NAME: Solutions

Math 55400 (Cowen) **Test 1** **March 1, 2017**

There are 6 pages, 5 questions, and 100 points on this test.

- Proofs should quote results from the class to justify assertions, as in 'By the rank-nullity theorem, we can see $\cdots$'.
- If you need to use a property of a vector space or transformation or use a theorem in your answer, explain what property or theorem you are using and indicate specifically why the hypotheses of the theorem are satisfied.
- Explain your answers for each question in such a way that your reasoning can be followed!!

NOTE! Throughout this test, unless otherwise specified, assume that you are working over a field $\mathbb{F}$.

(20 points) **1.** A matrix A is called *a rank one matrix* if the dimension of the range of A is 1.

(a) Find a 4×4 real matrix A that is a rank one matrix such that the first row of A is $(-1\ \ 2\ \ 1\ \ 3)$ and the first column of A is $\begin{pmatrix} -1 \\ 3 \\ 5 \\ 1 \end{pmatrix}$

Since A has rank 1, $\dim \mathcal{R}(A) = 1$. Since the range of a matrix is the subspace spanned by the columns, this subspace is 1-dimensional. In particular, every column of A is a multiple of the first column

so $$A = \begin{pmatrix} -1 & 2 & 1 & 3 \\ 3 & -6 & -3 & -9 \\ 5 & -10 & -5 & -15 \\ 1 & -2 & -1 & -3 \end{pmatrix}$$

(b) Let C be an $n \times p$ matrix over $\mathbb{F}$ that is a rank one matrix.
Prove: If B is $m \times n$ matrix, then the matrix BC is either 0 or is also a rank one matrix.

Block the $n \times p$ matrix C by columns $C = (C_1\ C_2 \cdots C_p)$ where C_j is the j^{th} column. Since C is rank 1, there is a non-zero column, call it u. Every column of C is a multiple of u, so $C = (a_1 u\ \ a_2 u\ \cdots\ a_p u)$ where $a_1, a_2, \ldots a_p \in F$ and one of these is 1. Regarding B as a 1×1 block matrix,

$BC = B(a_1 u\ \ a_2 u\ \ a_3 u\ \cdots\ a_p u) = (a_1 Bu\ \ a_2 Bu\ \ a_3 Bu\ \cdots\ a_p Bu)$

so every column of BC is a multiple of Bu so BC is rank 1 if $Bu \neq 0$ and $BC = 0$ if $Bu = 0$

(c) In the situation of part (b) above, when is $BC = 0$?

From (b) we see $BC = 0$ if and only if $Bu = 0$, that is $u \in \mathcal{N}(B)$.

(20 points) **2.** Let Q be an $m \times n$ matrix over $\mathbb{F}$. Let P and R be invertible matrices over $\mathbb{F}$ where P is $m \times m$ and R is $n \times n$.

(a) Prove that the nullspace of PQ is the nullspace of Q.

Use double containment as a proof strategy.

Suppose $v \in \mathcal{N}(Q)$, so $Qv = 0$. Then $(PQ)v = P(Qv) = P.0 = 0$ and $v \in \mathcal{N}(PQ)$. Thus $\mathcal{N}(Q) \subset \mathcal{N}(PQ)$.

Let $w \in \mathcal{N}(PQ)$, so $(PQ)w = P(Qw) = 0$.

P is invertible, so This means $P^{-1}P(Qw) = P^{-1}0 = 0$ and $Qw = 0$ also. In other words $w \in \mathcal{N}(Q)$ and we see $\mathcal{N}(PQ) \subset \mathcal{N}(Q)$

Therefore $\mathcal{N}(PQ) = \mathcal{N}(Q)$.

(b) Prove that the range of QR is the range of Q.

Suppose $x \in F^m$ is in the range of QR.

Then there is y in F^n so that $x = (QR)y = Q(Ry)$

Now Ry is in F^n and $x = Q(Ry)$ so $x \in$ range Q.

This means $\mathcal{R}(QR) \subset \mathcal{R}(Q)$.

Suppose u in F^m is in $\mathcal{R}(Q)$. This means there is w in F^n so that $u = Qw$. Because R is invertible, there is a vector z in F^n so that $z = R^{-1}w$

Now $QRz = QRR^{-1}w = Qw = u$ so u is in $\mathcal{R}(QR)$ and $\mathcal{R}(Q) \subset \mathcal{R}(QR)$ and we see $\mathcal{R}(Q) = \mathcal{R}(QR)$.

(c) Prove that $\text{rank}(PQ) = \text{rank}(Q) = \text{rank}(QR)$.

Since $\mathcal{R}(Q) = \mathcal{R}(QR)$ and the rank of a matrix is the dimension of the ~~range~~ range, we see $\text{rank}(Q) = \text{rank}(QR)$.

Because $\mathcal{N}(PQ) = \mathcal{N}(Q)$ we see dimension $\mathcal{N}(PQ) =$ dimension $\mathcal{N}(Q)$.

Now the rank nullity theorem says $\dim \mathcal{N}(Q) + \text{rank}(Q) = n$

and $\dim \mathcal{N}(PQ) + \dim \mathcal{R}(PQ) = n$

Thus ~~dim~~ $\text{rank}(PQ) = n - \dim \mathcal{N}(PQ) = n - \dim \mathcal{N}(Q) = \text{rank}(Q)$

so $\text{rank}(PQ) = \text{rank}\, Q = \text{rank}(QR)$.

(20 points) **3.** In this question, all matrices have real number entries.

(a) Let G and H be $n \times n$ matrices such that $GH = HG$.
Let M denote the range of G and N denote the range of H.
Prove that $\mathcal{R}(GH)$, the range of GH, is a subset of $M \cap N$.

Both G and H map $\mathbb{R}^n$ into $\mathbb{R}^n$. If $v \in \mathbb{R}^n$ then $Hv \in \mathbb{R}^n$ and $(GH)(v) = G(Hv)$ so $(GH)(v)$ is in $\mathcal{R}(G) = M$.
Similarly $Gv \in \mathbb{R}^n$ and $(GH)(v) = (H\cancel{G}\ldots v) = H(Gv)$ so $(GH)(v)$ is in $\mathcal{R}(H) = N$.
This shows $(GH)(v) \in M \cap N$. Since this is true for all v in $\mathbb{R}^n$, $\mathcal{R}(GH) \subset M \cap N$.

(b) Let $S = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 1 & 4 \\ 1 & 3 & 2 \end{pmatrix}$

Find a basis for the range of S acting on $\mathbb{R}^3$.

Since $\mathcal{R}(S)$ is the column space of S, we know $\mathcal{R}(S)$ is spanned by $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$.
Clearly $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ are independent.
However $\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, so $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\}$ is a basis for $\mathcal{R}(S)$.

(c) For this S, find a 3×3 real matrix T so that the range of ST is *NOT* the range of TS and show that you are right by finding a vector in range(ST) that is not in range(TS) (or vice-versa).

Since the ideas of (a) point out that $\mathcal{R}(ST) \subset \mathcal{R}(S)$ and $\mathcal{R}(TS) \subset \mathcal{R}(T)$, if $\mathcal{R}(T)$ contains a vector not in $\mathcal{R}(S)$, $\mathcal{R}(ST) \neq \mathcal{R}(TS)$.

Since (b) shows $\mathcal{R}(S)$ is a two dimensional subspace of $\mathbb{R}^3$, it is easy to find a vector in $\mathbb{R}^3$ not in $\mathcal{R}(S)$.

For example $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not a linear combination of $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

Taking $T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $TS = \begin{pmatrix} 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $ST = \begin{pmatrix} 2 & 2 & 2 \\ -3 & -3 & -3 \\ 1 & 1 & 1 \end{pmatrix}$

we see $\mathcal{R}(T) = \left\{ \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} : r \in \mathbb{R} \right\}$ So $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is in $\mathcal{R}(ST)$ not in $\mathcal{R}(TS) = \mathcal{R}(T)$

(20 points) **4.** For $\mathcal{V}$ a finite dimensional vector space, a linear transformation E on $\mathcal{V}$ that satisfies $E^2 = E$ is called *a projection on* $\mathcal{V}$.

Recall that in an exercise, you proved that if a linear transformation T on a finite dimensional vector space is invertible, its inverse is a polynomial in T.

Let E be a projection on the finite dimensional vector space $\mathcal{V}$. Show that $3E - I$ is an invertible linear transformation by finding its inverse.

The exercise shows that the inverse of $3E-I$ is a polynomial in $3E-I$.

For example, if $p(z) = z^2+2z-5$ ✓!

then $p(3E-I) = (3E-I)^2+2(3E-I)-5I$

$= 9E^2-6E+I+6E-2I-5I = 9E^2 - 6I$

It is clear from this that, actually the inverse is a polynomial in E because every polynomial in $3E-I$ is a polynomial in E.

Furthermore if $q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$

then $q(E) = a_n E^n + a_{n-1} E^{n-1} + \cdots + a_2 E^2 + a_1 E + a_0 I$

But $E^2 = E$, so $E^3 = E^2 \cdot E = E \cdot E = E^2 = E$, etc

So actually $q(E) = (a_n + a_{n-1} + \cdots + a_2 + a_1)E + a_0 I$!!

These ideas suggest that to find the inverse of $3E-I$, we only need to check polynomials of the form $aE+bI$.

Suppose $aE+bI$ is the inverse of $3E-I$.

Then $I = (aE+bI)(3E-I) = 3aE^2 + 3bE - aE - bI^2$

This means $I = 3aE + 3bE - aE - bI = (2a+3b)E - bI$

We want to choose a and b real numbers so that $-b = 1$ and $2a+3b=0$. In other words $b=-1$ and $a = \frac{3}{2}$. Thus $(3E-I)^{-1} = (\frac{3}{2}E - I)$.

(20 points) **5.** Let $\mathcal{V}$ be a finite dimensional vector space over the field $\mathbb{F}$, let T be a linear transformation on $\mathcal{V}$, and let v be a vector in $\mathcal{V}$.

(a) Let $J = \{p \in \mathbb{F}[x] : p(T)v = 0\}$. Prove that J is an ideal in $\mathbb{F}[x]$.

First we prove J is a subspace of $F[x]$: Let p and q be in J and $a \in F$.
Then $(ap)(T)(v) = a\,p(T)v = a \cdot 0 = 0$ so $ap \in J$.
Also $(p+q)(T)(v) = (p(T)+q(T))(v) = p(T)v + q(T)v = 0+0=0$
and $p+q$ is in J, so J is a subspace of $F[x]$.
To see J is an ideal, suppose $f \in J$ and $g \in F[x]$.
Then $(fg)(T)(v) = (f(T)\,g(T))(v) = g(T)(f(T)v) = g(T)0 = 0$,
$F[x]$ is commutative
so $fg \in J$ also, and J is an ideal.

(b) Let $D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and let $v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

Find a monic polynomial of degree 2 in the ideal $K = \{q \in \mathbb{R}[x] : q(D)v = 0\}$.

If p is a monic polynomial of degree 2, then $p(x) = x^2 + ax + b$
where $a, b \in \mathbb{R}$.
We want $p \in K$, so we need $p(D) = D^2 + aD + bI$

$$D^2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$ so $p(D) = D^2 + aD + bI$

is $$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} + a\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This means $p(D)v =$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + a\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} + a\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5+3a+2b \\ 4+2a+b \\ 3+a \end{pmatrix}$$

For $p(D)v = 0$ we want
$3a+2b=-5$
$2a+b=-4$
$a=-3$

so $a=-3$ and $b=2$ works.
Thus $p(x) = x^2 - 3x + 2$ is a monic polynomial in K.

5. (continued) Recall: $D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and let $v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

and the ideal, K, is $K = \{q \in \mathbb{R}[x] : q(D)v = 0\}$.

(c) Show that there is no monic polynomial of degree 1 in the ideal K in part (b) and explain why this means that the polynomial you found in part (b) is the monic generator of K.

Similarly, if $g(x) = x + c$, a monic polynomial of degree 1, then $g(D)(v) = 0$ means

$$g(x) = D + cI \quad \text{and} \quad g(D)v = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3+2b \\ 2+b \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for any b.

So there are no monic polynomials of degree 1 in J. Since we know that every ideal in $\mathbb{R}[x]$ has a monic generator, and there are no monic generators of degree 1, we know the generator has degree ≥ 2. Since $p(x) = x^2 - 3x + 2$ is in J, it must be a monic polynomial of lowest degree in J, that is, it must be the monic generator of J.