Questions from Old First Tests for Math 51100

1. The following system has infinitely many solutions:

$$\left\{\begin{array}{rcrcrcrcrcl} v & + & 2w & - & x & + & 2y & - & z & = & 5 \\ & & w & + & 2x & - & y & - & 2z & = & -4 \\ 2v & + & 3w & + & 2x & + & y & & & = & 4 \\ v & & & + & x & & & + & 3z & = & 3 \end{array}\right.$$

Find the general solution and *find two solutions* explicitly.

2. The following system has infinitely many solutions:

$$\left\{\begin{array}{rcrcrcrcrcl} v & + & w & + & x & + & y & + & z & = & 0 \\ & & w & + & 2x & + & 3y & + & 4z & = & -1 \\ 9v & + & 8w & + & 7x & + & 6y & + & 5z & = & 1 \\ v & + & w & + & x & & & & & = & 1 \end{array}\right.$$

Find the general solution and *find two solutions* explicitly.

3. The following system has infinitely many solutions:

$$\left\{\begin{array}{rcrcrcrcrcl} v & - & 2w & + & x & - & 2y & - & z & = & -1 \\ 2v & - & w & - & 2x & + & y & & & = & 4 \\ & + & 3w & + & 2x & + & y & + & 2z & = & 2 \\ 3v & & & + & x & & & + & z & = & 5 \end{array}\right.$$

Find the general solution and *find two solutions* explicitly.

4. Let $\mathcal{W}$ be the column space of B, the subspace spanned by the columns of B, where

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 2 & -1 & -5 & 1 & -3 \\ -1 & 1 & 3 & 0 & 4 \\ 1 & 1 & -1 & 2 & 2 \end{pmatrix}$$

   (a) What is the dimension of $\mathcal{W}$?

   (b) Choose columns of B that form a basis for $\mathcal{W}$.

5. Is the vector $v = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ in the range of C, where $C = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & -1 & -2 & 1 \\ 3 & 1 & 0 & -2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$?

Explain why or why not!!

6. Consider the following two systems of equations.

$$(H)\begin{cases} -2a + b + c + e + 3f = 0 \\ 3a - b - 2c - 3d - 2e - 2f = 0 \\ -5a + 2b + 4c + 5d + 5e + 4f = 0 \\ 4a - b - 2c - 4d - 2e - 2f = 0 \\ -3a + b + 4c + 7d + 5e = 0 \end{cases}$$

and

$$(N)\begin{cases} -2a + b + c + e + 3f = 1 \\ 3a - b - 2c - 3d - 2e - 2f = 1 \\ -5a + 2b + 4c + 5d + 5e + 4f = 1 \\ 4a - b - 2c - 4d - 2e - 2f = 3 \\ -3a + b + 4c + 7d + 5e = 0 \end{cases}$$

The vectors $(-1, 0, 1, -1, 0, -1)$ and $(1, 0, -1, 1, 0, 1)$ are solutions of system (H).
The vectors $(2, 1, 0, 0, 1, 1)$ and $(1, -3, 2, -1, 1, 1)$ are solutions of system (N).
Use theorems you know (i.e. do *not* try to solve the systems completely!) to answer the following questions.

(a) Find two linearly independent solutions of (H) different from those given above.

(b) Find two solutions of (N) different from those given above.

7. Consider the following two systems of equations.

$$(H)\begin{cases} a - 2b + c + e + 3f = 0 \\ -a + 3b - 2c - 3d - 2e - 2f = 0 \\ 2a - 5b + 4c + 5d + 5e + 4f = 0 \\ -a + 4b - 2c - 4d - 2e - 2f = 0 \\ a - 3b + 4c + 7d + 5e = 0 \end{cases}$$

and

$$(N)\begin{cases} a - 2b + c + e + 3f = 3 \\ -a + 3b - 2c - 3d - 2e - 2f = -2 \\ 2a - 5b + 4c + 5d + 5e + 4f = 6 \\ -a + 4b - 2c - 4d - 2e - 2f = -1 \\ a - 3b + 4c + 7d + 5e = 3 \end{cases}$$

The vectors $(0, -1, 1, -1, 0, -1)$ and $(0, 1, -1, 1, 0, 1)$ are solutions of system (H).
The vectors $(1, 1, 0, 0, 1, 1)$ and $(-3, 0, 2, -1, 1, 1)$ are solutions of system (N).
Use theorems you know (i.e. do *not* try to solve the systems completely!) to answer the following questions.

(a) Find two linearly independent solutions of (H) different from those given above.
(b) Find two solutions of (N) different from those given above.

8. For each of the situations below, decide which of the statements in the box below are true about systems with this matrix as the coefficient matrix. *Include all correct responses.*

   (a) B is a 7×12 matrix.
   If the rank of B is 5, the system $BX = b$ ______________.

   If the rank of B is 7, the system $BX = b$ ______________.

   If the rank of B is 9, the system $BX = b$ ______________.

   (b) C is a 11×7 matrix.
   If the rank of C is 5, the system $CX = b$ ______________.

   If the rank of C is 7, the system $CX = b$ ______________.

   If the rank of C is 9, the system $CX = b$ ______________.

(i) has at *least* one solution for *every* vector b.
(ii) has no solutions for *some* vectors b.
(iii) has at *most* one solution for *every* vector b.
(iv) has infinitely many solutions for *some* vectors b.
(v) The given information is contradictory, no such system is possible.
(vi) Using (only) the information given does not permit us to conclude that *any* of the assertions (i) - (v) is necessarily true.

9. The matrix $C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{pmatrix}$

   and the matrix D is a 3×5 matrix with rows D_1, D_2, and D_3.

   (a) Write D as a block matrix, blocked in rows, and find CD as a block matrix, blocked in rows.
   (b) Suppose, now, that the rank of D is 2 and $D_2 = -2D_1$. Find a basis for the row space of D.
   (c) Suppose, as in part (b), that the rank of D is 2 and $D_2 = -2D_1$.
   Find a basis for the row space of CD.

10. Let $\mathcal{W}$ be the range of A where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & -5 & 2 & 3 \\ 1 & 0 & 3 & -1 & -2 \\ 1 & 2 & -1 & 1 & 0 \end{pmatrix}$$

   (a) Find a vector in $\mathbb{R}^4$, *NOT* one of the columns of A, that is in $\mathcal{W}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

   (b) Find a vector in $\mathbb{R}^4$, *NOT* one of the columns of A, that is *NOT* in $\mathcal{W}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

11. Let $p_1 = (1, -1, 1, 1, 1)$, let $p_2 = (1, 1, 0, -2, 2)$, and let $q = (5, 1, 2, -4, 8)$.

   (a) Write q as a linear combination of p_1 and p_2.

   (b) Let $r = (1, -5, 3, 7, -1)$. Write r as a linear combination of p_1, p_2, and q or show that it is not a linear combination of them.

   (c) What is the dimension of the subspace spanned by p_1, p_2, q, and r? ____________
   Explain your answer!

12. Let $\mathcal{U}$ be the column space of C, the subspace spanned by the columns of C, where

$$C = \begin{pmatrix} 1 & 1 & -1 & 3 & 0 \\ 2 & -1 & 4 & 3 & 5 \\ -1 & 1 & -3 & -1 & -4 \\ 1 & 1 & -1 & 3 & 2 \end{pmatrix}$$

   (a) Find a vector in $\mathbb{R}^4$, *NOT* one of the columns of C, that is in $\mathcal{U}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

   (b) Find a vector in $\mathbb{R}^4$, *NOT* one of the columns of C, that is *NOT* in $\mathcal{U}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

13. Let $w_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 1 \\ -5 \\ 3 \\ -1 \end{pmatrix}$, $w_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$, and $w_5 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

   (a) Explain how you can tell that $\{w_1, w_2, w_3, w_4, w_5\}$ is a linearly dependent set without doing any calculations.

   (b) Write one of the vectors as a linear combination of the rest.

14. Let $y_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $y_2 = \begin{pmatrix} 1 \\ -5 \\ 3 \\ -1 \end{pmatrix}$, $y_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$, $y_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, and $y_5 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

   (a) Explain how you can tell that $\{y_1, y_2, y_3, y_4, y_5\}$ is a linearly dependent set without doing any calculations.

   (b) Write one of the vectors as a linear combination of the rest.

15. Let $y_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $y_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -5 \end{pmatrix}$, $y_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $y_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, and $y_5 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.

   (a) Explain how you can tell that $\{y_1, y_2, y_3, y_4, y_5\}$ is a linearly dependent set without doing any calculations.

   (b) Write one of the vectors as a linear combination of the rest.

16. Suppose the vectors $\{u_1, u_2, u_3, u_4\}$ are a basis for $\mathcal{U}$ which is a subspace of $\mathbb{R}^7$.

(a) What is the dimension of $\mathcal{U}$? ____________

(b) Let $y_1 = 2u_1 - 3u_4$, $y_2 = u_1 + 2u_2 - u_3$, $y_3 = u_2 + 2u_3 - u_4$, and $y_4 = -u_1 + 4u_2 + 3u_3 + u_4$. Are y_1, y_2, y_3, and y_4 linearly independent? If so, explain how you know, including justifying how the computation you did relates to your answer.
If not, write one of these vectors as a linear combination of the other three.

(c) What is the dimension of $\text{span}\{y_1, y_2, y_3, y_4\}$? ____________

17. Let $\mathcal{U}$ be the range of B where

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ -1 & 1 & -5 & 3 & 2 \\ 1 & 2 & -1 & 0 & 1 \\ 1 & 0 & 3 & -2 & -1 \end{pmatrix}$$

(a) Find a vector in $\mathbb{R}^4$, *NOT* one of the columns of B, that is in $\mathcal{U}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

(b) Find a vector in $\mathbb{R}^4$, *NOT* one of the columns of B, that is *NOT* in $\mathcal{U}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

18. Let $\mathcal{W}$ be the range of A where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & -5 & 2 & 3 \\ 1 & 0 & 3 & -1 & -2 \\ 1 & 2 & -1 & 1 & 0 \end{pmatrix}$$

(a) Find a vector, *NOT* one of the columns of A, that is in $\mathcal{W}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

(b) Find a vector, *NOT* one of the columns of A, that is *NOT* in $\mathcal{W}$ or explain why this is not possible. (Explain how you know your answer is correct!!)

19. Let $q_1 = (0, -4, 2, 6, -2)$, let $q_2 = (1, -1, 1, 1, 1)$, and let $r = (5, 1, 2, -4, 8)$.

(a) Write r as a linear combination of q_1 and q_2.

(b) Let $p = (2, 0, 1, -1, 3)$. Write p as a linear combination of q_1, q_2, and r or show that it is not a linear combination of them.

(c) What is the dimension of the subspace spanned by q_1, q_2, r, and p? ____________
Explain your answer!

20. For each of the following, decide if the statement is *always true* or *always false* or *sometimes true, sometimes false* when the given condition is true.

(a) **Given:** A is a 6×6 matrix, b is in $\mathbb{R}^6$, $AX = 0$ has infinitely many solutions.
Statement: The equation $AX = b$ has infinitely many solutions.

always true | always false | sometimes true, sometimes false

(b) **Condition:** B is a 6×6 matrix, b is in $\mathbb{R}^6$, and $\det(B) = 0$.
Statement: The equation $BX = b$ has no solutions.

always true | always false | sometimes true, sometimes false

(c) **Given:** C is a 6×6 matrix with $\det(C) = 5$.
Statement: The equation $BX = b$ has infinitely many solutions.

always true | always false | sometimes true, sometimes false

(d) **Given:** D is a 6×6 matrix, b is in $\mathbb{R}^6$, and the columns of D are linearly independent.
Statement: The equation $DX = b$ has no solutions.

always true | always false | sometimes true, sometimes false

(e) **Given:** E is a 6×8 matrix, b is in $\mathbb{R}^6$, $\mathcal{N}(E)$, the nullspace of E is 2–dimensional.
Statement: The equation $EX = b$ has infinitely many solutions.

always true | always false | sometimes true, sometimes false

(f) **Given:** F is a 7×5 matrix, b is in $\mathbb{R}^7$, $\mathcal{N}(F)$, the nullspace of F is 1–dimensional.
Statement: The equation $FX = b$ has infinitely many solutions.

always true | always false | sometimes true, sometimes false

(g) **Condition:** G is an $m \times n$ matrix and the vectors $w_1, w_2, \cdots, w_j$ are linearly independent in $\mathbb{R}^n$.
Statement: The vectors $Gw_1, Gw_2, \cdots, Gw_j$ are linearly independent.

always true | always false | sometimes true, sometimes false

(h) **Given:** The vectors v_1, v_2, v_3, v_4, v_5, v_6 span $\mathbb{R}^5$.
Statement: The vectors v_1, v_2, v_3, v_4, v_5, v_6 are linearly independent.

always true | always false | sometimes true, sometimes false

21. Suppose the vectors $\{v_1, v_2, v_3, v_4\}$ are a basis for $\mathcal{V}$ which is a subspace of $\mathbb{R}^7$.

(a) What is the dimension of $\mathcal{V}$? ____________

(b) Let $x_1 = v_1 + 2v_2 - v_3$, $x_2 = 2v_1 + v_3 - 2v_4$, $x_3 = v_2 + 2v_3 - v_4$, and $x_4 = -v_1 + 4v_2 + 2v_3$. Are x_1, x_2, x_3, and x_4 linearly independent? If so, explain how you know, including justifying how the computation you did relates to your answer.
If not, write one of these vectors as a linear combination of the other three.

(c) What is the dimension of $\text{span}\{x_1, x_2, x_3, x_4\}$? ____________

22. Let $r_1 = (1, -1, 1, 1, 1)$, let $r_2 = (1, -5, 3, 7, -1)$, and let $p = (5, 1, 2, -4, 8)$.

   (a) Write p as a linear combination of r_1 and r_2.

   (b) Let $q = (2, 0, 1, -1, 3)$. Write q as a linear combination of r_1, r_2, and p or show that it is not a linear combination of them.

   (c) What is the dimension of the subspace spanned by r_1, r_2, p, and q? ____________
   Explain your answer!

23. Is the vector $v = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}$ in the nullspace of C, where $C = \begin{pmatrix} 1 & 1 & 2 & 1 \\ -1 & -1 & -1 & 1 \\ 6 & 1 & 3 & -2 \\ -1 & 1 & 1 & 1 \end{pmatrix}$?

   Explain why or why not!!

   Find a vector in the range of C whose first coordinate is 4, that is, find a vector w in range C

   such that $w = \begin{pmatrix} 4 \\ ? \\ ? \\ ? \end{pmatrix}$

24. Is the vector $v = \begin{pmatrix} 1 \\ -5 \\ 2 \\ 1 \end{pmatrix}$ in the nullspace of D, where $D = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -2 & -1 & -2 & 1 \\ 7 & 1 & 0 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$?

   Explain why or why not!!

   Find a vector in the range of D whose first coordinate is 3, that is, find a vector w in range D

   such that $w = \begin{pmatrix} 3 \\ ? \\ ? \\ ? \end{pmatrix}$

25. Suppose the vectors $\{u_1, u_2, u_3, u_4\}$ are a basis for $\mathbb{R}^4$.
   Let $y_1 = u_1 + 2u_2 - u_3$, $y_2 = u_2 + 2u_3 - u_4$, $y_3 = -2u_1 + u_2 + 2u_3 + 2u_4$ and $y_4 = 3u_1 + 5u_2 - 5u_3 + u_4$.
   Decide if $\{y_1, y_2, y_3, y_4\}$ are linearly independent or linearly dependent.

   If they are linearly dependent, write one of the vectors as a linear combination of the others.

26. (a) Suppose the vectors $\{x_1, x_2, x_3, x_4\}$ are a basis for $\mathbb{R}^4$.
   Let $y_1 = 2x_1 - 3x_4$, $y_2 = x_1 + 2x_2 - x_3$, $y_3 = x_2 + 2x_3 - x_4$ and $y_4 = x_1 - 7x_2 + x_3 - 5x_4$.

   (b) Decide if $\{y_1, y_2, y_3, y_4\}$ are linearly independent or linearly dependent.

   If they are linearly dependent, write one of the vectors as a linear combination of the others.

27. Suppose the vectors $\{v_1, v_2, v_3, v_4\}$ are a basis for $\mathcal{V}$ which is a subspace of $\mathbb{R}^7$.

(a) What is the dimension of $\mathcal{V}$? ______________

(b) Let $w_1 = v_1 - 3v_2 + v_4$, $w_2 = v_1 + 2v_2 - v_3$, $w_3 = 2v_1 + v_3 - 2v_4$, and $w_4 = v_2 + 2v_3 - v_4$. Are w_1, w_2, w_3, and w_4 linearly independent? If so, explain how you know, including justifying how the computation you did relates to your answer.
If not, write one of these vectors as a linear combination of the other three.

(c) What is the dimension of span$\{w_1, w_2, w_3, w_4\}$? ______________

28. Suppose C and D are 4×4 matrices and v_1, v_2, v_3, and v_4 are four linearly independent vectors in $\mathbb{C}^4$. Suppose, also, that

$$Cv_1 = 2v_1 \qquad Cv_2 = 0 \qquad Cv_3 = 2v_3 \qquad Cv_4 = -v_4$$

and

$$Dv_1 = v_1 - v_3 \qquad Dv_2 = 3v_2 \qquad Dv_3 = v_1 + 2v_3 \qquad Dv_4 = 2v_4$$

(a) Prove that for any vector w in $\mathbb{C}^4$, we have $CDw = DCw$.
(b) Find a basis for the range of DC.

29. Suppose K and L are 4×4 matrices and y_1, y_2, y_3, and y_4 are four linearly independent vectors in $\mathbb{C}^4$. Suppose, also, that

$$Ky_1 = -y_1 \qquad Ky_2 = 2y_2 \qquad Ky_3 = 0 \qquad Ky_4 = 2y_4$$

and

$$Ly_1 = 2y_1 \qquad Ly_2 = y_2 + y_4 \qquad Ly_3 = 3y_3 \qquad Ly_4 = y_2 - 2y_4$$

(a) Prove that for any vector z in $\mathbb{C}^4$, we have $KLz = LKz$.
(b) Find a basis for the range of LK.

30. An $n \times n$ matrix Q is called a *projection* if $Q^2 = Q$.

(a) Show that if P is a projection then $I - P$ is also a projection.

(b) Show that if P is a projection then $(I - 2P)^{-1} = I - 2P$.

31. (a) Prove the following theorem:

Let A be an $m \times n$ matrix. If B is an $n \times m$ matrix so that $BA = I$, then $\mathcal{N}(A) = (0)$ (that is, the null space of A is the zero subspace).

(b) Prove the following theorem:
(You may use the result of part (a) even if you did not answer part (a)!)

Let A be an $m \times n$ matrix.
If B is an $n \times m$ matrix so that $BA = I$, then the columns of A are linearly independent.