1. (Compare to Problem 4 on Midterm 2!)
(a) Give a counter-example to the assertion of 4(b), that is, show that there are sets A and B with $m^*(A+B) > m^*(A) + m^*(B)$.
(b) Is it true that for all subsets A and B of $\mathbb{R}$, that $m^*(A+B) \geq m^*(A) + m^*(B)$?

2. Suppose D is a dense set of real numbers, that is, for each open interval, I, the set $D \cap I$ is non-empty. Prove that if f is a real valued function on $\mathbb{R}$ such that for each α in D the set $\{x : f(x) > \alpha\}$ is measurable, then f is a measurable function.

3. Show that the set of measurable simple functions defined on the interval $[a, b]$ is an algebra, that is, show that if c is a real number and φ and ψ are simple functions, then $c\varphi$, $\varphi + \psi$, and $\varphi\psi$ are also simple functions.

4. Let $[a, b]$ be a finite closed interval of $\mathbb{R}$ and suppose g is a measurable, real-valued function on $[a, b]$. Show that for each $\epsilon > 0$, there is a measurable simple function φ on $[a, b]$ so that $m(\{x : |g(x) - \varphi(x)| \geq \epsilon\}) < \epsilon$. That is, measurable functions can be approximated by simple functions on most of their domain!

5. Let $[a, b]$ be a finite closed interval of $\mathbb{R}$ and suppose φ is a measurable simple function. Show that for each $\epsilon > 0$, there is a continuous function f on $[a, b]$ so that $m(\{x : |f(x) - \varphi(x)| \geq \epsilon\}) < \epsilon$. That is, simple functions can be approximated by continuous functions on most of their domain!

6. Define f on the unit interval by

$$f(x) = \begin{cases} 1 & \frac{3}{2^{n+2}} < x \leq \frac{1}{2^n} \text{ for } n = 0, 1, 2, \cdots \\ 0 & \frac{1}{2^{n+1}} < x \leq \frac{3}{2^{n+2}} \text{ for } n = 0, 1, 2, \cdots \\ 0 & x = 0 \end{cases}$$

Show that f is measurable, that f is Lebesgue integrable on $[0, 1]$, find $\int f$, and explain why this is a very easy problem given our definition of the Lebesgue integral.

7. In class we waved our hands over the Theorem that all Riemann integrable functions are Lebesgue integrable and the integrals are the same. Show that the function $f(x) = x^2$ is Lebesgue integrable on the interval $[0, 1]$ and use the definition of the Lebesgue integral as we have given it to find the value of the integral.

8. Show that the Cantor function is Lebesgue integrable on the interval $[0, 1]$ and use the definition of the Lebesgue integral as we have given it to find the value of the integral.