NAME: Solution B

Math 163 (Cowen) **Test 1** **February 7, 2007**

There are 5 pages, 6 questions, and 100 points on this test. **The test finishes at 10:10am!** Follow the instructions for each question and show enough of your work that I can understand what you are doing.

(10 points) **1.** Write the domain and range of each of the following functions, and tell which is which.

(a) $f(t) = 3\sqrt{t-4}$

Domain: $t \geq 4$ or $[4, \infty)$
Range: $f(t) \geq 0$ or $[0, \infty)$

(b) $g(s) = 5 + 2\cos 3s$

Domain: $-\infty < s < \infty$ or $(-\infty, \infty)$
Range: $3 \leq g(s) \leq 7$ or $[3, 7]$

Here's how to get the range: you have to know that the range of cosine is $-1 \leq \cos(x) \leq 1$ for *any* x. Thus:

$$-1 \leq \cos(3s) \leq 1$$

$$\text{so} \quad -2 \leq 2\cos(3s) \leq 2$$

$$\text{and} \quad 5 - 2 \leq 5 + 2\cos(3s) \leq 5 + 2$$

(c)

$$h(v) = \begin{cases} 1 + v^2 & \text{if } -4 \leq v \leq -1 \\ 3 - v & \text{if } -1 < v \leq 2 \\ (v-2)^2 & \text{if } v > 2 \end{cases}$$

Domain: $-4 \leq v < \infty$ or $[-4, \infty)$
Range: $0 < h(v) < \infty$ or $(0, \infty)$

(10 points) **2.** Let $f(s) = \sqrt{s+5}$ and let $g(t) = \sin(2t)$.

(a) Find an expression for $f \circ g(x)$.

$$f \circ g(x) = f(g(x)) = f(\sin(2x)) = \sqrt{\sin(2x) + 5}$$

(b) Find an expression for $g \circ f(w)$.

$$g \circ f(w) = g(f(w)) = g(\sqrt{w+5}) = \sin(2\sqrt{w+5})$$

(25 points) **3.** Calculate each of the following limits, as a real number, $+\infty$, or $-\infty$, using a convenient method. If the limit does not exist, say so and explain why.

(a) $\displaystyle \lim_{x \to -2} x^3 - 5x + \frac{3}{x+1} = ??$

This function is continuous except for $x = -1$, so we find the limit by evaluating at $x = -2$:

$$\lim_{x \to -2} x^3 - 5x + \frac{3}{x+1} = (-2)^3 - 5(-2) + \frac{3}{-2+1} = -8 + 10 - 3 = -1$$

(Question 3, cont'd)

(b) $\lim_{t\to 4} \frac{t^2-16}{t^2-4t} = ??$

This function is not defined for $t = 4$ because both the numerator and the denominator of the fraction are zero at 4. To evaluate the limit, we hope we can do some algebra to simplify and then evaluate the limit:

$$\lim_{t\to 4} \frac{t^2-16}{t^2-4t} = \lim_{t\to 4} \frac{(t-4)(t+4)}{t(t-4)} = \lim_{t\to 4} \frac{(t+4)}{t} = \frac{8}{4} = 2$$

(c) $\lim_{s\to 9^+} \frac{s}{3-\sqrt{s}} = ??$

This function is not defined for $s = 9$ but the numerator of the fraction is not zero, so we expect the limit not to exist or to be infinite. For s near 9, but larger than 9, $\sqrt{s}$ is near 3 but larger than 3, so the numerator is nearly 9 and the denominator is a very small negative number and the quotient is a very negative number:

$$\lim_{s\to 9^+} \frac{s}{3-\sqrt{s}} = -\infty$$

(d) $\lim_{r\to 3} \frac{\frac{1}{r+2}-\frac{1}{5}}{r-3} = ??$

This function is not defined for $r = 3$ because both the numerator and the denominator of the fraction are zero at 3. To evaluate the limit, we hope we can do some algebra to simplify and then evaluate the limit:

$$\lim_{r\to 3} \frac{\frac{1}{r+2}-\frac{1}{5}}{r-3} = \lim_{r\to 3} \frac{\frac{5}{5(r+2)}-\frac{r+2}{5(r+2)}}{r-3} = \lim_{r\to 3} \frac{\frac{5-(r+2)}{5(r+2)}}{r-3} = \lim_{r\to 3} \frac{\frac{3-r}{5(r+2)}}{r-3} = \lim_{r\to 3} \frac{1}{r-3}\frac{3-r}{5(r+2)}$$

$$= \lim_{r\to 3} \frac{-1}{5(r+2)} = \frac{-1}{5(3+2)} = -\frac{1}{25}$$

(e)

$$h(v) = \begin{cases} 1+v^2 & \text{if } -4 \le v \le -1 \\ 3-v & \text{if } -1 < v \le 2 \\ (v-2)^2 & \text{if } v > 2 \end{cases}$$

$\lim_{v\to 2} h(v) = ??$

Because the definition of the function changes descriptions at $v = 2$, we need to evaluate the left and right limits separately, then compare:

$$\lim_{v\to 2^-} h(v) = \lim_{v\to 2^-} 3-v = 3-2 = 1$$

and

$$\lim_{v\to 2^+} h(v) = \lim_{v\to 2^+} (v-2)^2 = (2-2)^2 = 0$$

Thus, $\lim_{v\to 2} h(v)$ does not exist because the left and right limits are different.

(20 points) **4.** A boy throws a rock off the Golden Gate Bridge. The height of the rock (in feet) above the water after t seconds is approximately $h = 240 + 24t - 16t^2$.

(a) Find the average velocity of the rock between the time he throws it and two seconds later.

The height at $t = 0$ is $h = 240 + 0 - 0 = 240'$; the height at $t = 2$ is $h = 240 + 24 \cdot 2 - 16(2)^2 = 240 + 48 - 64 = 224'$. Thus, the average velocity is

$$\frac{224 - 240}{2 - 0} = -8 \text{ feet per second}$$

(b) Using the limit definition of instantaneous velocity, find the instantaneous velocity of the rock two seconds after he throws it. *Show your work!*

The instantaneous velocity is computed by the limit of the average velocities over shorter and shorter time intervals at $t = 2$ seconds:

$$\text{velocity} = \lim_{t\to 2} \frac{h(t) - h(2)}{t - 2} = \lim_{t\to 2} \frac{240 + 24t - 16t^2 - 224}{t - 2} = \lim_{t\to 2} \frac{16 + 24t - 16t^2}{t - 2}$$

$$= \lim_{t\to 2} \frac{-(t-2)(16t+8)}{t - 2} = \lim_{t\to 2} -(16t + 8) = -(32 + 8) = -40 \text{ feet per second}$$

(20 points) **5.** Let $f(x) = \sqrt{3x+1}$.

(a) Using the limit definition of derivative, find $f'(5)$. *Show your work!*

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{\sqrt{3(5+h)+1} - \sqrt{3\cdot 5 + 1}}{h} = \lim_{h\to 0} \frac{\sqrt{16+3h} - \sqrt{16}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{16+3h} - 4}{h} = \lim_{h\to 0} \frac{\sqrt{16+3h} - 4}{h} \frac{\sqrt{16+3h} + 4}{\sqrt{16+3h} + 4} = \lim_{h\to 0} \frac{16 + 3h - 16}{h(\sqrt{16+3h} + 4)}$$

$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{16+3h} + 4)} = \lim_{h\to 0} \frac{3}{\sqrt{16+3h} + 4} = \frac{3}{\sqrt{16} + 4} = \frac{3}{8}$$

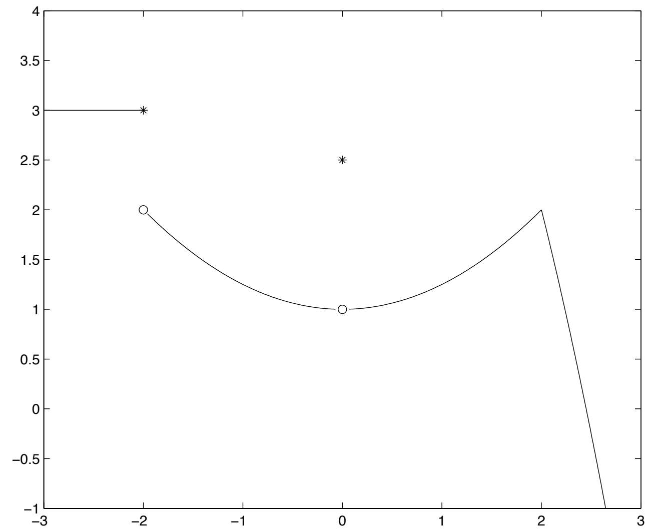
(b) Using your work in part (a), find an equation for the line tangent to the graph of $y = \sqrt{3x+1}$ at the point $(5, 4)$.

The line has slope $f'(5) = \frac{3}{8}$ and passes through $(5, 4)$, so an equation for the tangent line is

$$\frac{y - 4}{x - 5} = \frac{3}{8}$$

This is equivalent to the prettier equation

$$y = \frac{3}{8}x + \frac{17}{8}$$



6. For the function h whose graph is given above, answer the following questions.
(15 points) For questions asking for evaluation of a limit, if the limit does not exist, say so and explain why.

(a) $\lim_{x\to -2} h(x) = ??$

The graph of h has a jump at $x = -2$. From looking at the graph, $\lim_{x\to -2^-} h(x) = 3$ and $\lim_{x\to -2^+} h(x) = 2$. Thus, $\lim_{x\to -2} h(x)$ does not exist because the limit from the left and the limit from the right are different. Note that the value of h at $x = -2$ is irrelevant for computing the limit!

(b) $\lim_{x\to 0} h(x) = ??$

The graph of h has a jump at $x = 0$. From looking at the graph, $\lim_{x\to 0^-} h(x) = 1$ and $\lim_{x\to 0^+} h(x) = 1$. Thus, $\lim_{x\to 0} h(x) = 1$ because the limit from the left and the limit from the right are both 1. Note that the value of h at $x = 0$ is irrelevant for computing the limit!

(c) $\lim_{x\to 2} h(x) = ??$

From looking at the graph, $\lim_{x\to 2} h(x) = 2$.

(d) At which points is h discontinuous? In each case, explain why.

The function h is discontinuous at $x = -2$ because $\lim_{x\to -2} h(x)$ does not exist, so it is not equal to $h(-2) = 3$.
The function h is discontinuous at $x = 0$ because $\lim_{x\to 0} h(x) = 1$ which is not equal to $h(0) = 2.5$.

(e) At which points is h not differentiable? In each case, explain why.

The function h is not differentiable at $x = -2$ because it is discontinuous at $x = -2$.
The function h is not differentiable at $x = 0$ because it is discontinuous at $x = 0$.
The function h is not differentiable at $x = 2$ because the graph has a corner at $x = 2$ and there is not a single line tangent to the graph there. More formally, $\lim_{d\to 0} \dfrac{h(2+d) - h(2)}{d}$, which defines the derivative of h at 2, does not exist because the limits from the left (which is positive) and right (which is negative) are different.