1. Let $\mathbf{Q}_0$ be the set of rational numbers in $[0, 1]$ and suppose $\{I_n\}_{n=1}^N$ is a *finite* collection of open intervals covering $\mathbf{Q}_0$. Prove that $\sum_{n=1}^N \ell(I_n) > 1$. (That is, the definition of outer measure critically depends on using a countably infinite cover.)

2. Suppose A and B are subsets of $\mathbb{R}$. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

3. A set E is called a G_δ set if it is the intersection of a countable number of open sets. Note that all G_δ sets are in the Borel σ-algebra.

Prove that every compact set in $\mathbb{R}$ is a G_δ set. (It is also true that every closed set in $\mathbb{R}$ is a G_δ set, so you may choose to do that if you find it more convenient.)

4. Show that if A is a subset of $\mathbb{R}$, there is a G_δ set E so that $A \subset E$ and $m^*(A) = m^*(E)$.

5.

(a) Prove directly from the definition of outer measure that $m^*(\mathbf{C}) = 0$, where $\mathbf{C}$ is the Cantor set. (That is, do not use the fact that the Cantor set is measurable.)

(b) Using measurability, find an easier proof that $m^*(\mathbf{C}) = 0$.

6. Find a sequence of measurable sets $(E_n)_{n=1}^\infty$ so that $E_1 \supset E_2 \supset E_3 \supset \cdots$, and $m(E_n) = \infty$ for all n, but that $\cap_{n=1}^\infty E_n = \emptyset$.