- Write your answers on the test paper!
- For decimal approximations, it is enough to give 2 decimal places.
- Show enough of your work that your reasoning can be followed.
- There are 7 pages, 7 questions, and 100 points on this test.

(10 points) 1. Find all solutions of the following system:

$$\begin{cases} v + w + x + y + z = 0 \\ w + 2x + 3y + 4z = 3 \\ 5v + 6w + 7x + 8y + 9z = 3 \\ v + w + x = -1 \end{cases}$$

If there is a unique solution or no solution, say so. If there are infinitely many solutions, find the general solution and *find three solutions* explicitly.

(15 points) 2. Consider the following two systems of equations.

$$(H)\begin{cases} a + b + 2c - d + e - f = 0 \\ 2a + b - c + d - 2e = 0 \\ 2a + b - c + d - 2f = 0 \\ a + 3b - 2c + 3d - e - f = 0 \\ 2a + b + d - 2e - f = 0 \end{cases}$$

and

$$(N)\begin{cases} a + b + 2c - d + e - f = 3 \\ 2a + b - c + d - 2e = 6 \\ 2a + b - c + d - 2f = 2 \\ a + 3b - 2c + 3d - e - f = 3 \\ 2a + b + d - 2e - f = 6 \end{cases}$$

The vector $(1, -1, 1, 2, 1, 1)$ is a solution of system (H).
The vector $(2, 1, 1, 0, -1, 1)$ is a solution of system (N).
Use theorems you know (i.e. do *not* try to solve the systems completely!) to answer the following questions.

(a) Find two other non-trivial solutions of (H).

(b) Find two other solutions of (N).

(15 points) 3. Let $v_1 = (1, -1, 2)$, $v_2 = (-1, 2, -1)$, $v_3 = (2, -1, 5)$, and $v_4 = (1, 1, 5)$.

(a) Explain how you can tell that $\{v_1, v_2, v_3, v_4\}$ is a linearly dependent set without doing any calculations.

(b) Write one of the vectors as a linear combination of the rest.

(15 points) 4. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & -1 & -5 & 1 & -3 \\ -1 & 1 & 3 & 0 & 4 \\ 1 & 1 & -1 & 2 & 2 \end{pmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Choose columns of A that form a basis for the range of A, $\mathcal{R}(A)$. .

(c) What is the dimension of the range of A', $\mathcal{R}(A')$?

(d) What is the dimension of the nullspace of A', $\mathcal{N}(A')$?

(15 points) 5. The matrix C is

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{pmatrix}$$

and the matrix D is a 3×5 matrix with rows D_1, D_2, and D_3.

(a) Write D as a block matrix, blocked in rows, and find CD as a block matrix, blocked in rows.

(b) Suppose, now, that the rank of D is 2 and $D_2 = -2D_1$. Find a basis for the row space of D.

(c) Suppose, as in part (b), that the rank of D is 2 and $D_2 = -2D_1$. Find a basis for the row space of CD.

(15 points) 6. Let $p_1 = (1, -1, 1, 1, 1)$, let $p_2 = (1, 1, 0, -2, 2)$, and let $q = (5, 1, 2, -4, 8)$.

(a) Write q as a linear combination of p_1 and p_2.

(b) Let $r = (1, -5, 3, 7, -1)$. Write r as a linear combination of p_1, p_2, and q or show that it is not a linear combination of them.

(c) What is the dimension of the subspace spanned by p_1, p_2, q, and r? ______________
Explain your answer!

(15 points) 7. (a) Prove the following theorem:

Let A be an $m \times n$ matrix. If B is an $n \times m$ matrix so that $BA = I$, then $\mathcal{N}(A) = (0)$ (that is, the null space of A is the zero subspace).

(b) Prove the following theorem:
(You may use the result of part (a) even if you did not answer part (a)!)

Let A be an $m \times n$ matrix.
If B is an $n \times m$ matrix so that $BA = I$, then the columns of A are linearly independent.