant for T_z if and only if M is the range of T_ψ for some bounded analytic function ψ on $\mathbb{D}$ for which $|\psi(e^{i\theta})| = 1$ almost everywhere.

For a bounded analytic function, ψ, on $\mathbb{D}$, the matrix for T_ψ, w.r.t. basis $\{z^n\}$, is lower triangular and is constant along diagonals:

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

where $\psi(z) = \sum_{j=0}^{\infty} a_j z^j$.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set

$\{A\}' = \{B \in \mathcal{B}(\mathcal{H}) : AB = BA\}$

For example, for T_z on H^2,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \ddots \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} & \cdots \\ b_{10} & b_{11} & b_{12} & \cdots \\ b_{20} & b_{21} & b_{22} & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ b_{00} & b_{01} & b_{02} & \cdots \\ b_{10} & b_{11} & b_{12} & \ddots \end{pmatrix}$$

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This means that $b_{0j} = 0$ for $j \geq 1$ and $b_{i,j} = b_{i+1,j+1}$ for $i, j \geq 0$

In particular, letting $a_i = b_{i0}$, we see the matrix is lower triangular and is constant along diagonals:

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

This is T_ψ for $\psi(z) = \sum_{j=0}^{\infty} a_j z^j$ where $\|\psi\|_\infty = \|T_\psi\|$.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set

$\{A\}' = \{B \in \mathcal{B}(\mathcal{H}) : AB = BA\}$

We have seen for T_z on H^2,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

This means we can rephrase Beurling's Theorem about the invariant subspaces of the unilateral shift S:

Beurling's Theorem. Let M be a closed subspace of ℓ^2.

Then M is an invariant subspace for the unilateral shift if and only if M is the range of some bounded operator T on ℓ^2 such that $ST = TS$.

By the 1970's, there was interest in the more general question,

For ψ in H^∞ and T_ψ an operator on H^2, what is $\{T_\psi\}'$?

It turned out the descriptions of $\{T_\psi\}'$ involve a different class of operators:

composition operators.

Let Ω be a domain in $\mathbb{C}$ or $\mathbb{C}^N$ and suppose $\mathcal{H}$ is a Hilbert space of

analytic functions on Ω.

If φ is an analytic map of Ω into itself,

the *composition operator* C_φ is the operator on $\mathcal{H}$ given by

$$C_\varphi f = f \circ \varphi$$

Goal: relate the function–theoretic properties of φ to the operator–theoretic

properties of C_φ.

Some well known examples of Hilbert spaces of analytic functions are:

For example, Hardy spaces

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int_0^{2\pi} |f_r|^2 \frac{d\theta}{2\pi} < \infty\}$$

$$H^2(\mathbf{B}_N) = \{f \text{ analytic in } \mathbf{B}_N : \sup_{0<r<1} \int_{\partial \mathbf{B}_N} |f_r|^2 \, d\sigma_N < \infty\}$$

and Bergman spaces

$$A^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \int_{\mathbb{D}} |f(z)|^2 \frac{dA}{\pi} < \infty\}$$

$$A^2(\mathbf{B}_N) = \{f \text{ analytic in } \mathbf{B}_N : \int_{\mathbf{B}_N} |f(z)|^2 \, d\nu_N < \infty\}$$

If φ is an analytic map of Ω into itself,

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Goal: relate the function–theoretic properties of φ to the operator–theoretic properties of C_φ.

Theorem. *If φ is an analytic map of the disk into itself, then C_φ is bounded on $H^2(\mathbb{D})$ and*

$$\|C_\varphi\| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2}$$

On medium sized spaces, like H^2 and A^2, all composition operators are bounded.

Definition: If A is a bounded operator on a Hilbert space $\mathcal{H}$, the *spectrum of* A, denoted $\sigma(A)$, is

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \quad \text{does not have a continuous inverse}\}$$

This is a generalization of the eigenvalues of an $n \times n$ matrix, but in infinite dimensional spaces, there are often points in $\sigma(A)$ that are *not* eigenvalues of A.

For any bounded operator, $\sigma(A)$ is a non-empty, compact subset of the plane.

For example, it is easy to prove that the unilateral shift, S, on ℓ^2 does *not* have any eigenvalues, but

$$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

Spectra of composition operators are related to the nature of the fixed points of φ.

We will say b in the closed disk is a fixed point of φ if

$$\lim_{r \to 1^-} \varphi(rb) = b$$

If b is a fixed point of φ in the closed disk, then

$$\lim_{r \to 1^-} \varphi'(rb)$$

exists and we denote it by $\varphi'(b)$.

Theorem (Denjoy, Wolff, 1926).

If φ is an analytic map of the disk into itself, not an automorphism, then there is a unique fixed point a in the closed disk for which $|\varphi'(a)| \leq 1$. Moreover,

$$\lim_{n \to \infty} \varphi_n(z) = a$$

for all z in the open disk, uniformly on compact sets.

The point a of the theorem above will be called the Denjoy-Wolff point of φ.

Model for iteration of analytic functions mapping unit disk into itself.

Maps of the disk into itself are like linear fractional maps.

Let φ *be an analytic map of the unit disk* $\mathbb{D}$ *into itself, not an automorphism of the disk.*
Suppose that either φ *does not have a fixed point in* $\mathbb{D}$ *or that* $\varphi'(a) \neq 0$ *for the fixed point* a *in* $\mathbb{D}$.
Then there is a domain Δ*, either the plane or a half-plane, an automorphism* Φ *of* Δ *onto* Δ*, and a mapping* σ *of* $\mathbb{D}$ *into* Δ *such that*

$$\sigma \circ \varphi = \Phi \circ \sigma$$

Four distinct cases in the model:

If φ has a fixed point in $\mathbb{D}$:

- *(plane/dilation)* $\Delta = \mathbb{C}$, $\Phi(z) = \alpha z$

If φ has no fixed points in $\mathbb{D}$:

- *(half-plane/dilation)* $\Delta = \{\text{Re } z > 0\}$, $\Phi(z) = \alpha z$
- *(plane/translation)* $\Delta = \mathbb{C}$, $\Phi(z) = z + 1$
- *(half-plane/translation)*

  $\Delta = \{\text{Im } z > 0\}$, $\Phi(z) = z \pm 1$

Some applications of the model:

- Better understanding of iteration of the function φ, including questions about embeddability of the discrete semi-group of iterates of φ into a continuous semi-group

- Determination of the functions ψ mapping the disk into the disk that satisfy

$$\psi \circ \varphi = \varphi \circ \psi$$

- Determination of the eigenvectors and eigenvalues of composition operators on spaces of analytic functions on the disk

- Determination of the spectrum of composition operators on spaces of analytic functions on the disk

Spectra of C_φ:

C_φ is invertible if and only if φ is an automorphism

C_φ is compact (or power compact) implies $|a| < 1$ and

$$\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\varphi'(a)^n : n = 1, 2, \cdots\}$$

$|a| < 1$ and smoothness hypotheses

$$\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\varphi'(a)^n : n = 1, 2, \cdots\} \cup \{z : |z| \leq \rho\}$$

$|a| = 1$, $\varphi'(a) < 1$

$$\sigma(C_\varphi) = \{z : |z| \leq \varphi'(a)^{-1/2}\}$$

Less is known:

$|a| = 1$, $\varphi'(a) = 1$, half-plane translation

$|a| = 1$, $\varphi'(a) = 1$, plane translation

Problem: Explain the circular symmetry of the spectra of C_φ:

(Cowen, 1983)

If φ is a map of the disk into itself with $|a| = 1$ and $\varphi'(a) < 1$, then on $H^2(\mathbb{D})$,

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \varphi'(a)^{-1/2}\}$$

Moreover, C_φ is similar to $e^{i\theta} C_\varphi$ for each real number θ.

Some omitted topics:

- Adjoints of C_φ.
- Topology of the set of composition operators.
- Cyclicity, hypercyclicity, etc. of C_φ and C_φ^*.
- Normality, subnormality, hyponormality of C_φ and C_φ^*.
- Similarity and unitary equivalence of C_φ and C_ψ.

Composition operators in several variables

Still many mysteries in several variables... even boundedness is problematic.

Wogen (1988) gave necessary and sufficient conditions for a smooth map to give a bounded operator on $H^2(\mathbf{B}_N)$.

For example,

$$\varphi(z_1, z_2) = (\frac{5}{9} + \frac{5}{9}z_1 - \frac{1}{9}z_1^2 + \frac{1}{6}z_2^2,\ \frac{1}{5}z_2^2)$$

is a map of $\mathbf{B}_2$ into $\mathbf{B}_2$ that gives unbounded composition operator on $H^2(\mathbf{B}_2)$.

On the other hand, some things carry over to several variables.

Theorem (MacCluer, 1984)

If C_φ is compact on $H^2(\mathbf{B}_N)$, then φ has an attractive fixed point a in $\mathbf{B}_N$. Moreover, the spectrum of C_φ is

$$\sigma(C_\varphi) = \{0\} \cup \{1\} \cup \{\textit{all products of eigenvalues of } \varphi'(a)\}$$

If φ is an analytic map of $\mathbf{B}_N$ into itself, $\varphi(0) = 0$, and φ is not unitary on a slice, then

$$\sigma(C_\varphi) \supset \{\lambda : |\lambda| \leq \rho\}$$

where ρ is computed in terms of the essential spectral radius of C_φ and a constant depending on the local behavior of φ.

Some broad areas for investigation:

What can you say about the spectrum of C_φ if φ has no fixed point in $\mathbf{B}_N$?

What effect do degeneracies of φ have on the structure of C_φ? For example, if $\varphi(\mathbf{B}_N) \subset \mathbf{B}_N \cap \{(w_1, 0)\}$ and C_φ is bounded, what is the structure of C_φ?

For example, if

$$\varphi(z_1, z_2) = (2z_1z_2, 0) \quad \text{or} \quad \varphi(z_1, z_2) = (z_1^2 + z_2^2, 0)$$

then C_φ is unbounded, but if

$$\varphi(z_1, z_2) = (z_1z_2, 0)$$

then C_φ is compact.

Similarly, what if $\varphi(z_1, z_2) = (z_1\psi(z_1, z_2), z_2)$ which is unitary on the slice $z_1 = 0$, or what if $\varphi(z_1, z_2) = (\psi_1(z_1), \psi_2(z_2))$?

We need a better understanding of maps of the ball into the ball, for example, we need to have a substitute for the "Model for Iteration" for several variables.

What is a 'nice' class of functions of $\mathbf{B}_N$ into itself? Given φ, can we find a 'nice' map that is 'like' φ.

Most important reason to study composition operators:

They are operators that have very different structures

than other better known classes of operators!

I believe their structures better represent the structures of generic operators.

For example, composition operators can be used to study general operators:

Definition: An operator U is called a *universal operator* on H^2, if for every continuous T on a separable, infinite dimensional Hilbert space, there is a subspace M of H^2 and a number $\alpha > 0$ so that the restriction of U to M is similar to αT.

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Let f be the function $f(z) = (z + 1)/2$.

Theorem:(Cowen, Gallardo, 2012)

There are an analytic function, ψ, on the disk and an analytic map, φ, of the disk into itself so that T_ψ^* is a universal operator and, for $W_{f,\varphi} = T_f C_\varphi$, the operator $W_{f,\varphi}^*$ is a compact operator commuting with T_ψ^*.

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Corollary:

If A is any bounded operator on an infinite-dimensional Hilbert space $\mathcal{H}$, then there is a non-zero compact operator R so that the range of R is invariant for A.

Thank You!

Slides available: `http://www.math.purdue.edu/~cowen`

Of course, the points $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ depend on α, so we might write them as $\alpha = \beta_1(\alpha), \beta_2(\alpha), \cdots, \beta_n(\alpha)$.

In fact (!), if B is a finite Blaschke product of order n and α is a point of the disk that is *NOT* one of the $n(n-1)$ points of the disk for which $B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

the maps $\alpha \mapsto \beta_j(\alpha)$ *are just the* n *branches of the analytic function* $B^{-1} \circ B$ *that is defined and arbitrarily continuable on the disk with the* $n(n-1)$ *exceptional points removed.*

Theorem: (Cowen, 1974)

For B *a finite Blaschke product, the branches of* $B^{-1} \circ B$ *form a group whose normal subgroups are associated with compositional factorizations of* B *into compositions of two Blaschke products.*

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Theorem: (C. & Wahl, 2012). *For B a Blaschke product, and $W \sim B^{-1} \circ B$:*

If T is a bounded operator on A^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in A^2,

$$(Tf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta,\alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{1}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and $\psi_1, \psi_2, \cdots, \psi_n$ is a basis for $\left((B - B(\alpha_0))\, H^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta,\alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \ \text{ has a zero of order } m \text{ at } \alpha_0 \tag{2}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has properties (4) at each zero of B', then (3) defines a bounded linear operator on A^2 with T in $\{T_B\}'$.