Adjoints of Rational Composition Operators on the Hardy Space

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joint work with Eva Gallardo Gutiérrez, Zaragoza, Spain

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If $\mathcal{H}$ is a Hilbert space of complex-valued analytic functions on the domain Ω in $\mathbb{C}$ or $\mathbb{C}^N$ and φ is an analytic map of Ω into itself, the *composition operator* C_φ on $\mathcal{H}$ is the operator given by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{for } f \text{ in } \mathcal{H}$$

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Goal:

relate the properties of φ as a function with properties of C_φ as an operator.

Today, I'll consider the Hardy Hilbert space, H^2,

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which is also described as

$$H^2 = \{f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

and for f in H^2,

$$\|f\|^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2$$

For H^2, the Littlewood subordination theorem implies that

C_φ is bounded for all functions φ that are analytic and map $\mathbb{D}$ into itself.

That theorem plus some easy change of variable calculations to find the norm of composition operators induced by automorphisms of the disk, yield the following estimate of the norm for composition operators on H^2:

$$\|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{2}}$$

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This is the sort of result we seek, connecting the properties of the operator C_φ with the analytic and geometric properties of φ.

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While the whole story is not known, much progress has been made $\cdots$

If A is an operator on a Hilbert space $\mathcal{H}$,

A^*, the *adjoint of* A, is the operator on $\mathcal{H}$ that satisfies

$$\langle Av, w\rangle = \langle v, A^*w\rangle$$

for all vectors v and w in $\mathcal{H}$.

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It has been difficult to identify the adjoint operator, C_φ^*.

Some special formulas have been useful!

For α in $\mathbb{D}$, let $K_\alpha(z) = \dfrac{1}{1 - \overline{\alpha} z}$

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K_α is called the *kernel for evaluation at* α for H^2.

For f in H^2,

$$\langle f, C_\varphi^* K_\alpha \rangle = \langle C_\varphi f, K_\alpha \rangle = \langle f \circ \varphi, K_\alpha \rangle = f(\varphi(\alpha)) = \langle f, K_{\varphi(\alpha)} \rangle$$

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Since this is true for all f in H^2, this means

$$C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$$

for all α in the open disk.

This calculation can be used to prove the following result on adjoints:

Theorem. [C, 1988]

If $\varphi(z) = \dfrac{az+b}{cz+d}$ *is a non-constant linear fractional map of the unit disk into itself, then*

$$C_\varphi^* = T_g C_\sigma T_h^*$$

where $\sigma(z) = \dfrac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$, $g(z) = \dfrac{1}{-\overline{b}z + \overline{d}}$, *and* $h(z) = cz + d$.

Extensions.

The inner case.

Consequences.

C_φ^* can be explicitly described as an integral operator,

and it is a "folk theorem".

For z in $\mathbb{D}$,

$$(C_\varphi^* f)(z) = \langle C_\varphi^* f, K_z \rangle = \langle f, C_\varphi K_z \rangle = \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})} z} \frac{d\theta}{2\pi}$$

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In her 1997 thesis, Rebecca Wahl used these ideas to get information about the kernels of C_φ^* for $\varphi(z) = \dfrac{z^2}{az+b}$ and to show that hyponormality of C_φ and C_φ^* are impossible in this case.

We were able to find a formula for C_φ^* for $\varphi(z) = (z + z^2)/2$:

$$(C_\varphi^* f)(z) = \frac{z + \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} f\left(\frac{z + \sqrt{z^2 + 8z}}{4}\right) - \frac{z - \sqrt{z^2 + 8z}}{2\sqrt{z^2 + 8z}} f\left(\frac{z - \sqrt{z^2 + 8z}}{4}\right)$$

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BUT, this does not make sense because $\sqrt{z^2+8z}$ has a singularity at $z=0$.

On the other hand, the formula as a whole DOES make sense for every f in H^2 and defines $C_\varphi^* f$ as a single-valued analytic function!

Definition.

Let K be a finite subset of $\mathbb{D}$. Suppose σ is an n-valued function that is arbitrarily continuable in $\mathbb{D} \setminus K$ and takes values in $\mathbb{D}$. Suppose ψ is an m-valued analytic function defined and arbitrarily continuable in $\mathbb{D} \setminus K$ with values in $\mathbb{C}$ and bounded. *Plus compatibility condition. Plus finiteness condition.*

Then the *multiple valued weighted composition operator* $W_{\psi,\sigma}$ is defined by

$$(W_{\psi,\sigma} f)(z) = \sum_{\textit{all branches}} \psi(z) f(\sigma(z))$$

for z in $\mathbb{D} \setminus K$ and extended to $\mathbb{D}$.

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In the example, $\psi(z) = \dfrac{z \pm \sqrt{z^2 + 8z}}{\pm 2\sqrt{z^2 + 8z}}$ and $\sigma(z) = \dfrac{z \pm \sqrt{z^2 + 8z}}{4}$

Definition. (Compatibility condition.)

Suppose K, σ, and ψ are as in the preceding definition. Choose b a base point in $\mathbb{D} \setminus K$. We say (σ, ψ) *is a compatible pair* if for any loop at b in $\mathbb{D} \setminus K$ along which each branch of σ continues to itself, then every branch of ψ continues to itself on that path.

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For example, if $\sigma(z) = \sqrt[3]{z}$ and $\psi(z) = \sqrt{z}$ in $\mathbb{D} \setminus \{0\}$, then (σ, ψ) is *not* a compatible pair.

In particular, if (σ, ψ) is a compatible pair, the multiplicity of ψ divides the multiplicity of σ.

Definition. (Finiteness condition.)

Suppose K, σ, and ψ are as in the earlier definition. For any point c in K, and any branch σ_j of σ, the cluster set of σ_j near c does not intersect the boundary of $\mathbb{D}$ and for each branch ψ_j of ψ

$$\lim_{z \to c} \psi_j(z)(z-c) = 0$$

These conditions ensure that $(W_{\psi,\sigma}f)(z)$ is bounded on every compact subset of $\mathbb{D}$ and therefore that all the singularities at points of K are removable so that $W_{\psi,\sigma}f$ is analytic in $\mathbb{D}$.

Moreover, if ψ is bounded, that is, there is $M < \infty$ so that

$$\limsup_{|z|\to 1^-} |\psi_j(z)| \leq M$$

for each branch of ψ, then $W_{\psi,\sigma}$ is bounded on H^2 and

$$\|W_{\psi,\sigma}\| \leq M\sqrt{n\sum \frac{1+|\sigma(0)|}{1-|\sigma(0)|}}$$

Notation.

For g analytic on $\mathbb{D} \setminus \{0\}$, let

$$\widetilde{g}(z) = \overline{g\left(\frac{1}{\overline{z}}\right)}$$

which is analytic on $1 < |z| < \infty$.

Theorem.

Let φ *be a rational map of* $\mathbb{D}$ *into itself. Then for any* f *in* H^2

$$(C_\varphi^* f)(z) = (BW_{\psi,\sigma} f)(z)$$

where B *is the backward shift operator and* $W_{\psi,\sigma}$ *is multiple valued weighted composition operator induced by* $\sigma(z) = \dfrac{1}{\widetilde{\varphi^{-1}}}$ *and*

$$\psi(z) = \frac{\widetilde{(\varphi^{-1})'}}{\widetilde{\varphi^{-1}}}$$

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$$\psi(z) = \frac{\widetilde{(\varphi^{-1})'}}{\widetilde{\varphi^{-1}}}$$

Corollary.

f is in $\ker C_\varphi^*$ if and only if

$$(W_{\psi,\sigma} f)(z) = \sum \psi(z) f(\sigma(z)) = \sum \psi(0) f(\sigma(0))$$

Based on the paper by Cowen and Gallardo Gutiérrez

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http://www.math.iupui.edu/˜ccowen/Downloads.html