Math 300: Number Facts We Know and Number Assumptions

The Numbers

We know about four kinds of numbers: integers, rational numbers, real numbers, and complex numbers, we know about the operations of addition, subtraction, multiplication, and (for rational, real, and complex numbers) division, and we know some ways to represent the numbers. We also know about the order properties of the real numbers.

integers: The integers are the counting numbers (natural numbers), zero, and their negatives.

rational numbers: Rational numbers are those numbers that can be expressed as the quotient of two integers. For example, $\frac{-3}{4}$ and $2.625 = 2\frac{5}{8} = \frac{21}{8}$ are rational, but $\sqrt{2}$ and $\sqrt{5}$ are not rational, as we have seen. All integers are rational numbers: the integer n can be written as $n = \frac{n}{1}$.

real numbers: Real numbers are those numbers that can be expressed in decimal form (i.e. base 10) with a finite number of digits to the left of the decimal point. For example, $\frac{-3}{4} = -.75$ and $2\frac{2}{3} = \frac{8}{3} = 2.6666\cdots$ are real numbers. Every rational number is a real number.

complex numbers: Complex numbers are those numbers that can be expressed as $a + bi$, where $i = \sqrt{-1}$ and a and b are real numbers. Every real number is a complex number: if r is a real number, $r = r + 0i$.

We know how to do arithmetic with any of these kinds of numbers, and we know their properties. Suppose a, b, and c are numbers of the same type, either a, b, and c are integers, or a, b, and c are rational numbers, or a, b, and c are real numbers, or a, b, and c are complex numbers, then the following laws are satisfied:

commutative: Addition and multiplication are commutative. For all numbers a and b,

$$a + b = b + a \quad \text{and} \quad ab = ba$$

associative: Addition and multiplication are associative. For all numbers a, b, and c,

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a(bc) = (ab)c$$

distributive: Multiplication distributes over addition. For all numbers a, b, and c,

$$a(b + c) = ab + ac$$

identities: The number 0 is the identity for addition and the number 1 is the identity for multiplication. For all numbers a,

$$a + 0 = a \quad \text{and} \quad a1 = a$$

inverses: For each number, a, there is an additive inverse for a, that is, there is a number "$-a$" such that $a + (-a) = 0$. For each rational number, a, or each real number a, if $a \neq 0$, there is a multiplicative inverse for a, that is, there is a number "a^{-1}" (also denoted "$\frac{1}{a}$") such that $aa^{-1} = a\frac{1}{a} = 1$.

We also know about the order properties of the real numbers. For all real numbers a and b, either $a < b$ or $a = b$ or $a > b$. Furthermore, if $c > 0$ and $a < b$, then $ac < bc$, but if $c < 0$ and $a < b$, then $ac > bc$.

Some Important Facts about the Numbers

For the time being, we will accept the following facts about numbers without proof, but we may, later, decide to prove some of them.

Theorem 1 *If a and b are numbers such that $ab = 0$, then either $a = 0$ or $b = 0$.*

Theorem 2 *If a and b are integers such that ab is divisible by the prime number p, then either a is divisible by p or b is divisible by p.*

Theorem 3 (Fundamental Theorem of Arithmetic) *Each integer greater than 1 is a product of prime numbers. Further, any two decompositions of such an integer into products of primes differ only in the order of the factors.*

Theorem 4 *If m and n are positive integers, there are unique non-negative integers q and r, with $0 \leq r < n$ such that*

$$m = nq + r$$

Theorem 5 *If a and b are rational numbers such that $a < b$, then there is a rational number c such that $a < c$ and $c < b$.*

Corollary 6 *There is no smallest, positive rational number. There is no largest, positive rational number.*

Theorem 7 *If a and b are real numbers such that $a < b$, then there is a rational number c such that $a < c$ and $c < b$.*

Corollary 8 *There is no smallest, positive real number. There is no largest, positive real number.*