The following results are true and you can use them in your solutions of the homework problems, if you wish, even though we will not have time to go over them in class.

**Lemma.** *If $A$ is in $\mathcal{B}(X)$ for $X$ a Banach space, then $\lim_{n\to\infty}\|A^n\|^{1/n}$ exists.*

*Proof.* Note that $0 \leq \|A^n\|^{1/n} \leq (\|A\|^n)^{1/n} = \|A\|$ so $\liminf_{n\to\infty}\|A^n\|^{1/n}$ and $\limsup_{n\to\infty}\|A^n\|^{1/n}$ are both finite.

For $m$, $k$, and $j$ positive integers, we have

$$\|A^{mk+j}\|^{1/(mk+j)} \leq \|A^m\|^{k/(mk+j}\|A\|^{j/(mk+j)}$$

Keeping $m$ fixed temporarily, for each $j = 0, 1, 2, \cdots, m-1$, this implies

$$\limsup_{k\to\infty}\|A^{mk+j}\|^{1/(mk+j)} \leq \lim_{k\to\infty}\|A^m\|^{k/(mk+j}\|A\|^{j/(mk+j)} = \|A^m\|^{1/m}$$

If we have a sequence $n_\ell$ such that $\lim_{\ell\to\infty}\|A^{n_\ell}\|^{1/n_\ell} = \limsup_{n\to\infty}\|A^n\|^{1/n}$, for a fixed $m$, since there are infinitely many $n_\ell$'s and only finitely many integers $j$ with $0 \leq j \leq m-1$, for at least one of these $j$'s, call it $j_0$, infinitely many of the $n_\ell$'s satisfy $n_\ell = mk + j_0$ for some $k$, we have

$$\limsup_{k\to\infty}\|A^{mk+j_0}\|^{1/(mk+j_0)} = \limsup_{k\to\infty}\|A^{mk+j}\|^{1/(mk+j)} = \limsup_{n\to\infty}\|A^n\|^{1/n}$$

That is, for each $m$, there is a $j_m$ so that

$$\limsup_{n\to\infty}\|A^n\|^{1/n} \leq \limsup_{k\to\infty}\|A^{mk+j_m}\|^{1/(mk+j_m)} \leq \|A^m\|^{1/m}$$

Thus, since this is true for each $m$, we have

$$\limsup_{n\to\infty}\|A^n\|^{1/n} \leq \liminf_{m\to\infty}\|A^m\|^{1/m}$$

but this means $\lim_{n\to\infty}\|A^n\|^{1/n}$ exists. $\square$

## Holomorphic (or Riesz) Functional Calculus

**Definition.** Let $T$ be in $\mathcal{B}(X)$ for $X$ a Banach space. If $f$ is a complex valued function defined on an open set $\Omega \supset \sigma(T)$ such that $f$ is holomorphic on each component of $\Omega$, we say $f$ is *locally analytic* on $\sigma(T)$. Denote the algebra of such functions by $\mathcal{A}(T)$.

Let $f$ be locally analytic on $\sigma(T)$ with domain $\Omega$ and let $\gamma$ be a cycle (i.e. a finite set of closed curves $\gamma_j$ for $j = 1, 2, \cdots, n$) in $\Omega$ such that for all points $\lambda$ of $\sigma(T)$, the index $\mathrm{ind}_\gamma(\lambda) = 1$ and for all points $\mu$ in $\mathbb{C} \setminus \Omega$, $\mathrm{ind}_\gamma(\mu) = 0$. Then we define $f(T)$ by

$$f(T) = \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda I - T)^{-1}\,d\lambda$$

**Theorem 5. (Spectral Mapping Theorem.)** *The map of $\mathcal{A}(T)$ into $\mathcal{B}(X)$ given by $f \mapsto f(T)$ is an algebra homomorphism of $\mathcal{A}(T)$ into $\mathcal{B}(X)$ such that*

- $1 \mapsto I$ *and* $z^k \mapsto T^k$
- *For complex numbers $\alpha$ and $\beta$ and $f$ and $g$ in $\mathcal{A}(T)$, $(\alpha f + \beta g)(T) \mapsto \alpha f(T) + \beta g(T)$ and $f \cdot g \mapsto f(T)g(T)$*
- $\sigma(f(T)) = f(\sigma(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}$
- *If $S$ is in $\mathcal{B}(X)$ such that $ST = TS$, then $Sf(T) = f(T)S$ for each $f$ in $\mathcal{A}(T)$*
- *If $f_n$, for $n = 1, 2, 3, \cdots$ and $f$ are in $\mathcal{A}(T)$ with a common domain $\Omega$ and the $f_n$ converge uniformly on compact subsets of $\Omega$ to $f$, then $f_n(T)$ converges to $f(T)$ in norm.*
- *Suppose $f$ is analytic on the disk $D_r = \{\lambda : |\lambda| < r\}$ and $f(z) = \sum_{k=0}^\infty a_k z^k$. If $T$ is in $\mathcal{B}(X)$ with $\sigma(T) \subset D_r$, then $f(T) = \sum_{k=0}^\infty a_k T^k$*
- *If $f$ is locally analytic on $\sigma(T)$ and $g$ is locally analytic on $f(\sigma(T))$, then $g \circ f$ is locally analytic on $\sigma(T)$ and $(g \circ f)(T) = g(f(T))$*

**Definition.** If $T$ is in $\mathcal{B}(X)$ for $X$ a Banach space and $M$ is a closed subspace of $X$ such that $Tx$ is in $M$ for each $x$ in $M$, we say $M$ *is an invariant subspace for* $T$. We say $M$ is a *non-trivial invariant subspace for* $T$ if $M$ is an invariant subspace for $T$ and $M \neq (0)$ and $M \neq X$.

**Theorem 6. (Riesz Projection Theorem.)** *Suppose $T$ in $\mathcal{B}(X)$ for $X$ a Banach space satisfies $\sigma(T) = K_1 \cup K_0$ where $K_1$ and $K_0$ are non-empty, disjoint, compact sets. If $\Omega_1$ and $\Omega_0$ are disjoint open sets for which $\Omega_1 \supset K_1$ and $\Omega_0 \supset K_0$, define $f$ by $f(z) = 1$ for $z \in \Omega_1$ and $f(z) = 0$ for $z \in \Omega_0$. Then*

- *$f$ is locally analytic on $\sigma(T)$ and $P = f(T)$ is a bounded projection that commutes with $T$*
- *If $M_1 = PX$ and $M_0 = (I - P)X$, then $M_1$ and $M_0$ are closed invariant subspaces for $T$ and $X = M_1 \oplus M_0$*
- *$\sigma(T|_{M_1}) = K_1$ and $\sigma(T|_{M_0}) = K_0$*

Note: problems 84, 85, 89, and 94 are related to these ideas.