NAME: Solutions

Math 55400 (Cowen) **Final Exam** **May 3, 2017**

There are 9 pages, 8 questions, and 160 points on this test.

- Proofs should quote results from the class to justify assertions, as in 'By the rank-nullity theorem, we can see $\cdots$'.
- If you need to use a property of a vector space or transformation or use a theorem in your answer, explain what property or theorem you are using and indicate specifically why the hypotheses of the theorem are satisfied.
- Explain your answers for each question in such a way that your reasoning can be followed!!

NOTE! Throughout this test, unless otherwise specified, assume that you are working over a field $\mathbb{F}$.

1. Let A be an $m \times n$ matrix over the field $\mathbb{F}$ and let B be an $n \times p$ matrix over $\mathbb{F}$.

(a) Prove: If $v_1, v_2, \cdots, v_j$ are vectors in $\mathbb{F}^n$ such that $\{v_1, v_2, \cdots, v_j\}$ is a linearly dependent set, then $\{Av_1, Av_2, \cdots, Av_j\}$ is a linearly dependent set in $\mathbb{F}^m$.

Proof Since $\{v_1, v_2, \cdots v_j\}$ is a linearly dependent set, there are $a_1, a_2, \cdots a_j$ not all zero so that

$$a_1v_1 + a_2v_2 + \cdots + a_jv_j = 0$$

Thus $A(a_1v_1 + a_2v_2 + \cdots + a_jv_j) = A0 = 0$

By linearity, we get $a_1Av_1 + a_2Av_2 + \cdots + a_jAv_j = 0$, also.

Since not all a_i's are zero this means $Av_1, Av_2, \cdots Av_j$ are linearly dependendent. //

(b) Use the result of part (a) to show that if the range of B has dimension k, then the dimension of the range of AB is no more than k, that is, $\text{rank}(AB) \le \text{rank}(B)$.

Proof Let $j > k$ and $w_1, w_2, \cdots w_j$ in $\mathcal{R}(AB)$ be given.

There $v_1, v_2, \cdots v_j$ in F^p so that

$w_1 = ABv_1$, $w_2 = ABv_2$, $\cdots$ $w_j = ABv_j$.

In particular, $Bv_1, Bv_2, \cdots Bv_j$ are vectors in $\mathcal{R}(B)$

and because $\text{rank}(B) = k$ and $j > k$

we know $Bv_1, Bv_2 \cdots Bv_j$ are linearly dependent.

By part (a), this means that

$w_1 = A(Bv_1)$, $w_2 = A(Bv_2)$, $\cdots$ $w_j = A(Bv_j)$

are linearly dependent. Since every collection of vectors in $\mathcal{R}(AB)$ that includes more than k vectors, is dependent, $\dim \mathcal{R}(AB) \le k$. //

(20 points) **2.** Let S and T be linear operators on the finite dimensional vector space $\mathcal{V}$ such that $ST = TS$. Show that if p is any polynomial in $\mathbb{F}[x]$, the nullspace of $p(S)$ is an invariant subspace for T.

Clearly $SI = IS$ for I the identity operator on $\mathcal{V}$.

Since $ST = TS$

we see $S^2T = S(ST) = S(TS) = (ST)S = (TS)S = TS^2$,

$S^3T = S(S^2T) = S(TS^2) = (ST)S^2 = (TS)S^2 = T(SS^2) = TS^3$

and so on, so we see

$S^nT = TS^n$ for every non-negative integer n.

Since $p(S)$ is a linear combination of monomials in S,

this means $p(S)T = T\,p(S)$ for every polynomial in $F[x]$.

Let p be a polynomial in $F[x]$ and let

$\mathcal{W}$ be the nullspace of $p(S)$ in $\mathcal{V}$.

If $w \in \mathcal{N}(p(S)) = \mathcal{W}$,

then $p(S)(Tw) = (p(S)T)w = T(p(S)w) = T0 = 0$

so $Tw \in \mathcal{N}(p(S)) = \mathcal{W}$ also.

This says $\mathcal{W}$ is an invariant subspace for T. //

(20 points) **3.** Let C and D be $n \times n$ matrices over the field $\mathbb{F}$.

(a) Prove that if $I - CD$ is invertible, then $I - DC$ is also invertible and

$$(I - DC)^{-1} = I + D(I - CD)^{-1}C$$

Since $I-CD$ is invertible, $(I-CD)^{-1}$ exists as an $n \times n$ matrix over F. Let $A = I + D(I-CD)^{-1}C$.

Then $A(I-DC) = (I + D(I-CD)^{-1}C)(I-DC)$

$= I(I-DC) + D(I-CD)^{-1}C(I-DC)$

$= I - DC + D(I-CD)^{-1}(C - CDC)$

$= I - DC + D(I-CD)^{-1}(I-CD)C$

$= I - DC + DIC = I - DC + DC = I$

Similarly, $(I-DC)A = (I-DC)(I + D(I-CD)^{-1}C) = I - DC + D(I-CD)(I-CD)^{-1}C$

$= I - DC + (D - DCD)(I-CD)^{-1}C$

$= I - DC + DIC = I - DC + DC = I.$

Since $(I-DC)A = I = A(I-DC)$, A is the inverse of $I-DC$, that is, $A = (I-DC)^{-1}$ as required.

(b) Use this result to show that CD and DC have the same eigenvalues over the field $\mathbb{F}$.

Part (a) says if 1 is not an eigenvalue of CD, then 1 is not and eigenvalue of DC. The contrapositive of this is, if 1 is an eigenvalue of DC then 1 is an eigenvalue of CD.
Reversing the roles of D and C, this gives 1 is an eigenvalue of CD if and only if 1 is an eigenvalue of DC.

Suppose $\lambda \neq 0$ is a number in F, and λ is an eigenvalue of CD, with eigenvector v, that is $(CD)v = \lambda v$.
Then v is an eigenvector for $(\frac{1}{\lambda} CD)$ with eigenvalue 1:

$$\left(\tfrac{1}{\lambda} CD\right)v = \tfrac{1}{\lambda} CDv = \tfrac{1}{\lambda}(\lambda v) = 1v.$$

Part (a) says, then, that 1 is an eigenvalue, for $\lambda \neq 0$, of $(\frac{1}{\lambda}C)D$ then 1 is an eigenvalue of $D(\frac{1}{\lambda}C) = \frac{1}{\lambda}DC$ and therefore that λ is an eigenvalue of DC.

This shows the non-zero eigenvalues of CD and DC are the same.

If 0 is not an eigenvalue of CD, then CD is invertible and $\mathcal{R}(CD) = F^n$ and $\mathcal{N}(CD) = (0)$. Since $\mathcal{R}(CD) \subset \mathcal{R}(C)$, we see $\mathcal{R}(C) = F^n$ also and the rank-nullity theorem shows $\mathcal{N}(C) = (0)$ so C is invertible.

Since $\mathcal{N}(CD) \supset \mathcal{N}(D)$ we see $\mathcal{N}(D) = (0)$ and the rank-nullity theorem show $\mathcal{R}(D) = F^n$ also, and this means D is invertible.

In particular, then 0 not an eigenvalue of CD implies C and D are invertible and DC is invertible so 0 is not an eigenvalue of DC either. Thus CD and DC have all eigenvalues the same.

(20 points) **4.** The complex number 1 is the only eigenvalue of the matrix $G = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(a) Find the characteristic polynomial and the minimal polynomial for G.

The characteristic polynomial of G is $(x-1)^4$ because 1 is the only complex number that is an eigenvalue and G is a 4×4 matrix so the characteristic polynomial of G has degree 4; $(x-1)^4$ is the only 4th degree polynomial whose only complex root is 1.

$G-I = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$ so clearly $\mathcal{R}(G-I) = \text{span}\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\\0\end{pmatrix}\right\} \neq (0)$

So minimal polynomial of G is not $p(x) = x-1$.

On the other hand $(G-I)^2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and minimal polynomial is $(x-1)^2$.

(b) Find a matrix J in Jordan Canonical Form that is similar to G. Explain your how you know that your answer is correct. (While one possible solution is finding a matrix S so that $S^{-1}GS$ is in Jordan form, you aren't required to find an S if you can determine J in some other way.)

Because the minimal polynomial of G is $(x-1)^2$ there must be at least one block of form $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

There are two Jordan forms for 4×4 matrices with only eigenvalue 1:

$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. The first of these has $\text{rank}(J_1 - I) = 1$ and the second has $\text{rank}(J_2 - I) = 2$

so, since $\text{rank}(G-I)$ is 2, J_2 is the Jordan form for G.

(20 points) **5.** Let R be an $n \times n$ matrix with entries in $\mathbb{C}$ that has n distinct eigenvalues. Prove that if Q is an $n \times n$ matrix with $QR = RQ$, then there is a polynomial p for which $Q = p(R)$.

Since R is an $n \times n$ complex matrix with n distinct eigenvalues, we know that R is diagonalizable; say S is a matrix whose columns are eigenvectors for R so that $S^{-1}RS = D$ where D is the diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ along the diagonal.

Now $S^{-1}QRS = (S^{-1}QS)(S^{-1}RS) = (S^{-1}QS)D$

and $S^{-1}RQS = (S^{-1}RS)(S^{-1}QS) = D(S^{-1}QS)$.

Since $QR = RQ$, we see that the matrix $C = S^{-1}QS$ satisfies $CD = DC$. Writing $C = (c_{ij})_{i=1\, j=1}^{n \quad n}$ for C and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ we see

$$CD = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \begin{pmatrix} c_{11}\lambda_1 & c_{12}\lambda_2 & \cdots & c_{1n}\lambda_n \\ c_{21}\lambda_1 & c_{22}\lambda_2 & & c_{2n}\lambda_n \\ \vdots & \vdots & & \vdots \\ c_{n1}\lambda_1 & c_{n2}\lambda_2 & & c_{nn}\lambda_n \end{pmatrix}$$

and
$$DC = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} c_{11}\lambda_1 & c_{12}\lambda_1 & \cdots & c_{1n}\lambda_1 \\ c_{21}\lambda_2 & c_{22}\lambda_2 & \cdots & c_{2n}\lambda_2 \\ \vdots & & & \\ c_{n1}\lambda_n & c_{n2}\lambda_n & \cdots & c_{nn}\lambda_n \end{pmatrix}$$

Thus for $i \neq j$ since $\lambda_i \neq \lambda_j$ comparing the ij entry of CD which is $c_{ij}\lambda_j$ with the ij entry of DC which is $c_{ij}\lambda_i$ we get $c_{ij}\lambda_i = c_{ij}\lambda_j$ or $c_{ij}(\lambda_i - \lambda_j) = 0$ and $c_{ij} = 0$ for $i \neq j$, so that C is also diagonal!

Since $\lambda_1, \ldots \lambda_n$ are distinct, there is a polynomial p of degree n so that $p(\lambda_i) = c_{ii}$ so $C = p(D)$. Since $R = SDS^{-1}$ and $Q = SCS^{-1}$

$p(R) = p(SDS^{-1}) = Sp(D)S^{-1} = SCS^{-1} = Q.$

(20 points) **6.** Let $H = \begin{pmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$

The characteristic polynomial for H is $(x+3)^2(x-6)$.
The vector $(1,2,2)$ is an eigenvector for the eigenvalue 6.

Find a unitary matrix U so that $U'HU$ is a diagonal matrix.

$$(H+3I) = \begin{pmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

Thus $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathcal{N}(H+3I)$ if and only if $a+2b+2c=0$.

Actually this is not surprising because $H=H'$, so the vectors in $\mathcal{N}(H+3I)$ must be orthogonal to vectors in the ~~eigenspace~~ $\mathcal{N}(H-6I)$ ~~of~~, which is spanned by $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, so $\left\langle \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = 0$, for $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathcal{N}(H+3I)$. $\mathcal{N}(H+3I) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

of course $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ is not an orthnormal (or even orthogonal) basis, but $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ are orthogonal so we want a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{C}^3$ so that $\left\langle \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = 0 = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$

so $\begin{cases} a+2b+2c = 0 \\ -2a+b = 0 \end{cases}$ this gives $b = 2a$ and $a+2(2a)+2c=0$ and $c = -\frac{5}{2}a$

so $a=2$ $b=4$ $c=-5$ satisfies $\left\langle \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix} \right\rangle = 0$

and we have an orthogonal basis for $\mathbb{C}^3$ consisting of eigenvectors for H.

and $u = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$, $v = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}$ $w = \begin{pmatrix} 2/3\sqrt{5} \\ 4/3\sqrt{5} \\ -\frac{\sqrt{5}}{3} \end{pmatrix}$ is an orthonormal basis and

for $U = \begin{pmatrix} \frac{1}{3} & -2/\sqrt{5} & 2/3\sqrt{5} \\ \frac{2}{3} & 1/\sqrt{5} & 4/3\sqrt{5} \\ \frac{2}{3} & 0 & -\sqrt{5}/3 \end{pmatrix}$ satisfies $U'HU = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

(20 points) **7.** (a) Prove that if S is an $n \times n$ complex matrix such that $S' = S$, then the eigenvalues of S^2 are non-negative real numbers.

Since $S' = S$, the eigenvalues of S are real numbers. If λ is an eigenvalue of S with eigenvector v, then $S^2 v = \lambda^2 v$, and the eigenvalues of S^2 are all squares of real numbers, that is they are non-negative real numbers.

(b) Let T be an $n \times n$ complex matrix such that $T' = T$. Prove that if the eigenvalues of T are non-negative real numbers, then there is a matrix S with $S' = S$ such that $S^2 = T$ and the eigenvalues of S are all non-negative real numbers.

Suppose $T = T'$ and the eigenvalues of T are non-negative real numbers. Now there is a unitary matrix U so that $U'TU = D$ where D is a diagonal matrix with eigenvalues, that is, diagonal entries, $\lambda_1, \lambda_2, \ldots \lambda_n$ where the $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of T.

Because each $\lambda_j \geq 0$, there is a non-negative real number μ_j so that $\mu_j^2 = \lambda_j$ (i.e. $\mu_j = \sqrt{\lambda_j}$).

Let E be the diagonal matrix with entries $\mu_1, \mu_2, \ldots \mu_n$. Clearly $E^2 = D$. Now let $S = UEU'$. We see $S' = (UEU')' = U''E'U' = UEU' = S$

and $S^2 = (UEU')(UEU') = UE^2U' = UDU' = U(U'TU)U' = T$.

(It is easy to see that the eigenvalues of S along with multiplicities are uniquely determined.)

(20 points) **8.** The vectors $u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ $u_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ and $u_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ span the subspace $\mathcal{U}$ in $\mathbb{R}^4$.

These vectors are expressed in terms of the usual, Euclidean basis for $\mathbb{R}^4$, $\mathcal{E}_4 = \{e_1, e_2, e_3, e_4\}$, in the sense that $u_1 = e_1 - e_2 + e_3$.

(a) Choose a vector u_4 so that $\mathcal{B} = \{u_1, u_2, u_3, u_4\}$ is a basis for $\mathbb{R}^4$ and show that it is a basis.

Clearly $u_2 \neq \alpha u_1$ for any α, so $\{u_1, u_2\}$ is linearly independent,
$u_3 \neq \alpha u_1 + \beta u_2 = \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 2\beta \\ 0 \\ 0 \\ -\beta \end{pmatrix}$ because to make second coordinate of u_3 be 0,
means $\alpha = 0$ and $u_3 \neq \beta u_2$ for any β, so $\{u_1, u_2, u_3\}$ is linearly independent.
Choose $u_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ [Of course there are many other choices possible!]
For $u_4 = \alpha u_1 + \beta u_2 + \gamma u_3$, then from second coordinate, we must have $\alpha = -1$
and $\gamma = \alpha = -1$ (from third coordinate) and $\beta = -1$ from 4^{th} coordinate, but
$u_4 \neq -u_1 - u_2 - u_3$ so $\{u_1, u_2, u_3, u_4\}$ are linearly independent.

(b) Find linear functionals f_1, f_2, f_3, and f_4 that form a dual basis for $\mathcal{B}$ (chosen in (a)), and thus form a basis for the dual space of $\mathbb{R}^4$. To explain precisely what linear functionals you mean, tell what the values of f_j acting on the basis $\mathcal{E}_4$ are, that is, give the values $f_j(e_1)$, $f_j(e_2)$, $f_j(e_3)$, and $f_j(e_4)$, for $1 \leq j \leq 4$.

(b) Find linear functionals f_1, f_2, f_3, and f_4 that form a dual basis for $\mathcal{B}$ (chosen in (a)), and thus form a basis for the dual space of $\mathbb{R}^4$. To explain precisely what linear functionals you mean, tell what the values of f_j acting on the basis $\mathcal{E}_4$ are, that is, give the values $f_j(e_1)$, $f_j(e_2)$, $f_j(e_3)$, and $f_j(e_4)$, for $1 \le j \le 4$.

Having chosen $u_4 = (0,1,0,0)$, we want to find f_1, f_2, f_3, f_4 So that $f_i(u_j) = 1$ if $i=j$ $f_i(u_j)=0$ if $i \neq j$.
In particular, we want $f_1(u_1)=1$, $f_1(u_2)=f_1(u_3)=f_1(u_4)=0$
In terms of the standard basis, $\{e_1,e_2,e_3,e_4\}$, letting $f_1(e_1)=a$, $f_1(e_2)=b$, $f_1(e_3)=c$, $f_1(e_4)=d$, then $f_1(u_1) = a-b+c=1$, and we get the system

$$\begin{cases} a-b+c = 1 \\ 2a - d = 0 \\ a - c + d = 0 \\ b = 0 \end{cases} \quad \text{so} \quad \begin{aligned} a &= 1/4 \\ b &= 0 \\ c &= 3/4 \\ d &= 1/2 \end{aligned}$$

for f_2 we get

$$\begin{cases} a-b+c = 0 \\ 2a - d = 1 \\ a - c + d = 0 \\ b = 0 \end{cases} \quad \text{so} \quad \begin{aligned} a &= \tfrac{1}{4} \\ b &= 0 \\ c &= 3/4 \\ d &= -1/2 \end{aligned}$$

Continuing, and summarizing, we get

$f_1(e_1) = 1/4$	$f_2(e_1) = 1/4$	$f_3(e_1) = \frac{1}{4}$	$f_4(e_1) = 1/4$
$f_1(e_2) = 0$	$f_2(e_2) = 0$	$f_3(e_2) = 0$	$f_4(e_2) = 1$
$f_1(e_3) = \frac{3}{4}$	$f_2(e_3) = -1/4$	$f_3(e_3) = -\frac{1}{4}$	$f_4(e_3) = \frac{3}{4}$
$f_1(e_4) = \frac{1}{2}$	$f_2(e_4) = -1/2$	$f_3(e_4) = \frac{1}{2}$	$f_4(e_4) = \frac{1}{2}$

8. (continued)

(c) Find the annihilator of the subspace $\mathcal{U}$, that is, find $\mathcal{U}^o$ the subspace of the dual of $\mathbb{R}^4$ that annihilates the vectors in $\mathcal{U}$. Express your answer in terms of the vectors in the basis $\mathcal{B}^* = \{f_1, f_2, f_3, f_4\}$.

Since $\mathcal{U} = \text{span}\{u_1, u_2, u_3\}$, we want to see which f in the dual of $\mathbb{R}^4$ satisfy $f(u) = 0$ for all u in $\mathcal{U}$.

Suppose f in $\mathcal{U}^0$ is $f = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$ where f_1, f_2, f_3, f_4 ~~form~~ the dual basis from part (b)

$f \in \mathcal{U}^0$ if and only if $f(u_1) = f(u_2) = f(u_3) = 0$ because u_1, u_2, and u_3 are in $\mathcal{U}$ and every vector in $\mathcal{U}$ is a linear combination of them.

In particular if $f = \alpha f_1 + \beta f_2 + \gamma f_3 + \delta f_4$

then $f(u_1) = \alpha f_1(u_1) + \beta f_2(u_1) + \gamma f_3(u_1) + \delta f_4(u_1)$
$= \alpha \cdot 1 + \beta \cdot 0 + \gamma \cdot 0 + \delta \cdot 0 = \alpha$

for $f(u_1) = 0$, then $\alpha = 0$

Similarly, $f(u_2) = 0 \Rightarrow \beta = 0$

and $f(u_3) = 0 \Rightarrow \gamma = 0$

Our conclusion then is $\mathcal{U}^0 = \text{span}\{f_4\}$.