1. Let n_0, n_1, n_2, n_3, $\cdots$ be an increasing sequence of distinct integers with $n_0 = 1$. Given a series $\sum_{n=1}^{\infty} a_n$, we define a new series $\sum_{m=1}^{\infty} b_m$ by letting $b_m = \sum_{j=n_{m-1}}^{n_m - 1} a_j$ for each positive integer m. The series $\sum_{m=1}^{\infty} b_m$ is just the series $\sum_{n=1}^{\infty} a_n$ with parentheses inserted to group the terms.

(a) Suppose $\sum_{n=1}^{\infty} a_n$ is a series such that $\lim_{n\to\infty} a_n = 0$ and suppose there is a number M so that $n_j - n_{j-1} \leq M$ for each positive integer j. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{m=1}^{\infty} b_m$ converges and that when they both converge, they converge to the same sum.

(b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ such that for the sequence of integers $n_j = 2j - 1$, the corresponding series $\sum_{m=1}^{\infty} b_m$ converges.

Dfn. A *rearrangement of the series* $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{m=1}^{\infty} c_m$ where $c_m = a_{\pi(m)}$ for some permutation π of the positive integers, that is, π is a one-to-one map of the positive integers onto the positive integers. Thus, a rearrangement of a series is a new series with the same terms, but in different order.

2. Show that if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement of the series, $\sum_{m=1}^{\infty} c_m$, also converges absolutely and $\sum_{m=1}^{\infty} c_m = \sum_{n=1}^{\infty} a_n$.

3. Suppose π is a permutation of the positive integers such that there is a number M so that $|\pi(j) - j| \leq M$. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series and $\sum_{m=1}^{\infty} c_m$ is the rearrangement determined by π, then $\sum_{m=1}^{\infty} c_m$ also converges and $\sum_{m=1}^{\infty} c_m = \sum_{n=1}^{\infty} a_n$.

4. Let $\mathcal{A}_0 = \{f \in C([0,1]) : f(0) = 0\}$. (It is not hard to show that $\mathcal{A}_0$ is a closed subalgebra of $C([0,1])$, and you may use that fact without proof, if you wish.) Let $\mathcal{P}_0$ denote the algebra of polynomials such that $p(0) = 0$. Show that $\mathcal{P}_0$ is dense in $\mathcal{A}_0$, that is, given f in $\mathcal{A}_0$, show that for each $\epsilon > 0$, there is a polynomial p in $\mathcal{P}_0$ so that $|f(x) - p(x)| < \epsilon$ for all x in $[0,1]$.

5. Let $\mathcal{A}_{00} = \{f \in C([0,1]) : f(0) = f'(0) = 0\}$.

(a) Show that $\mathcal{A}_{00}$ is an algebra (or 'function algebra' in the terminology of Pugh), that is, show that $\mathcal{A}_{00}$ is closed under addition, scalar multiplication, and function multiplication.

(b) Let $\mathcal{P}_{00}$ denote the algebra of polynomials such that $p(0) = p'(0) = 0$. Show that $\mathcal{P}_{00}$ is dense in $\mathcal{A}_{00}$, that is, given f in $\mathcal{A}_{00}$, show that for each $\epsilon > 0$, there is a polynomial p in $\mathcal{P}_{00}$ so that $|f(x) - p(x)| < \epsilon$ for all x in $[0,1]$.

(c) Is $\mathcal{A}_{00}$ a closed subalgebra of $C([0,1])$?