Dfn. The *Cantor set*, $\mathbf{C}$, is the set of real numbers in $[0,1]$ that *can be* represented in base 3 by a ternary expansion using only the digits 0 and 2. For example, $\frac{1}{3} = (.1000\cdots)_3 = (.0222\cdots)_3$ and $\frac{2}{3} = (.2000\cdots)_3 = (.1222\cdots)_3$ are both in $\mathbf{C}$, but $\frac{7}{12} = (.1202020\cdots)_3$ is not.

Dfn. The *Cantor function*, $c(x)$, is the function on $[0,1]$ defined by

$$c(x) = \begin{cases} (.\frac{d_1}{2}\frac{d_2}{2}\frac{d_3}{2}\cdots)_2 & x \in \mathbf{C},\ x = (.d_1d_2d_3\cdots)_3, \text{where each } d_j \neq 1 \\ c(x_\ell) = c(x_r) & x_\ell = \max\{y \in \mathbf{C} : y < x\} \text{ and } x_r = \min\{y \in \mathbf{C} : x < y\} \end{cases}$$

That is, for x in the Cantor set, $c(x)$ has binary expansion obtained by dividing each digit in the ternary expansion of x by 2. For example, $c(\frac{1}{3}) = c(\frac{7}{12}) = c(\frac{2}{3}) = \frac{1}{2}$.

1. Show that the Cantor function is continuous.
(**Hint:** Use ternary expansions with δ and binary expansions with ϵ.)

2. Suppose s_n, for $n = 1, 2, 3, \cdots$ is a sequence of real numbers and $\lim_{n\to\infty} s_n = S$. For each positive integer n, let σ_n be defined by

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + s_3 + \cdots + s_n)$$

Prove that $\lim_{n\to\infty} \sigma_n = S$ also.

3. Decide whether each of the following series converges or diverges, and prove your answer.

(a) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$$

(c) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{2}}$$

(d) $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

4. Show that the series

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots$$

converges, but the Ratio and Root tests do not apply.