NAME: Some Solutions

Math 165 (Cowen) **Test 5** **27 April 2009**

There are 6 pages, 9 questions, and 100 points on this test. **The test finishes at 1:15pm!** Follow the instructions for each question and show enough of your work that I can understand what you are doing.

(10 points) **1.** Find the area in first quadrant (as a number) bounded by the curve $y = 9x - x^3$.

$\left.\begin{array}{l} y = 9x - x^3 \\ y = 0 \end{array}\right\} \Rightarrow$ intercepts are at

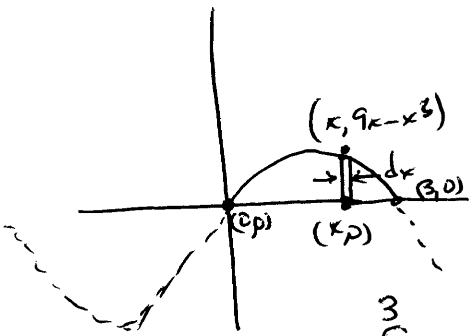
$9x - x^3 = 0$

$x(9 - x^2) = 0$

$x(3 - x)(3 + x) = 0$

So curve crosses at

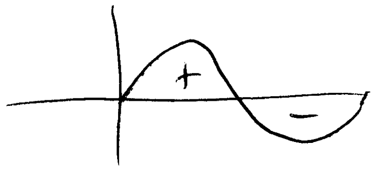
$x = 0,\ x = 3,\ x = -3$



$$\text{Area} = \int_{x=0}^{3} \left((9x - x^3) - 0\right) dx$$

$$= \frac{9}{2}x^2 - \frac{x^4}{4}\Big|_0^3 = \left(\frac{9}{2}(3)^2 - \frac{(3)^4}{4}\right) - \left(\frac{9}{2}0^2 - \frac{0^4}{4}\right) = \frac{81}{2} - \frac{81}{4} = \underline{\underline{\frac{81}{4}}}$$

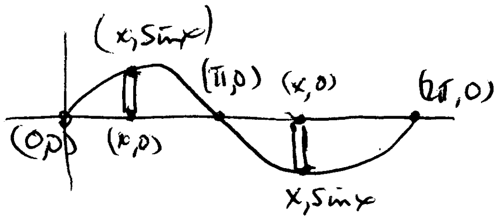
(10 points) **2.**(a) In terms of areas, what does the integral $\int_0^{2\pi} \sin(x)\, dx$ represent?



The area between $y = \sin x$ and $y = 0$ with the area above the axis counted positive and the area below counted as negative: the net area above the axis.

(b) Find the area, as a positive number, *between* the curve $y = \sin(x)$ and the x-axis (that is, $y = 0$), between $x = 0$ and $x = 2\pi$. (For example, this would represent the area of carpet needed to cover a part of the floor in the shape of the region between the x-axis and the curve.)

The actual area is $\displaystyle\int_0^{\pi} (\sin x - 0)\, dx + \int_{\pi}^{2\pi} (0 - \sin x)\, dx$



$$= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi}$$

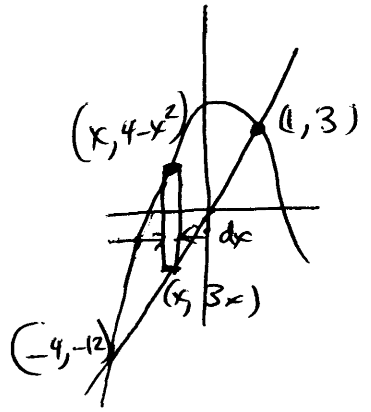
$$= (-\cos\pi) - (-\cos 0) + \cos 2\pi - \cos\pi$$

$$= -(-1) + (1) + (1) - (-1) = 4$$

(20 points) **3.**(a) Use integration based on vertical rectangles to find the area, as a number, between the curves $x^2 + y = 4$ and $3x - y = 0$. Show the integral(s) explicitly, as well as the calculations, used to find your answer.

$y = 4 - x^2$ a parabola opening downward

$y = 3x$ a line through the origin



intersections: $3x = y = 4 - x^2$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = -4$ and $x = 1$

$y = -12$ $\quad y = 3$

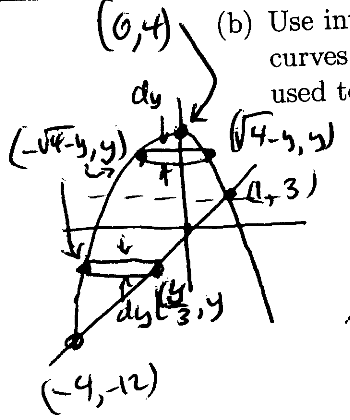
$$\text{Area} = \int_{-4}^{1} \left((4-x^2) - 3x\right) dx = 4x - \frac{x^3}{3} - \frac{3}{2}x^2 \Big|_{-4}^{1} =$$

$$\left(4(1) - \frac{1^3}{3} - \frac{3}{2}1^2\right) - \left(4(-4) - \frac{(-4)^3}{3} - \frac{3}{2}(-4)^2\right)$$

$$= 4 - \frac{1}{3} - \frac{3}{2} + 16 - \frac{64}{3} + \frac{3}{2}16 = 4 - \frac{65}{3} - \frac{3}{2} + 16 + 24 =$$

$$20\frac{5}{6}$$

(b) Use integration based on horizontal rectangles to find the area, as a number, between the curves $x^2 + y = 4$ and $3x - y = 0$. Show the integral(s) explicitly, as well as the calculations, used to find your answer.



dy means we need x in terms of y

$x^2 = 4 - y$

$x = \sqrt{4-y}$ right half $\qquad x = -\sqrt{4-y}$ left half

$3x = y$

$x = \frac{y}{3}$

Two integrals because sometimes rectangles have end on line, sometimes ends both on parabola

$$\int_{-12}^{3} \left(\frac{y}{3} - (-\sqrt{4-y})\right) dy + \int_{3}^{4} \sqrt{4-y} - (-\sqrt{4-y})\, dy =$$

$$= \int_{-12}^{3} \frac{y}{3} + (4-y)^{1/2} dy + \int_{3}^{4} 2(4-y)^{1/2} dy = \frac{y^2}{6} + \frac{2}{3}\left(-(4-y)^{3/2}\right)\Big|_{-12}^{3} + 2\frac{2}{3}\left(-(4-y)^{3/2}\right)\Big|_{3}^{4} =$$

$$= \left[\left(\frac{3^2}{6} + \frac{2}{3}(-1^{3/2})\right) - \left(\frac{(-12)^2}{6} + \frac{2}{3}(-(16)^{3/2})\right)\right] + \left[\frac{4}{3}(-0^{3/2}) - \frac{4}{3}(-(1)^{3/2})\right] =$$

$$= \frac{9}{6} - \frac{2}{3} - \frac{144}{6} + \frac{2}{3}64 + 0 + \frac{4}{3} = \frac{3}{2} + \frac{130}{3} - 24 = 43\frac{1}{3} - 22\frac{1}{2} = 20\frac{5}{6}$$

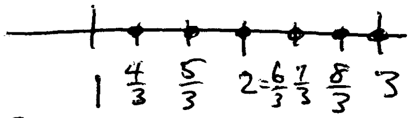
(10 points) **4.** Find an approximate value for the integral

$$\int_1^3 \frac{3x}{x+2}\,dx$$

by calculating a Riemann sum using 6 subintervals and the right end points of each of the subintervals for evaluating the function.

• dots

$\frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$ length of each subinterval



$f(x) = \frac{3x}{x+2}$

$$\text{Riemann sum} = \frac{1}{3}\left[\left(\frac{3\left(\frac{4}{3}\right)}{\frac{4}{3}+2}\right)+\left(\frac{3\left(\frac{5}{3}\right)}{\frac{5}{3}+2}\right)+\left(\frac{3\left(\frac{6}{3}\right)}{\left(\frac{6}{3}+2\right)}\right)+\left(\frac{3\left(\frac{7}{3}\right)}{\frac{7}{3}+2}\right)+\left(\frac{3\left(\frac{8}{3}\right)}{\frac{8}{3}+2}\right)+\left(\frac{3\left(\frac{9}{3}\right)}{\frac{9}{3}+2}\right)\right]$$

(10 points) **5.** Charles is thinking about the integral

$$\int_0^2 \frac{r}{4r^4-4r^2+2}\,dr = \int_0^2 \frac{r}{(2r^2-1)^2+1}\,dr$$

and decides it would be easier if he made a substitution. Do the substitution $p = 2r^2-1$, that is, find a definite integral with the new variable p that gives the same calculation as the integral above after making the substitution $p = 2r^2-1$. (Do NOT evaluate either integral.)

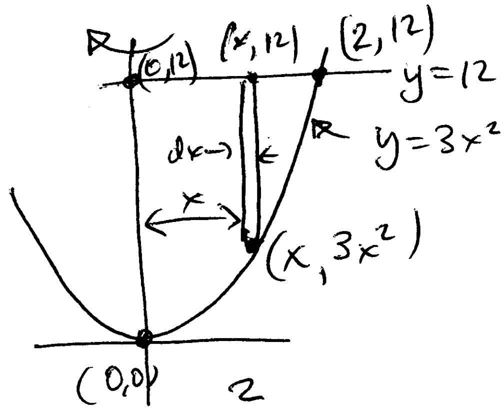
$$\int_{r=0}^{r=2} \frac{r}{(2r^2-1)^2+1}\,dr = \int_{-1}^{7} \frac{\frac{1}{4}dp}{p^2+1} = \int_{-1}^{7} \frac{dp}{p^2+1}$$

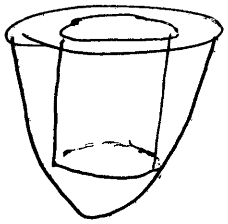
$p = 2r^2-1$ $\qquad r=0 \quad p=-1$

$dp = 4r\,dr$ $\qquad r=2 \quad p=7$

$\frac{1}{4}dp = r\,dr$

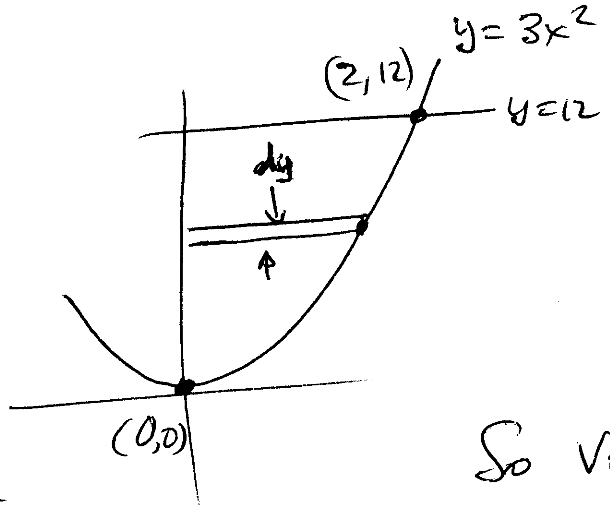
(10 points) **6. Set up** an integral (you do NOT need to evaluate the integral) to find the volume (a paraboloid) obtained by revolving, around the y-axis, the region in the plane bounded by $y = 3x^2$ and $y = 12$ and having $x \geq 0$. (Note that the regions in problems 6 and 7 are the same.)

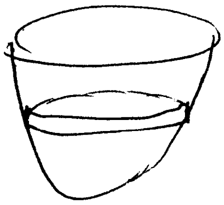




$$\int_{x=0}^{2} 2\pi x \left(12-3x^2\right) dx$$

OR





$y = 3x^2$

$\frac{y}{3} = x^2$

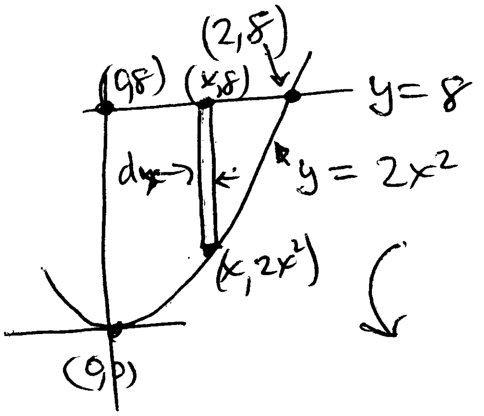
$x = \sqrt{\frac{y}{3}}$ or $x = -\sqrt{\frac{y}{3}}$

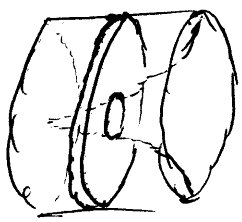
right half

So Volume is

$$\int_{y=0}^{12} \pi \left(\sqrt{\frac{y}{3}}\right)^2 dy$$

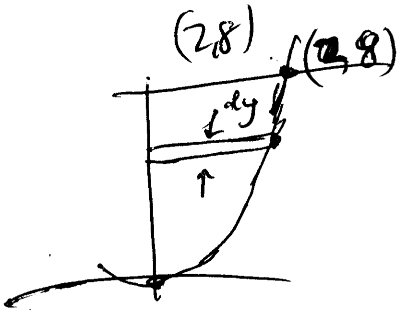
(10 points) **7. Set up** an integral (you do NOT need to evaluate the integral) to find the volume (a cylinder with hole in the center) obtained by revolving, around the x-axis, the region in the plane bounded by $y = 2x^2$ and $y = 8$ and having $x \geq 0$. (Note that the regions in problems 6 and 7 are the same.)

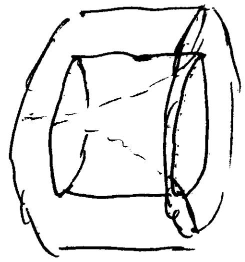




$$\text{Volume (washers)} = \int_{x=0}^{2} \pi\left(8^2 - (2x^2)^2\right) dx$$

OR





$y = 2x^2$

$\frac{y}{2} = x^2$

$x = \sqrt{\frac{y}{2}}$

$$\text{Volume (cylinders)} = \int_{y=0}^{8} 2\pi y\left(\sqrt{\frac{y}{2}} - 0\right) dy$$

(10 points) **8.** Find the function f that satisfies $f'(x) = 6x + \sqrt{2x+1}$ and $f(4) = 50$.

$$f'(x) = 6x + (2x+1)^{1/2}$$

$$f(x) = 3x^2 + \frac{1}{2}\frac{2}{3}(2x+1)^{3/2} + C$$

$$f(x) = 3x^2 + \frac{1}{3}(2x+1)^{3/2} + C$$

$$50 = f(4) = 3(4^2) + \frac{1}{3}(2\cdot 4+1)^{3/2} + C = 48 + \frac{1}{3}27 + C$$

$$50 = 48 + 9 + C = 57 + C$$

$$C = -7$$

so $f(x) = 3x^2 + \frac{1}{3}(2x+1)^{3/2} - 7$

(10 points) **9.** $h(x) = \int_{2x}^{x^3+1} \sqrt{t^2+2}\, dt$

Find $h'(x)$.

we know (FTC) if $p(v) = \int_5^v \sqrt{t^2+2}\,dt$ then $p'(v) = \sqrt{v^2+2}$

$$h(x) = \int_{2x}^{x^3+1} \sqrt{t^2+2}\,dt = \int_{2x}^{5} \sqrt{t^2+2}\,dt + \int_{5}^{x^3+1} \sqrt{t^2+2}\,dt$$

so $h(x) = \int_5^{x^3+1} \sqrt{t^2+2}\,dt - \int_5^{2x} \sqrt{t^2+2}\,dt$

Using this result and the chain rule we get

$$h'(x) = \left(\sqrt{(x^3+1)^2+2}\right)(3x^2) - \sqrt{(2x)^2+2}\,(2)$$