Math 353 (Cowen) **Real vs Complex Eigenvectors and Reading Problem** **5 April 2011**

In class today, we faced the question "If A is a real matrix, does the null space of A in $\mathbb{R}^n$ have the same dimension as the nullspace of A in $\mathbb{C}^n$?" and the question "If A is a real matrix, λ is a real number, and v is a vector in $\mathbb{C}^n$ that is an eigenvector of A for λ, is there a vector w in $\mathbb{R}^n$ that is an eigenvector of A for λ?" The answer for both questions is YES!

Proof. Let v be a vector in $\mathbb{C}^n$ such that $Av = 0$. Then taking complex conjugates, we get

$$\overline{A}\overline{v} = \overline{Av} = \overline{0}$$

where $\overline{v}$ is the vector whose j^{th} component is $\overline{v_j}$ where v_j is the j^{th} component of v and $\overline{A}$ is the matrix whose entries are the conjugates of the entries of A. But since A is a real matrix and 0 is a real vector, we have

$$A\overline{v} = \overline{A}\overline{v} = \overline{Av} = \overline{0} = 0$$

In other words, if v is in the null space of A, then $\overline{v}$ is also in the nullspace of A. This means that $x = (v + \overline{v})/2$ and $y = (v - \overline{v})/(2i)$, which are both real vectors, are also in the null space of A. (The vectors x and y might be called the real and imaginary parts of v.) It follows easily, now, that the dimensions of the null space of A in $\mathbb{R}^n$ and $\mathbb{C}^n$ are the same.

Similarly, we can easily see that if $Av = \lambda v$ where λ and A are real, then $A\overline{v} = \lambda\overline{v}$, so the conjugates of eigenvectors for A are also eigenvectors and so are the real and imaginary parts. Thus, A has real eigenvectors.

You may have seen this argument in Math 26200 in connection with the solution of the differential equation $y'' + y = 0$ which has complex solutions e^{ix} and e^{-ix} or real solutions $\cos(x)$ and $\sin(x)$. $\square$

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8. Read the paragraphs on the last page, then answer the following questions:

(a) Explain why (fifth line from the bottom of the proof) that the range of V is M.

Let M be the subspace with (orthonormal) basis $v_1 = (.5, .5, .5, .5)$ and $v_2 = (.5, -.5, -.5, .5)$
Let G be the 4×4 matrix

$$G = \begin{pmatrix} 5 & 4 & 1 & -1 \\ 4 & -11 & 2 & -3 \\ 1 & 2 & 0 & 1 \\ -1 & -3 & 1 & -2 \end{pmatrix}$$

(b) Find the minimum value of $\langle Gx, x \rangle$ for x in M with $\|x\| = 1$.

(c) Find a vector x in M with $\|x\| = 1$ and $\langle Gx, x \rangle$ equal to the minimum value found in (b).

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Reading for Problem 8

(It is OK to remove this page from the exam!!)

Suppose A is an $k \times k$ matrix such that $A = A'$. If we want to minimize $<Ay, y>$ for y in $\mathbb{C}^k$ with $\|y\| = 1$, we may proceed as follows: There is an orthonormal basis $u_1, u_2, \cdots, u_k$ for $\mathbb{C}^k$ consisting of eigenvectors for A, say $Au_j = \lambda_j u_j$. The eigenvalues of A are real numbers and we assume that they have been arranged so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

For y in $\mathbb{C}^k$ with $\|y\| = 1$, there are scalars α_j so that

$$y = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$$

and

$$\begin{aligned} 1 = \|y\|^2 = \, <y, y> \, &= \, <\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k, \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k> \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} \overline{\alpha_i} \alpha_j <u_i, u_j> \, = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_k|^2 \end{aligned}$$

Now

$$<Ay, y> \, = \, <A\left(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k\right), \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k>$$

$$= \, <\alpha_1 \lambda_1 u_1 + \alpha_2 \lambda_2 u_2 + \cdots + \alpha_k \lambda_k u_k, \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k> \, = \lambda_1 |\alpha_1|^2 + \lambda_2 |\alpha_2|^2 + \cdots + \lambda_k |\alpha_k|^2$$

But since $\lambda_j \geq \lambda_k$, this shows

$$<Ay, y> \, \geq \lambda_k |\alpha_1|^2 + \lambda_k |\alpha_2|^2 + \cdots + \lambda_k |\alpha_n|^2 = \lambda_k \left(|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_k|^2\right) = \lambda_k$$

On the other hand, $\|u_k\| = 1$ and $<Au_k, u_k> \, = \, <\lambda_k u_k, u_k> \, = \lambda_k$, so we see that λ_k is the minimum value of $<Ay, y>$ for $\|y\| = 1$.

Now for M a k–dimensional subspace of $\mathbb{C}^n$ and B an $n \times n$ Hermitian matrix, suppose we wish to minimize $<Bx, x>$ such that x is in M and $\|x\| = 1$. Choose $v_1, v_2, \cdots, v_k$ an orthonormal basis for M, and let V be the $n \times k$ matrix with these columns. Since V has orthonormal columns, $V'V = I$, and if y is a vector in $\mathbb{C}^k$

$$\|Vy\|^2 = \, <Vy, Vy> \, = \, <V'Vy, y> \, = \, <y, y> \, = \|y\|^2$$

The fact that M is the range of V, implies that for x in M with $\|x\| = 1$, there is y in $\mathbb{C}^k$ with $Vy = x$ and $\|y\| = 1$. It follows that

$$<Bx, x> \, = \, <BVy, Vy> \, = \, <V'BVy, y>$$

so minimizing $<Bx, x>$ with x in M and $\|x\| = 1$ is the same as minimizing $<V'BVy, y>$ with y in $\mathbb{C}^k$ and $\|y\| = 1$. Since $(V'BV)' = V'BV$, we can find this minimum value as above.