Professor Carl Cowen **Math 54600** **Fall 09**

PROBLEMS

1. (Geometry in Inner Product Spaces)
(a) (*Parallelogram Law*) Show that in any inner product space

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(b) (*Polarization Identity*) Show that in any inner product space

$$<x,y> = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

which expresses the inner product in terms of the norm.
(c) Use (a) or (b) to show that the norm on $C([0,1])$ does not come from an inner product. (c.f. Exercise 1.13 of MacCluer.)

2. In a normed linear space, is the closed ball $\{x : \|x - x_0\| \leq r\}$ the closure of the open ball $\{x : \|x - x_0\| < r\}$? (Note that this is not always the case for a metric space, for example, it is not true for the integers with the usual metric!)

3. Suppose $\{B_n\}_{n=1}^{\infty}$ is a sequence of closed balls (not necessarily concentric) in a Banach space such that $B_1 \supset B_2 \supset B_3 \supset \cdots$. Prove that

$$\bigcap_{n=1}^{\infty} B_n \neq \emptyset.$$

* **4.** Is the corresponding statement true if we replace "closed balls" with "closed, bounded, non-empty convex sets" in problem 3?

5. (Two useful lemmas about metric spaces.) Let Ω be a metric space.
(a) Suppose (x_n) is a Cauchy sequence in Ω and (x_{n_k}) is a subsequence that converges to y. Then $y = \lim_{n\to\infty} x_n$.
(**Corollary.** Ω is complete if and only if every Cauchy sequence has a convergent subsequence.)
(b) Suppose (x_n) is a sequence in Ω. If every subsequence (x_{n_k}) has a subsequence $(x_{n_{k_j}})$ that converges to y, then $y = \lim_{n\to\infty} x_n$.

6. Show that $C([0,1])$, the collection of continuous (complex valued) functions on the interval $[0,1]$, is a Banach space with the supremum norm. (c.f. Exercise 1.2 of MacCluer.)

7. Let $C^1([0,1])$ be the collection of continuous (complex valued) functions on the interval $[0,1]$ that have a continuous first derivative.
(a) Show that $C^1([0,1])$ is not a Banach space with the supremum norm.
(b) But, show that $C^1([0,1])$ is a Banach space with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$.
(c.f. Exercise 1.3 of MacCluer.)

8. Show that the normed linear space obtained by putting the norm

$$\|f\| = \int_0^1 |f(t)|dt$$

on the set of continuous functions on $[0,1]$ is not a Banach space.

9. Show that ℓ^1 is a Banach space, that is, show that it is complete. (c.f. Exercise 1.4 of MacCluer.)

10. (Two more Banach spaces)

(a) Let c denote the vector space of convergent sequences of complex numbers. Show that if $x = (x_1, x_2, \cdots)$ then $\|x\| = \sup\{|x_j| : j = 1, 2, \cdots\}$ makes c into a Banach space.

(b) Let c_0 denote the vector space of sequences of complex numbers that converge to zero. With the norm as in (a), show that c_0 is a Banach space.

(c) Show that c is isomorphic to c_0. (compare with problem 40.)

11. Let X be a separable Banach space. A sequence of vectors $v_1, v_2, v_3, \cdots$ is an *(ordered) basis* for X if for each x in X there are unique scalars $\alpha_1, \alpha_2, \alpha_3, \cdots$ such that

$$x = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j v_j.$$

[Such bases are also called *Schauder bases*]

(a) Show that ke_k is a basis for ℓ^p, for $1 \le p < \infty$.

(b) Find a basis for c (problem 10).

(c) Show that $1, x, x^2, x^3, \cdots$ is not a basis for $C[0, 1]$.

***12.** Let X be a separable Banach space. A sequence of vectors $v_1, v_2, v_3, \cdots$ is called an *unconditional basis* for X if for each x in X there are unique scalars $\alpha_1, \alpha_2, \alpha_3, \cdots$ such that for any $\epsilon > 0$, there is a finite set F of positive integers such that if $\tilde{F}$ is any finite set of integers with $\tilde{F} \supset F$, then

$$\| \sum_{j\in\tilde{F}} \alpha_j v_j - x\| < \epsilon$$

(Note (see problem 38) that this is convergence in a net!)

(a) Show that an unconditional basis for X is an ordered basis.

(b) Show that e_k is an unconditional basis for ℓ^p for $1 \le p < \infty$.

(c) Let $M = \{x \in \ell^1 : \sum_{i=1}^{\infty} x_i = 0\}$. Show that if $v_k = e_k - e_{k+1}$, then $\{v_k\}_{k=1}^{\infty}$ is an ordered basis for M, but not an unconditional basis.

13. Prove that $C(\Omega)$ (example 10 from class) is a Frechet space.

14. A Banach space is *separable* if it has a countable dense subset. Prove that ℓ^p is separable for $1 \le p < \infty$, but that ℓ^∞ is not.

15. A subset E of a topological vector space is *bounded* if for every neighborhood V of 0, there is a number λ so that $\lambda V \supset E$. Show that if A and B are bounded sets, then so is $A + B$. (Here $A + B = \{a + b : a \in A \text{ and } b \in B\}$).

16. Suppose (X, μ) is a finite measure space. For f and g measurable functions on X, let

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|} d\mu.$$

(a) Show that ρ is a pseudometric on the space of measurable functions on X, and that $\rho(f,g) = 0$ if and only if $f = g$ almost everywhere $d\mu$.
(b) Show that $\rho(f_n, f) \to 0$ if and only if f_n converges to f in measure.
The metric space obtained by taking equivalence classes of functions equal almost everywhere is sometimes called L^0.
(c) Show that $L^0([0,1])$ is an F-space but is not locally convex.

17. Let X be a locally compact, normal, Hausdorff space (or take X to be the real numbers if you wish). Let $C_c(X)$ denote the set of continuous functions on X with compact support, that is,

$$C_c(X) = \{f \in C(X) : \text{ the closure of } \{x : |f(x)| > 0\} \text{ is compact}\}.$$

and let $C_0(X)$ denote the set of functions on X that “vanish at infinity”, that is,

$$C_0(X) = \{f \in C(X) : \text{ for all } \epsilon > 0, \{x : |f(x)| \geq \epsilon\} \text{ is compact}\}.$$

Show that if $\|f\| = \sup\{|f(x)| : x \in X\}$, then $C_0(X)$ is a Banach space and that $C_c(X)$ is dense in $C_0(X)$.

18. (See problem 8.) Let X be the normed linear space of continuous functions on [0,1] with the norm $\|f\| = \int_0^1 |f(t)|dt$. Show that the linear functional $\Lambda f = f(\frac{1}{2})$ is not bounded.

19. (On linear functionals and local convexity.)
(a) Show that if $T : X \to Y$ is linear and $K \subset Y$ is convex, then $T^{-1}(K)$ is convex.
(b) Prove that if X is a topological vector space whose only open convex sets are $\emptyset$ and X, then 0 is the only continuous linear functional on X.

20. (On linear functionals and local convexity, continued.) Suppose $0 < p < 1$.
(a) Find a separating family of continuous linear functionals for ℓ^p, that is, find a family of linear functionals $\mathcal{F}$ such that for any two vectors x and y in ℓ^p, there is f in $\mathcal{F}$ so that $f(x) \neq f(y)$.
(b) Give an example of a non-trivial open convex subset of ℓ^p.
(c) Is ℓ^p locally convex?

21. (a) Let X be a compact Hausdorff space. We say a linear functional Λ, on C(X) is a *positive linear functional* if $\Lambda f \geq 0$ whenever $f \geq 0$. Show that a positive linear functional on C(X) is bounded. [Hint: One way is to use two versions of the Riesz Representation Theorem from Math 545.]
(b) The result of part (a) is false if we only require X to be locally compact: give an example of a positive linear functional, Λ, on $C_c(\mathbf{R})$ and a positive function f in $C_0(\mathbf{R})$ such that there is a sequence f_n (in $C_c(\mathbf{R})$) converging to f uniformly and $\Lambda(f_n)$ is an unbounded sequence.

22. If X is a normed linear space, the *packing diameter* of X (an ad hoc term) is the least number α such that

$$\{x_j\}_{j=1}^{\infty} \subset \overline{B(0,1)} \Rightarrow \inf_{1 \le n < m < \infty} \{\|x_n - x_m\|\} \le \alpha.$$

(This is the largest diameter of infinitely many non-overlapping balls whose centers are in $\overline{B(0,1)}$.)
(a) Show that the packing diameter of ℓ^1 is 2.
(b) Show that the packing diameter of ℓ^∞ is 2.
(c) Find the packing diameter of ℓ^2.

23. *(counts as 2 problems!)*
Let Σ be the algebra of subsets E of the natural numbers $\mathbf{N}$. Let S be the space of all finitely additive, complex valued, set functions that are bounded: that is, μ in S means
(i) $\mu(\emptyset) = 0$,
(ii) $\sup\{|\mu(E)| : E \subset \mathbf{N}\} < \infty$,
(iii) $\mu(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$ whenever $E_1, \cdots, E_n$ are disjoint elements of Σ.
S is a linear space with the operations

$$(\alpha_1\mu_1 + \alpha_2\mu_2)(E) = \alpha_1\mu_1(E) + \alpha_2\mu_2(E)$$

for all $\mu_1, \mu_2 \in S$, all complex numbers α_1, α_2, and $E \in \Sigma$.
(a) Prove that for $\mu \in S$, the number

$$\|\mu\| = \sup\{\sum_{i=1}^n |\mu(E_i)| : \mathbf{N} = \cup_{i=1}^n E_i; E_i \text{ disjoint}\}$$

is finite, and that this norm makes S into a Banach space.
(b) Prove that S is isometrically isomorphic to $(\ell^\infty)'$.
(c) Let J be the canonical isometry of ℓ^1 into $(\ell^\infty)'$. Prove that $\mu \in J(\ell^1)$ if and only if μ is countably additive.

24. (See problem 10.) Show that the dual space of c is isometrically isomorphic to ℓ^1.

25. (Two embedding theorems from the geometric theory of Banach spaces.)
(a) Let X be $\mathbf{C}^2$ with the infinity norm, that is, $\|(x,y)\| = \max\{|x|, |y|\}$. Prove that there is no linear isometry T from X into ℓ^2.
(b) Let X be a separable Banach space. Find a linear isometry S from X into ℓ^∞. This shows that every separable Banach space is isometrically isomorphic to a closed subspace of ℓ^∞.

26. Classify the isometries of $\ell^p(n)$ for $1 \le p \le \infty$, with $p \ne 2$ and $n < \infty$. (I.e. Find necessary and sufficient conditions for a linear transformation on $\mathbf{C}^n$ to be isometric as a mapping of $\ell^p(n)$.)

27. (on Bi–holomorphic maps) Let B_j, for $j = 1, 2$, be bounded, open, balanced, convex sets in C^N, and let p_j be their Minkowski functionals. (So p_1 and p_2 are norms on C^N and B_1 and B_2 are their open unit balls.)

(a) (Schwarz Lemma) Writing D for the open unit disk and B_1' for the unit ball in the dual of the Banach space C^N with norm p_1, show that

$$\overline{B_1'} = \{\varphi'(0) : \varphi \text{ is holomorphic and } \varphi(B_1) \subset D\}$$

(b) Show that if f is a bi–holomorphic map of B_1 onto B_2, then f is a linear transformation.

(c) Show that the unit ball in C^N

$$\{z : |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 < 1\}$$

is not bi–holomorphically equivalent to the unit polydisk

$$\{z : |z_j| < 1, \text{ for } j = 1, 2, \cdots, N\}$$

***28.** Let X be a real, locally convex topological vector space. A set $K \subset X$ is called a *cone* if

(i) $x \in K$ and $\alpha \geq 0$ imply $\alpha x \in K$,

(ii) $x, y \in K$ implies $x + y \in K$,

(iii) $x \in K, x \neq 0$ implies $-x \notin K$.

(Examples: first quadrant in $\mathbf{R}^2$, or $\{f \in C[0,1] : f(t) \geq 0 \text{ for } 0 \leq t \leq 1\}$.)

A linear functional Λ is said to be *positive* (with respect to K) if $\Lambda x \geq 0$ for all $x \in K$. (See problem 21.)

Prove Krein's Theorem:

Let K be a cone in X (as above) and M a subspace of X such that M contains a point of the interior of K. If λ is a positive linear functional on M (with respect to $K \cap M$), then there is a positive linear functional Λ on X (with respect to K) such that $\Lambda(x) = \lambda(x)$ for all x in M.

(Note: This can be used to "invent" Lebesgue measure. Let X be the space of real valued Borel measurable functions on $[0,1]$ with essential sup norm, $M = C[0,1]$, and $K = \{f \in X : f(t) \geq 0 \text{ for } 0 \leq t \leq 1\}$. Let $\lambda(f) = \int_0^1 f(t)dt$, the *Riemann integral,* for f in M. Then Λ is the Lebesgue integral.)

29. (A linear algebra theorem.)

Let V be a complex vector space, let n be a positive integer, and let λ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be linear functionals on V. If ζ is a linear functional on V, $\ker(\zeta) = \{v \in V : \zeta(v) = 0\}$, as usual. Prove: λ is a linear combination of $\lambda_1, \lambda_2, \cdots, \lambda_n$ if and only if

$$\ker(\lambda) \supset \bigcap_{j=1}^{n} \ker(\lambda_j)$$

30. Let X be normed linear space. Show that if $\dim(X) = \infty$, then the weak topology on X is strictly weaker than the norm topology (i.e. show that every weak-open set is norm-open, but there are norm-open sets that are not weak-open).

31. Let X be a Banach space and let T be a linear operator on X. Show that T is norm continuous if and only if it is weakly continuous.

32. Show that if X is a reflexive Banach space and M is a closed subspace of X then X/M is reflexive.

33. Let X be a compact Hausdorff space. Show that $C(X)$ is reflexive if and only if X is finite.

34. (a) Show that the linear functional $\lambda(f) = f(1/2)$ is bounded on the space $C[0,1]$, and $\|\lambda\| = 1$.
(b) $C[0,1]$ is a closed subspace of $L^\infty([0,1])$ so there is Λ in $L^\infty([0,1])'$ so that $\|\Lambda\| = 1$ and $\Lambda(f) = \lambda(f)$ for f continuous. Show that this Λ is not in $J(L^1)$ where J is the canonical isometry of L^1 into $L^{\infty\prime}$. Conclude that L^∞ is not reflexive.

Definition. A normed linear space X is said to be *strictly convex* if $\|x + y\| = \|x\| + \|y\|$ implies x and y are linearly dependent.
(Note: It follows from the equality statement in Minkowski's inequality that L^p and ℓ^p are strictly convex for $1 < p < \infty$.

35. Show that X is strictly convex if and only if every x of norm 1 is an extreme point of $\overline{B(0,1)}$.

36. Let X be a reflexive space. Show that X' is strictly convex if and only if for every $x_0 \neq 0$, the linear functional $\lambda(tx_0) = t\|x_0\|$ on the subspace $M = \{tx_0 : t \in \mathbf{C}\}$ has a unique extension of norm 1 defined on all of X.

Definition: A *directed set* is a set J together with an order relation $\prec$ such that

- $j \prec j$ for each j
- $i \prec j$ and $j \prec k$ implies $i \prec k$
- for any i and j in J, there is k in J so that $i \prec k$ and $j \prec k$.

A *net* in a space X is a function whose values are in X and whose domain is a directed set. (As for sequences, the function is usually written x_j rather than $x(j)$.) We say the net x_j converges to y in X, that is,

$$\lim_{j \in J} x_j = y$$

if for every open set U containing y, there is j_U in J so that $j_U \prec j$ implies $x_j \in U$. The usual order on the natural numbers is a directed set whose nets are called "sequences". Indeed, the point of the definition of nets is to extend the ideas relating the topology of the real line and convergence of sequences to topologies in which there is not a countable base for the neighborhoods of points.

If x_j for j in J is a net, then y_k for k in the directed set K is a *subnet* of x_j if there is a function $\sigma : K \mapsto J$ so that $y_k = x_{\sigma(k)}$ (or as is more customary, $y_k = x_{j_k}$) and for each j in J, there is k_0 in K so that $k_0 \prec k$ implies $j \prec j_k$. (Thus, every subnet of a convergent net converges to the same limit!)

***37.** Find a directed set Γ and a net x_γ in ℓ^2 (or any other Banach space) such that the weak limit of x_γ is 0 but such that the limit of $\|x_\gamma\|$ is ∞.

38. (a) Let $\mathcal{F}$ denote the collection of non–empty, finite subsets of the natural numbers with the ordering $F \prec G$ if $F \subset G$. Show that $\mathcal{F}$ is a directed set.

(b) Let $\mathcal{P}$ denote the collection of finite subsets of $[0,1]$ that contain 0 and 1 with the ordering $P \prec Q$ if $P \subset Q$. Show that $\mathcal{P}$ is a directed set.

(c) Let x be a point of the topological space X and let $\mathcal{N}$ denote the collection of open sets of X containing x with the ordering $U \prec V$ if $U \supset V$. Show that $\mathcal{N}$ is a directed set.

(d) Let z_n be a sequence of complex numbers. For F in $\mathcal{F}$, let $s_F = \sum_{n\in F} z_n$. Prove:

$$\lim_{F\in\mathcal{F}} s_F \quad \text{exists}$$

if and only if

$$\sum_{n=1}^{\infty} |z_n| < \infty$$

and in this case,

$$\lim_{F\in\mathcal{F}} s_F = \sum_{n=1}^{\infty} z_n$$

(e) Let $f(x) = x^2$ for $0 \le x \le 1$. For $P \in \mathcal{P}$ with $P = \{x_0, x_1, \cdots, x_n\}$ where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$, let

$$\sigma_P = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$

Calculate

$$\lim_{P\in\mathcal{P}} \sigma_P$$

(Hint for (e): If you tell the familiar name for this limit and justify your assertion, you may simply write down the answer for (e) without further explanation.)

39. (Exercise on subnets)
Definition: We say z is a *cluster point* of the net x_j if for every neighborhood U of z and every k in the directed set J, there is j in J so that $k \prec j$ and $x_j \in U$.

(a) Prove that z is a cluster point of the net x_j if and only if there is a subnet of x_j that converges to z.

Let X be the set of pairs of non–negative integers with the (Hausdorff) topology described as follows: For $(m,n) \neq (0,0)$, the set $\{(m,n)\}$ is open. A set U containing $(0,0)$ is open if and only if, for all but a finite number of integers m, the set $\{n : (m,n) \notin U\}$ is finite.

(b) Show that no sequence in $X \setminus \{(0,0)\}$ converges to $(0,0)$.

(c) Find a sequence in $X \setminus \{(0,0)\}$ that has $(0,0)$ as a cluster point.

40. (a) Find the extreme points of the unit ball of ℓ^∞.

(b) Show that the unit ball of c_0 has no extreme points but that the unit ball of c does. Conclude that c and c_0 are not isometrically isomorphic. (See problem 10.)

41. (a) Show that the unit ball of $L^1([0,1])$ has no extreme points.
(b) Find the extreme points of the unit ball of ℓ^1.

42. Is the weak star topology on ℓ^1 induced by c the same as the weak star topology on ℓ^1 induced by c_0? (See problems 10 and 40.)

43. Let K be a convex, bounded, non-empty set in a Banach space. For x in K let

$$r_x = \sup\{\|x-y\| : y \in K\} \quad \text{and let} \quad \alpha = \inf\{r_x : x \in K\}.$$

Definition: x_0 is a *center* of K if $r_{x_0} = \alpha$.
(a) Find a convex, bounded, non-empty set K in $\ell^\infty(2)$ that has infinitely many centers.

(b) Prove that in a Hilbert space, every convex, bounded, non-empty set K has a unique center.
(This theorem is actually true in any uniformly convex Banach space.)

44. Let S be the linear functional on $C[0,1]$ given by

$$S(f) = \int_0^1 f(t)dt.$$

A *quadrature rule of order* n is a linear functional on $C[0,1]$ of the form

$$\Lambda(f) = \sum_{j=0}^{n} w_j f(t_j)$$

where $w_0, w_1, \cdots, w_n$ and $0 \le t_0 < t_1 < \cdots < t_n \le 1$ are given numbers. We say that the quadrature rule is *exact on the subspace* P if $\Lambda(f) = S(f)$ for f in P.

(Example: The *trapezoid rule* is $w_0 = w_n = 1/2n$, and $w_1 = w_2 = \cdots = w_{n-1} = 1/n$ and $t_j = j/n$ for $j = 1, 2, \cdots, n$ and P is the set of piecewise linear functions with nodes t_j. That is, $f \in P$ if and only if there are numbers a_j, b_j so that $f(t) = a_j t + b_j$ for $t_{j-1} \le t \le t_j$.)
(a) Suppose Λ_n is a sequence of quadrature rules exact on P_n with $P_1 \subset P_2 \subset P_3 \subset \cdots$ and $\bigcup_{n=1}^{\infty} P_n$ is dense in $C[0,1]$. Prove that $\Lambda_n \to S$ in the weak-star topology if and only if there is a number M so that $\sum_{j=0}^{n} |w_j^{(n)}| \le M$. (Hint: use ideas of uniform boundedness.)
(b) Prove that the piecewise linear functions with dyadic rational nodes (i.e. $j/2^n$ for $0 \le j \le 2^n$) are dense in $C[0,1]$ and conclude that if Λ_n is the trapezoid rule of order 2^n, then $\Lambda_n \to S$ weak-star.

45. Let D be the collection of finite, discrete probability measures on $[0,1]$, that is,

$$D = \{\sum_{i=1}^{n} \alpha_i \delta_{x_i} : 0 \le \alpha_i; \quad \sum_{i=1}^{n} \alpha_i = 1; \qquad \text{and } 0 \le x_i \le 1\},$$

where δ_x denotes the measure with point mass at x.
Let μ be a Borel probability measure on $[0,1]$.
Prove that there are $\mu_1, \mu_2, \cdots \in D$ such that

$$\int f d\mu = \lim_{k\to\infty} \int f d\mu_k$$

for all f in $C([0,1])$ by showing:
(a) $C([0,1])$ is separable.
(b) $P = \{\mu : \mu \text{ is a Borel probability measure}\}$ is weak-star compact, convex.
(c) The extreme points of P are δ_x, for x in $[0,1]$.
(d) If $\{f_n\}$ is dense in $C([0,1])$ and $\epsilon > 0$, can choose μ_k in D so that $\mu_k \in V_{f_1,f_2,\cdots,f_k;\epsilon;\mu}$, the weak-star neighborhood of μ determined by $f_1, f_2, \cdots, f_k$ and ϵ.

46. For $0 < p < 1$, the *dual Banach space* of ℓ^p is the Banach space of bounded linear functionals on ℓ^p, that is, let Λ be a bounded linear functional on ℓ^p and let

$$\|\Lambda\| = \sup\{|\Lambda(x)| : x \in \ell^p \text{ and } \|x\|_p \le 1\},$$

then the dual Banach space of ℓ^p is the space of bounded linear functionals on ℓ^p with this norm. For $0 < p < 1$, find the dual Banach space of ℓ^p.

47. For $0 < p < 1$, the *enveloping Banach space* of ℓ^p is the Banach space whose norm is the Minkowski functional of the convex hull of the unit ball of ℓ^p, that is, let B be the closed convex hull of $\{x \in \ell^p : \|x\|_p < 1\}$ and let $|||x||| = \inf\{t > 0 : t^{-1}x \in B\}$, then the enveloping Banach space of ℓ^p is the completion of ℓ^p with the norm $|||\cdot|||$.
For $0 < p < 1$, find the enveloping Banach space of ℓ^p.

48. In Exercises 46 and 47, the non–locally convex spaces ℓ^p, for $0 < p < 1$, are considered. If ℓ^p were a Banach space, it would be its own enveloping Banach space, the space in Exercise 46 would be its dual, and the second dual of the space in 47 would be the dual of the space in 46. Is there a duality relationship between the spaces in Exercises 46 and 47?

49. (Local Reflexivity) Let X be a Banach space, and let $\lambda > 1$, $z_1, z_2, \cdots, z_m$ in X'', and $y_1, y_2, \cdots, y_n$ in X' be given. Prove that there are $x_1, x_2, \cdots, x_n$ in X such that $z_i(y_j) = y_j(x_i)$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ and for any scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$,

$$\frac{1}{\lambda}\|\sum_{i=1}^{m} \alpha_i x_i\| \le \|\sum_{i=1}^{m} \alpha_i z_i\| \le \lambda\|\sum_{i=1}^{m} \alpha_i x_i\|$$

Definition Let X and Y be Banach spaces. We say X is *finitely represented* in Y, if for all $\lambda > 1$ and $x_1, x_2, \cdots, x_n$ in X there are $y_1, y_2, \cdots, y_n$ in Y such that for any scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$,

$$\frac{1}{\lambda}\|\sum_{i=1}^{n} \alpha_i x_i\| \leq \|\sum_{i=1}^{n} \alpha_i y_i\| \leq \lambda\|\sum_{i=1}^{n} \alpha_i x_i\|$$

50. (a) Prove that ℓ^1 is finitely represented in $L^1([0,1])$.
(b) Prove that $L^1([0,1])$ is finitely represented in ℓ^1.

51. For a vector space $\mathcal{W}$ and subspaces $\mathcal{U}$ and $\mathcal{V}$, we write $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ if $\mathcal{U} \cap \mathcal{V} = \{0\}$ and for each w in $\mathcal{W}$, there are u in $\mathcal{U}$ and v in $\mathcal{V}$ so that $w = u + v$. If X is a Banach space and Z is a closed subspace of X, we say Z *is complemented in* X if there is a closed subspace Y of X so that $X = Y \oplus Z$.
(a) Show that if Y and Z are closed subspaces of a Banach space X such that $X = Y \oplus Z$ then X/Z is isomorphic to Y.
(b) Show that if M is a closed subspace of a Banach space X such that $\dim(X/M)$ is finite, then M is complemented.

Note: For the purposes of homework grading, problems 1–51 will be deemed to be from the "first half of the course" and problems 52–109 will be deemed to be from the "second half of the course"

52. Suppose X, Y are Banach spaces and $A, B \in \mathcal{B}(X, Y)$ are such that range$(A) \supset$ range(B).
(a) If $\ker(A) = (0)$, show that there is $C \in \mathcal{B}(X)$ such that $B = AC$.
(b) Get the conclusion of (a) assuming only that $\ker(A)$ is complemented in X.

53. Show that if K is a closed convex subset of a Hilbert space H and x is in H, then there is a unique point y_0 in K such that

$$\|x - y_0\| = \inf\{\|x - y\| : y \in K\}.$$

54. Let H be a Hilbert space and P a (non-zero) projection on H. Prove that the following are equivalent (c.f. Exercise 2.18 of MacCluer.):
(i) P is an orthogonal projection.
(ii) $\|P\| = 1$.
(iii) $P = P^*$.

55. Show that, given any two vectors v and w in a Hilbert space with $\|v\| = \|w\|$ and for which $<v, w>$ is real, there is an invertible isometry U of the whole space so that $Uv = w$ and $Uw = v$. (An invertible isometry U of a Hilbert space is called unitary and $U^{-1} = U^*$). (Contrast this situation with problem 26.)

56. Suppose T is a bounded linear operator on a Hilbert space $\mathcal{H}$ and suppose further that the range of T is one-dimensional. (c.f. Exercise 2.6 of MacCluer.)
(a) Show that there are vectors x and y in $\mathcal{H}$ so that $Tz = <z, x> y$.
(b) What is T^*?
(c) When is $T = T^*$?

57. Suppose M is a dense linear manifold in the Banach space X, that is, M is a vector subspace of X whose closure is all of X, and suppose $T : M \mapsto Y$ is a linear transformation of M into the Banach space Y. (c.f. Exercise 2.2 of MacCluer.)
(a) Show that if there is a constant C with $C < \infty$ so that $\|Tx\| \le C\|x\|$ for x in M, then T extends to an operator $\widetilde{T} : X \mapsto Y$ such that $\|\widetilde{T}\| \le C$ (that is, for x in M, we have $\widetilde{T}x = Tx$, but $\widetilde{T}$ is defined on all of X).
(b) Show that if X and Y are Hilbert spaces and we have $<Tx, Ty> = <x, y>$ for each x and y in M, then T extends to an isometry $\widetilde{T}$ of X into Y.

58. (a) Let T be an operator on Euclidean n-dimensional space, $\ell^2(n)$. Show that T is unitarily equivalent to an operator S whose matrix is upper triangular, that is,

$$T = U^{-1}SU$$

where U is isometric and invertible (see problem 55.).
(b) Find an example of a bounded operator on ℓ^2 (infinite dimensional) that is not unitarily equivalent to any operator whose matrix is upper triangular.

59. Let U be an invertible isometry on a Banach space X. Show that $\sigma(U) \subset \{\lambda : |\lambda| = 1\}$.

60. Let X be a finite dimensional Banach space. Show that the set of invertible operators is dense in $\mathcal{B}(X)$.

61. Let S be the unilateral (right) shift on ℓ^2. Show that there is no sequence A_n of bounded operators that are invertible on ℓ^2 such that $\|A_n - S\| \to 0$. (Thus the set of invertible operators on ℓ^2 is not dense in $\mathcal{B}(\ell^2)$; compare with problem 60.)

62. Let H be a separable Hilbert space. Let *inv*: $\{\text{invertible operators}\} \to \mathcal{B}(H)$ be defined by $inv(T) = T^{-1}$ and *adj*: $\mathcal{B}(H) \to \mathcal{B}(H)$ be defined by $adj(T) = T^*$. For each of the topologies (weak operator, strong operator, uniform) on $\mathcal{B}(H)$ decide if *inv* and *adj* are continuous.

For n a positive integer, let $\ell^p(n)$ denote the vector space $\mathbf{C}^n$ with the norm $\|(x_1, x_2, \cdots, x_n)\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$. Let $e_1, e_2, e_3, \cdots$ be the standard unit vector basis for $\ell^p(n)$ or ℓ^p. An operator T on $\ell^p(n)$ or ℓ^p is associated with a matrix $A = (a_{ij})$ where $Te_j = \sum_i a_{ij} e_i$.

63. **(Schur Test)** Let $A = (a_{ij})_{i,j=1}^\infty$ be an infinite matrix with $a_{ij} \geq 0$ and such that there are scalars $p_i > 0$ and $\beta, \gamma > 0$ with

$$\sum_{i=1}^\infty a_{ij} p_i \leq \beta p_j, \qquad \sum_{j=1}^\infty a_{ij} p_j \leq \gamma p_i,$$

for all $i, j \geq 1$. Show that there is a bounded operator T on ℓ^2 whose matrix is A and that $\|T\|^2 \leq \beta\gamma$.

64. A matrix $A = (a_{ij})$ is called a *Hilbert-Schmidt matrix* if $\sum |a_{ij}|^2 < \infty$. Show that a Hilbert-Schmidt matrix is the matrix of a bounded operator on ℓ^2.

***65.** Show that the Hilbert matrix $a_{ij} = (i + j - 1)^{-1}$, for $i, j = 1, 2, 3, \cdots$ defines a bounded operator on ℓ^2. (Can you find its norm?)

66. (a) Using the "usual basis" for $\ell^1(\mathbf{N})$, we establish a correspondence between matrices and operators by

$$M = (a_{ij}) \quad \text{where} \quad Te_j = \sum_{i=1}^\infty a_{ij} e_i$$

Let m be defined by

$$m = \sup\{\sum_{i=1}^\infty |a_{ij}| : j = 1, 2, \cdots\}$$

Prove that if T is a bounded operator on ℓ^1, then for the matrix defined above, $m < \infty$ and $m = \|T\|$. Conversely, prove that if M is a matrix such that $m < \infty$, then the formula above defines a bounded linear operator T on ℓ^1 and $\|T\| = m$.

(b) State and prove a similar result for the Banach space c_0. (Hint: The dual of c_0 is ℓ^1.)

(c) State a partial result of this sort for ℓ^∞ and discuss the relationship with part (b) above.

67. For $p = 1, 2, \infty$, find $\|T\|$ where T is the operator on $\ell^p(2)$ whose matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

68. Give an example of a quasi-nilpotent operator that is neither nilpotent nor compact.

69. Consider $L^p = L^p([0,1])$ for $1 \le p < \infty$. Given f in L^∞, the operator $M_f : L^p \to L^p$ is defined by $(M_f(g))(x) = f(x)g(x)$ for g in L^p and $0 \le x \le 1$.
(a) Find $\|M_f\|$.
(b) When is M_f a projection?
(c) For $p \neq 2$, find M_f'.
(d) For $p = 2$, find M_f^*.

70. Let f be in $L^\infty([0,1])$ and let m denote Lebesgue measure. The *essential range of* f is the set

$$R_f = \{\lambda \in \mathbf{C} : \text{ for all } \epsilon > 0, \quad m(\{x : |f(x) - \lambda| < \epsilon\}) > 0\}.$$

(a) Show that if f is continuous, and $g = 0$ almost everywhere then

$$R_{f+g} = \{f(x) : 0 \le x \le 1\}.$$

(b) Let M_f be the multiplication operator on L^p (see problem 69).
Show that $\sigma(M_f) = R_f$.

71. Let $L^1 = L^1([0,1])$. Given f in L^1 define the operator $M_f : \mathcal{D} \to L^1$ (where $\mathcal{D}$ is a dense subspace of L^1) by $(M_f(g))(x) = f(x)g(x)$ as above.
(For example, $\mathcal{D}_0 = \{$ continuous functions on $[0,1]\}$ is a possible domain.)
(a) Find a domain $\mathcal{D}_f$ such that M_f is a closed operator.
(b) For which f is M_f bounded on the domain $\mathcal{D}_f$ you found in part (a)?

Definition: Let H be a complex Hilbert space. If A is a bounded operator on H, the *numerical range of* A is the set

$$W(A) = \{ <Ax, x> \ : \|x\| = 1\}$$

72. (a) Let A, B, C be the operators on $\ell^2(2)$ whose matrices with respect to the usual basis are given below. Find their numerical ranges. (i.e. describe the sets geometrically and prove your answer is correct.)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

(b) Show that $W(T) = \{0\}$ if and only if $T = 0$.
(c) Show that $W(T) \subset \mathbf{R}$ if and only if T is self-adjoint.
(Hint: For (b) and (c) it may be helpful to consider the vector $x + y$ and the effect of multiplying by i.)

73. Show that the numerical range of an operator is a convex set.

74. Show that if A is self-adjoint, the closure of $W(A)$ is the smallest closed interval containing $\sigma(A)$.

75. Let T be a bounded operator on a Hilbert space H.
Prove that T is compact if and only if $T(H)$ does not contain any closed infinite dimensional subspaces.

76. For $1 \leq p < q$, show that if $T : \ell^q \to \ell^p$ is bounded, then T is compact.

77. Prove that every Hilbert–Schmidt operator on ℓ^2 is compact. (see problem 64).

78. Suppose $\mathcal{H}$ is a Hilbert space and $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed subspaces so that $\mathcal{H}_1 + \mathcal{H}_2 = \mathcal{H}$ and $\mathcal{H}_1^\perp = \mathcal{H}_2$. If A is an operator on $\mathcal{H}$ then A can be represented as a block matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is an operator on $\mathcal{H}_1$, A_{12} is an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$, etc. If you are unfamiliar with such a decomposition, you should convince yourself that sums and products work appropriately.

(a) Let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1$. Use P to write A_{ij} in terms of A and conclude that A is bounded if and only if each of the A_{ij} are and that there is a constant C so that $\max \|A_{ij}\| \leq \|A\| \leq C \max \|A_{ij}\|$.

(b) Show that A^* is represented as

$$A^* = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$$

(c) Show that if

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

then $\sigma(A) \subset \sigma(A_{11}) \cup \sigma(A_{22})$.

(d) Show that if A is as in (c) and $\mathcal{H}_1$ is finite dimensional, then $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$.

(e) Find an example of A as in (c) for which $\sigma(A) \neq \sigma(A_{11}) \cup \sigma(A_{22})$.

79. D is a diagonal operator on ℓ^2 if there are numbers $\{d_j\}$ so that $De_j = d_j e_j$ for $j = 1, 2, 3, \cdots$ where $\{e_j\}$ is the usual orthonormal basis for ℓ^2. Let D be diagonal.

(a) Find necessary and sufficient conditions for D to be a bounded operator, and find $\|D\|$ in this case.

(b) Find $\sigma(D)$.

(c) Find necessary and sufficient conditions for D to be a compact operator.

(d) Give an example of a compact operator that is not Hilbert–Schmidt.

80. Prove that the set of Hilbert–Schmidt operators on ℓ^2 is an ideal; that is, show that if A is Hilbert–Schmidt and B is bounded, then AB and BA are Hilbert–Schmidt. (see problem 64).

81. Suppose H is a Hilbert space and that $\mathcal{J}$ is an ideal in $\mathcal{B}(H)$; that is, suppose that $\mathcal{J}$ is a subspace of $\mathcal{B}(H)$ such that if A is in $\mathcal{J}$ and B is bounded, then AB and BA are in $\mathcal{J}$.

(a) Show that $\mathcal{J}$ contains all finite rank operators on H.

(b) Show that the smallest closed ideal in $\mathcal{B}(H)$ is the set of compact operators on H.

(c) Conclude from (b) and problem 79 that the ideal of Hilbert–Schmidt operators is not closed.

82. An alternate definition of the Hardy space $H^2(D)$ is

$$H^2 = \{f \text{ analytic in } D : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

It is not difficult to show that H^2 is a Hilbert space with inner product

$$<f, g> = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

and that the polynomials are dense in the resulting Hilbert space.

(a) Show that if α is in the open disk then $f \mapsto f(\alpha)$ is a continuous linear functional on H^2 and find the "reproducing kernel" function $K_\alpha(z)$ in H^2 such that $<f, K_\alpha> = f(\alpha)$ for all f in H^2.

(b) Show that if α is in the open disk then $f \mapsto f'(\alpha)$ is a continuous linear functional on H^2 and find the kernel function $K'_\alpha(z)$ in H^2 such that $<f, K'_\alpha> = f'(\alpha)$ for all f in H^2.

83. Let V be the Volterra operator on $L^2([0, 1])$,

$$(Vf)(x) = \int_0^x f(t)dt.$$

(We have seen that V is a compact quasi-nilpotent operator.)

(a) Find $\sigma(V^*V)$.

(b) Find $\|V\|$.

84. (a) Let N be nilpotent of order n in $\mathcal{B}(X)$. Suppose f and g are locally analytic on $\{0\} = \sigma(N)$. Show that $f(N) = g(N)$ if and only if $f^{(k)}(0) = g^{(k)}(0)$ for $k = 0, 1, \cdots, n-1$.

(b) Use (a) to show that if T is an operator on a finite dimensional space, then for any f locally analytic on $\sigma(T)$, there is a polynomial p, of degree at most n such that $p(T) = f(T)$. Find the condition that guarantees $p(T) = f(T)$.

85. (a) Let A be a bounded operator, f be locally analytic on $\sigma(A)$, and S be invertible. Show $f(S^{-1}AS) = S^{-1}f(A)S$.

(b) Let D be diagonal (see problem 79), with $De_j = d_j e_j$ and let f be locally analytic on $\sigma(D)$. Show that $f(D)$ is a diagonal operator with $f(D)e_j = f(d_j)e_j$.

(c) Use (a) and (b) to find e^A where A has matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

86. Prove: If A is a bounded operator and $A = A^*$ then

$$U = (A + i)^{-1}(A - i)$$

is unitary. (U is called the Cayley transform of A.) Find and prove a converse of the form "If U is unitary and ... then ... is self-adjoint and bounded."

87. Let A be a strictly positive operator (i.e. A is positive and invertible). Show that there is a *unique* strictly positive operator B with $B^2 = A$. (This is also true for positive operators but a slightly different proof may be needed.)

88. (a) Let S be an invertible operator on a Hilbert space. Prove that S^* is also invertible and $(S^*)^{-1} = (S^{-1})^*$.
Corollary. If S is invertible and self-adjoint, then S^{-1} is also self-adjoint.
(b) Suppose $T : H_1 \to H_2$ is a bounded operator between Hilbert spaces such that $\ker(T) = (0)$ and T has closed range. Prove that T^*T is invertible.
(c) Suppose T is as in (b). Prove that

$$T(T^*T)^{-1}T^*$$

is the orthogonal projection of H_2 onto range(T).
(d) Use (c) to find the matrix for the orthogonal projection of $\ell^2(4)$ onto the subspace spanned by $v_1 = (1, 1, -1, 2)$ and $v_2 = (1, -1, 2, -1)$.
(Hint: Consider $T : \ell^2(2) \to \ell^2(4)$ defined by $Te_1 = v_1$ and $Te_2 = v_2$.)

89. (a) Suppose X and Y are Banach spaces and $A : X \to Y$ and $B : Y \to X$ are bounded operators. Show that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.

(b) Give an example to show that $\sigma(AB)$ and $\sigma(BA)$ need not be equal.

(c) Suppose Ω is an open set that contains $\sigma(AB)$ and $\sigma(BA)$ and suppose f is analytic in Ω. Show that $Bf(AB) = f(BA)B$.

(d) Suppose T is an operator on a Hilbert space $\mathcal{H}$ and $\|T\| \leq 1$ (that is, T is a *contraction*). Prove that

$$U = \begin{pmatrix} T & \sqrt{I - TT^*} \\ \sqrt{I - T^*T} & -T^* \end{pmatrix}$$

is unitary on $\mathcal{H} \oplus \mathcal{H}$ (U is called a *unitary dilation* of T). (Here, the $\sqrt{\ }$ is as in problem 87 extended to cover the positive, not necessarily strictly positive, case.)

90. Let P and Q be self-adjoint projections. If $\|P - Q\| < 1$ then dim(im(P))= dim(im(Q)) and dim(ker(P))= dim(ker(Q)). (The conclusion is equivalent to the existence of a unitary operator U so that $P = U^{-1}QU$. In $\mathbf{R}^n$, $\|P - Q\| = \sin\theta$ where θ is the angle between im(P) and im(Q).)

91. Let φ be defined by

$$\varphi(x) = \begin{cases} x + 1 & -1 \leq x \leq -\frac{1}{2} \\ 4x^2 & |x| < \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then the multiplication operator M_φ (see problem 69) on $L^2([-1, 1])$ is self-adjoint and has spectrum $\sigma(M_\varphi) = [0, 1]$.
Letting E denote the projection valued spectral measure of M_φ, find the projections $E(\{0\}), E(\{\frac{1}{3}\}), E(\{\frac{1}{2}\}), E(\{1\})$, and $E([0, \alpha))$ for $0 < \alpha \leq 1$.

92. On ℓ^2 let $e_0, e_1, e_2, \cdots$ be the usual orthonormal basis and let T be the operator given by

$$Te_j = \begin{cases} e_1 & \text{for } j = 0 \\ e_{j-1} + e_{j+1} & \text{for } j > 0. \end{cases}$$

Find $\|T\|$ and $\sigma(T)$.

* **93.** Find the projection valued spectral measure for the operator T in problem 92.

94. Let

$$T = \begin{pmatrix} 5 & 4 \\ -2 & -1 \end{pmatrix}$$

Then $\sigma(T) = \{1, 3\}$. Use the definition of the Riesz functional calculus to explicitly compute the Riesz Projection associated with $\{1\}$. ("Explicitly" means, choose appropriate neighborhoods, cycles, integrals and use only the Cauchy theorem for complex valued analytic functions and the computation of the inverse of $(\lambda I - T)^{-1}$.)
Explain geometrically what this answer means.

95. (a) Give an example of a bounded non–compact operator T such that T^2 is compact.
(b) Let T be a bounded operator on an infinite dimensional Banach space such that T^2 is compact. Prove: Either the spectrum of T is a finite set containing 0 or it is a countable set whose only limit point is 0. Moreover, if $\lambda \neq 0$ is in the spectrum, then λ is an eigenvalue of T of finite multiplicity.

96. Let T be a compact operator on a Hilbert space, $\mathcal{H}$. Let S be the self-adjoint compact operator $S = (T^*T)^{1/2}$. The eigenvalues of S are non-negative, and we name them so that there is an orthonormal basis of eigenvectors f_n with $Sf_n = s_n f_n$ and $s_1 \geq s_2 \geq s_3 \geq \cdots$. (The s_n are called the *singular values* of T.) Let $\mathcal{R}_n$ denote the set of continuous operators on $\mathcal{H}$ of rank n. Prove that $\|T\| = s_1$ and, more generally, that the distance of T from $\mathcal{R}_n$ is s_{n+1}.

97. Prove that if A is a bounded self–adjoint operator on the Hilbert space $\mathcal{H}$, for any $\epsilon > 0$, there a bounded self–adjoint operator F on $\mathcal{H}$ so that $\|F - A\| < \epsilon$ and the spectrum of F is a finite set.

98. Suppose A and B are positive semidefinite bounded operators on the Hilbert space $\mathcal{H}$ (that is, A and B are self–adjoint and their spectra are contained in $\{x : x \geq 0\}$). Find a necessary and sufficient condition that AB be positive semidefinite.

For the following two problems, let $\ell^2(n)$ denote n–dimensional complex Euclidean space, that is, $\mathbf{C}^n$ with the usual Hilbert space norm.

* **99.** Find the dual of $\mathcal{B}(\ell^2(n), \ell^2(m))$.

* **100.** Find the extreme points of the unit ball of $\mathcal{B}(\ell^2(n))$.

Krein Spaces

On $\mathbf{C}^n$ let $<x,y>$ denote the Euclidean inner product, $<x,y> = \sum_{i=1}^n x_i\overline{y_i}$.

Definition An *indefinite scalar product* on $\mathbf{C}^n$ is a function from $\mathbf{C}^n \times \mathbf{C}^n$ to $\mathbf{C}$ such that
(a) $[x,y] = \overline{[y,x]}$ (antisymmetry)
(b) $[\alpha_1 x_1 + \alpha_2 x_2, y] = \alpha_1[x_1,y] + \alpha_2[x_2,y]$ (sesquilinearity)
(c) $[x,y] = 0$ for all y in $\mathbf{C}^n \Longrightarrow x = 0$. (non-degeneracy).

(Note that we have not(!) assumed $[x,x] \geq 0$.)

101. (a) Let J be a self-adjoint (with respect to the Euclidean inner product) invertible matrix on $\mathbf{C}^n$, show that $[x,y] = <Jx,y>$ is an indefinite scalar product.
(b) Let $[\cdot,\cdot]$ be an indefinite scalar product on $\mathbf{C}^n$; show that there is an invertible self-adjoint matrix J so that $[x,y] = <Jx,y>$.
(c) Let $[x,y] = \sum_{i=1}^n x_{n-i+1}\overline{y_i}$. Show that $[\cdot,\cdot]$ is an indefinite scalar product by finding the matrix J corresponding to this scalar product. (J is called the *sip* matrix of size n.)

Definition A vector space $\mathcal{K}$ with an indefinite scalar product $[\cdot,\cdot]$ is called a *Krein space* if there is an inner product $<\cdot,\cdot>$ on $\mathcal{K}$ that makes $\mathcal{K}$ into a Hilbert space and a bounded, invertible, self–adjoint operator J on this Hilbert space so that $[x,y] = <Jx,y>$.

Definition If M is a subspace of $\mathbf{C}^n$, define the *orthogonal companion* of M, denoted $M^{[\perp]}$, by

$$M^{[\perp]} = \{y : [y,x] = 0 \text{ for all } x \in M\}.$$

102. (a) Find an example in which $M + M^{[\perp]} \neq \mathbf{C}^n$.
(b) Prove that, always, $\dim(M) + \dim(M^{[\perp]}) = n$.

Definition A subspace M is called *non-degenerate* if the only x in M such that $[y,x] = 0$ for all y in M is $x = 0$.

103. Prove that M is non-degenerate if and only if $M + M^{[\perp]} = \mathbf{C}^n$.

Definition A set of vectors $x_1, x_2, \cdots$ is said to be *orthonormal* with respect to $[\cdot,\cdot]$ if

$$[x_i, x_j] = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j. \end{cases}$$

104. Prove that if M is a non-degenerate subspace, then M has an orthonormal basis.

Definition If Q is a Hermitian matrix on $\mathbf{C}^n$, the *signature* of Q is the number of positive eigenvalues minus the number of negative eigenvalues (with multiplicity).

105. Prove: If M is a $[\cdot,\cdot]$-nondegenerate subspace of $\mathbf{C}^n$ and $e_1, e_2, \cdots, e_k$ is an $[\cdot,\cdot]$-orthonormal basis for M, then

$$\sum_{i=1}^{k} [e_i, e_i] = \text{ signature}(PJ|_M)$$

where P is the $<\cdot,\cdot>$-orthogonal projection of $\mathbf{C}^n$ onto M and J is the Hermitian matrix associated with $[\cdot,\cdot]$.
Corollary $\sum_{i=1}^{k} [e_i, e_i]$ does not depend on the choice of basis.

Definition A subspace M is called *positive* if $[x,x] > 0$ for all $x \neq 0$ in M,
non-negative if $[x,x] \geq 0$ for all x in M,
and *neutral* if $[x,x] = 0$ for all x in M.

106. (a) Prove that if $[x,y] = <Jx,y>$, then the maximal dimension of a non-negative subspace is the number of positive eigenvalues of J (with multiplicity).
(b) Let J be the sip matrix of size n; find a positive subspace of maximal dimension.

107. Prove that a non-negative subspace is the direct sum of a positive subspace and a neutral subspace.

108. (a) Prove that if $[x,y] = <Jx,y>$, then the maximal dimension of a neutral subspace is the minimum of the number of positive eigenvalues and the number of negative eigenvalues of J.
(b) Let J be the sip matrix of size n; find a neutral subspace of maximal dimension.

Definition If $[\cdot,\cdot]$ is a scalar product on $\mathbf{C}^n$ and A is an $n \times n$ matrix, the [*adjoint*] of A, denoted $A^{[*]}$, is the matrix such that

$$[Ax, y] = [x, A^{[*]}y].$$

109. (a) If $[x,y] = <Jx,y>$, find a formula relating A^*, J and $A^{[*]}$.
(b) Show that for finite dimensional subspaces of a Krein space $\left(M^{[\perp]}\right)^{[\perp]} = M$.
(c) Show that image$(A^{[*]})$ = (kernel$(A))^{[\perp]}$ and kernel$(A^{[*]})$ = (image$(A))^{[\perp]}$.