1. (Compare to Problem 4 on Homework 4) Suppose A is a subset of the $\mathbb{R}$ with $m^*(A) < \infty$ and suppose E is a G_δ set such that $E \supset A$ and $m^*(E) = m^*(A)$. Prove that A is measurable if and only if $m^*(E \setminus A) = 0$.

An alternate construction of the σ-algebra of measurable sets uses the concept of *inner measure*. In this construction for $\mathbb{R}$, outer measure is defined as we have defined it for subsets of $\mathbb{R}$ and the outer measure of open sets and compact sets is declared to be the *measure* of these sets, without proving any additional properties of measure (in contrast to outer measure) at this time, that is, if E is open or compact, then we write $m(E)$ instead of (only) $m^*(E)$.

Definition. If A is any subset of $\mathbb{R}$, the *inner measure of* A, denoted $m_*(A)$, is defined by

$$m_*(A) = \sup\{m(K) : K \subset A \text{ and } K \text{ is compact}\}$$

2. Suppose A is a subset of $\mathbb{R}$. Show that A is closed if and only if $A \cap [-n, n]$ is compact for every positive integer n.

3. (a) Show that for any subset A of $\mathbb{R}$, we have $m_*(A) \leq m^*(A)$.
(b) Suppose U is an open subset of $\mathbb{R}$. Show that $m_*(U) = m^*(U)$.

4. Suppose A and B are disjoint subsets of $\mathbb{R}$. Show that $m_*(A \cup B) \geq m_*(A) + m_*(B)$.

5. A set E is called an F_σ set if it is the union of a countable number of closed sets. Note that all F_σ sets are in the Borel σ-algebra.
(a) Prove that every open set in $\mathbb{R}$ is an F_σ set.
(b) Show that if A is a subset of $\mathbb{R}$, there is an F_σ set E so that $E \subset A$ and $m_*(A) = m_*(E)$.

6. Suppose $m^*(A) < \infty$. Prove that the set A is measurable if and only if $m_*(A) = m^*(A)$.

Solution to Problem 5b. Suppose first that $m_*(A) < \infty$. For each positive integer n, there is a compact set K_n with $K_n \subset A$ and $m(K_n) > m_*(A) - \frac{1}{n}$. Let $E = \cup_{n=1}^{\infty} K_n$. Then E is an F_σ set, $E \subset A$, and for each positive integer n,

$$m_*(A) - \frac{1}{n} \le m(K_n) \le m_*(E) \le m_*(A)$$

Since this is true for each positive integer n, we must have $m_*(A) \le m_*(E) \le m_*(A)$, or $m_*(E) = m_*(A)$ as we wished to prove. (Remark: notice that because E is measurable, we have

$$m_*(A) - \frac{1}{n} \le m(K_n) \le m(E) \le m_*(A)$$

so that $m(E) = m_*(A)$.)

If $m_*(A) = \infty$, an analogous argument works with K_n satisfying $K_n \subset A$ and $m(K_n) > n$.

Solution to Problem 6.

First, notice that the revision $m^*(A) < \infty$ is important:

Suppose P is a non measurable subset of $[0,1)$ as we constructed Tuesday. Let $A = P \cup [3, \infty)$. Then $m_*(A) = m^*(A) = \infty$, but A is not measurable because if it were, $A \cap [0,1) = P$ would be measurable, which it is not.

To prove the equivalence, first suppose $m_*(A) = m^*(A) < \infty$. By problem 5b (and the remark) above and problem 4 on Homework 4, there are an F_σ set F and a G_δ set G so that $F \subset A \subset G$ and $m(F) = m_*(A) = m^*(A) = m(G)$. Now, F, G, and $G \setminus F = G \cap F^c$ are all measurable and $G = F \cup (G \cap F^c)$. Since the latter two sets are disjoint, we have $m(G) = m(F) + m(G \cap F^c) = m(G) + m(G \cap F^c)$ which means $m(G \cap F^c) = 0$. Now $A \cap F^c \subset G \cap F^c$, so $m^*(A \cap F^c) \le m^*(G \cap F^c) = 0$, so actually $A \cap F^c$ is measurable and $m(A \cap F^c) = 0$. This means that $A = F \cup (A \cap F^c)$ is measurable and $m(A) = m(F) = m(G)$.

Conversely, suppose A is measurable and $m(A) = m^*(A) < \infty$. For each positive integer n, let $A_n = A \cap [-n, n]$ so that each A_n is measurable, $A_1 \subset A_2 \subset A_3 \subset \cdots$ and

$$A = \cup_{k=1}^{\infty} A_k = A_n \cup (\cup_{k=n}^{\infty} (A_{k+1} \setminus A_k))$$

where the latter union is a union of disjoint measurable sets. This means that, for each n, we have $m(A) = m(A_n) + \sum_{k=n}^{\infty} m(A_{k+1} \setminus A_k)$, so $m(A) = \lim_{n \to \infty} m(A_n)$. We can use A_n to get a compact subset of A_n, hence a subset of A, whose measure is close to $m(A_n)$, that is, close to $m(A)$, which gives an estimate for the inner measure of A that is nearly $m(A_n)$.

The set $[-n, n] \setminus A_n$ is measureable, so we can find a covering of $[-n, n] \setminus A_n$ by open intervals (I_j) so that $\sum \ell(I_j) < m^*([-n, n] \setminus A_n) + \frac{1}{n} = m([-n, n] \setminus A_n) + \frac{1}{n} = 2n - m(A_n) + \frac{1}{n}$. Now, let $K_n = [-n, n] \setminus (\cup I_j) = [-n, n] \cap (\cap I_j^c)$. Clearly, K_n is compact, $K_n \subset A_n$ and $m(K_n) \ge m(A_n) - \frac{1}{n}$. Thus, $K_n \subset A$ for each positive integer n, so $m(K_n) \le m(A)$ and $\lim_{n \to \infty} m(K_n) = \lim_{n \to \infty} m(A_n) = m(A)$. This means that $m_*(A) = m(A) = m^*(A)$, as we were to prove.